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Photon Transport Theory*

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A first order, momentum-configuration space transport equation for photons is derived for low energy (nonrelativistic) systems. The derivation is first order in the sense that the transition probabilities characterizing photon scattering emission and absorption are computed only to the first nonvanishing order by conventional perturbation methods.

The present approach provides an essentially axiom-deduction development of the theory of radiative transfer (albeit via several ill-evaluated approximations) within the context of which various processes and their interrelationships may be investigated. Most of these processes have hitherto been studied only phenomenologically and usually piecemeal. Specific application to photon scattering, cyclotron radiation, recombination radiation, de-excitation radiation, and bremsstrahlung is made in the text.

The derivation of an H-theorem for photon-particle systems is sketched; and contact is made with the usual statistical mechanical treatment of the equilibrium states of such systems.

It is also shown that some aspects of collective particle behavior can be introduced quite naturally into the description of photon transport in the fully ionized plasma.

I. INTRODUCTION

It is the purpose of this paper to present in considerable detail some of the formal aspects of "first-order" photon transport theory. By "first order" we imply that an explicit calculation of the effects of specific physical processes on photon balance shall be restricted to first- (nonvanishing) order perturbation theory. This (as well as some other more subtle considerations to be discussed in detail later) seems to suggest that the validity of the subsequent analysis increases as the particle densities in the systems of interest decrease and as the importance of collective (coherent) particle behavior decreases. However, at this stage, it is perhaps unwise to attempt to formulate so simple a criterion of validity, as any such attempt is apt to be too stringent. For example, the equation whose derivation and implications are the concern of the present investiga-

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tion has been employed extensively in the study of radiation transport in stellar systems (*1*) as well as in fission reactors (*2*). Thus we shall concern ourselves very little with such questions, but rather shall present in clearly stated operational terms a set of *sufficient* conditions in the context of which the equation of interest is expected to be useful.

Some of the material to be presented herein was initially, but sketchily, developed in an earlier work (*3*) (hereafter referred to as I), particularly that of Section II in which we present the basic statement of our approach to the problem and a derivation of an equation of photon balance. This inclusion of repetitive detail is for the purpose of completeness as well as to illuminate some subtleties that the earlier treatment glossed over.

In Section III we discuss briefly and in somewhat general terms some aspects of the thermodynamics of systems of interacting particles and photons. In Section IV we obtain explicit formulas for the transition probabilities (or cross-sections) germane to the description of photon balance in partially or completely ionized gases in the presence of externally applied, constant, uniform magnetic fields. In particular, it will be interesting to note that the results of Drummond and Rosenbluth's (*4*) calculations of cyclotron radiation losses from hot plasmas are contained nicely in the present "first-order" treatment, as well as estimates of de-excitation and electron-ion recombination radiation losses. The emission and absorption of radiation by bremsstrahlung is also accounted for in the sense of the Born approximation, as well as photon scattering—which in the present nonrelativistic treatment reduces to Thomson scattering. Finally, in Section V, it is shown that some aspects of the effect of collective particle behavior upon photon transport enters the theory quite naturally when dealing with fully ionized plasmas. Specifically, it is found that—in such instances—the photons of momentum $\hbar k$ propagate between successive events with phase velocity $c(1 + \omega_p^2/2c^2k^2)$, where $\omega_p^2 = 4\pi ne^2/m$ is the usual "plasma frequency."

It is to be emphasized that none of the results of the present treatment are original though the approach to radiation transport problems as developed herein is, so far as the authors know, new.

II. DERIVATION OF A PHOTON BALANCE RELATION

The Hamiltonian to be employed in the present investigation is the same as the one presented in (I), i.e.,

$$H = T^{\gamma} + T^{p} + H^{pe} + V + H^{p\gamma 1} + H^{p\gamma e} + H^{p\gamma 2}, \tag{1}$$

where

$$T^{\gamma} = \int d^3x\, [2\pi c^2 \mathbf{P}^2 + (1/8\pi)\, (\nabla \times \mathbf{A})^2], \tag{2a}$$

$$T^p = \sum_\sigma \frac{\hbar^2}{2m_\sigma} \int d^3x\ (\nabla\psi_\sigma^+)\cdot(\nabla\psi_\sigma), \tag{2b}$$

$$H^{pe} = \sum_\sigma \left[\frac{ie_\sigma\hbar}{m_\sigma c} \int d^3x\ \psi_\sigma^+ \mathbf{A}^e\cdot\nabla\psi_\sigma + \frac{e_\sigma^2}{2m_\sigma c^2}\int d^3x\ A^{e^2}\psi_\sigma^+\psi_\sigma\right], \tag{2c}$$

$$V = \sum_{\sigma\sigma'} \frac{e_\sigma e_{\sigma'}}{1+\delta_{\sigma\sigma'}} \int d^3x\ d^3x'\ \frac{\psi_\sigma^+(x)\psi_{\sigma'}^+(x')\psi_\sigma(x)\psi_{\sigma'}(x')}{|x-x'|}, \tag{2d}$$

$$H^{p\gamma 1} = \sum_\sigma \frac{ie_\sigma\hbar}{m_\sigma c}\int d^3x\ \mathbf{A}\cdot\psi_\sigma^+\nabla\psi_\sigma, \tag{2e}$$

$$H^{p\gamma e} = \sum_\sigma \frac{e_\sigma^2}{m_\sigma c^2}\int d^3x \psi_\sigma^+\psi_\sigma\ \mathbf{A}^e\cdot\mathbf{A}, \tag{2f}$$

$$H^{p\gamma 2} = \sum_\sigma \frac{e_\sigma^2}{2m_\sigma c^2}\int d^3x\ \psi_\sigma^+\psi_\sigma\ A^2, \tag{2g}$$

For some subsequent purposes it will be convenient to regroup some of the terms in the Hamiltonian as follows:

$$T^p + H^{pe} = \sum_\sigma \frac{1}{2m_\sigma}\int d^3x\ (\mathbf{\Pi}^{\sigma*}\psi_\sigma^+)\cdot(\mathbf{\Pi}^\sigma\psi_\sigma), \tag{3a}$$

$$H^{p\gamma 1} + H^{p\gamma e} = -\sum_\sigma \frac{e_\sigma}{2m_\sigma c}\int d^3x\ [(\mathbf{\Pi}^{\sigma*}\psi_\sigma^+)\cdot\mathbf{A}\psi_\sigma + \mathbf{A}\cdot\psi_\sigma^+(\mathbf{\Pi}^\sigma\psi_\sigma)], \tag{3b}$$

where we have introduced the notation

$$\mathbf{\Pi}^\sigma = -i\hbar\nabla - (e_\sigma/c)\mathbf{A}^e. \tag{4}$$

In (2) (or 3), ψ_σ is a wave operator for particles of kind σ; $\mathbf{A}^e$ is the divergenceless vector potential for an externally applied electromagnetic field, and $\mathbf{A}$ and $\mathbf{P}$ are the canonically conjugate wave operators for the photon field (*5*). Note that $\nabla\cdot\mathbf{A} = \nabla\cdot\mathbf{P} = 0$.

For calculational purposes it is convenient to transform to momentum space for the photons according to

$$\mathbf{A} = \sqrt{\frac{2\pi\hbar c}{V}}\sum_{\mathbf{k}\lambda}\frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{k}}\zeta_\lambda^+(\mathbf{k}), \tag{5a}$$

$$\mathbf{P} = i\sqrt{\frac{\hbar}{8\pi c V}}\sum_{\mathbf{k}\lambda}\sqrt{k}e^{-i\mathbf{k}\cdot\mathbf{x}}\zeta_\lambda^-(\mathbf{k}), \tag{5b}$$

where

$$\zeta_\lambda^\pm(\mathbf{k}) = \alpha_\lambda^+(\mathbf{k})\ \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \pm \alpha_\lambda(-\mathbf{k})\ \boldsymbol{\varepsilon}_\lambda(-\mathbf{k}). \tag{6}$$

The $\alpha_\lambda(\mathbf{k})$ and $\alpha_\lambda^+(\mathbf{k})$ are destruction and creation operators for photons of momentum $\mathbf{k}$ and polarization λ, and $\boldsymbol{\varepsilon}_\lambda(\mathbf{k})(\lambda = 1, 2)$ are the unit polarization vectors of the photon field. The volume of quantization is designated by V and the sum over $\mathbf{k}$ is the usual sum over the integers permitted by the requirement that $\mathbf{A}$ and $\mathbf{P}$ be periodic on the boundaries of V. The creation and destruction operators for photons obey the commutation rules

$$[\alpha_\lambda(\mathbf{k}), \alpha_{\lambda'}^{+}(\mathbf{k}')]_- = \delta_{\lambda\lambda'}\delta(\mathbf{k} - \mathbf{k}'), \tag{7}$$

whereas the wave operators for the particles will obey the rules

$$[\psi_\sigma(\mathbf{x}), \psi_{\sigma'}^{+}(\mathbf{x}')]_\mp = \delta_{\sigma\sigma'}\delta(\mathbf{x} - \mathbf{x}'), \tag{8}$$

depending on whether ψ_σ represents a boson or fermion field. We employ the same notation for Dirac and Kronecker deltas, letting the context reveal which interpretation of the symbol is appropriate in a given case.

Again as in I, we introduce a singlet photon density according to the definition

$$\chi_\lambda(\mathbf{x}, \mathbf{k}, t) = (8/V) \sum_{\mathbf{q}} e^{-2i\mathbf{x}\cdot\mathbf{q}}(F, \rho_\lambda(\mathbf{k}, \mathbf{q})F), \tag{9}$$

where

$$\rho_\lambda(\mathbf{k}, \mathbf{q}) = \alpha_\lambda^{+}(\mathbf{k} + \mathbf{q})\alpha_\lambda(\mathbf{k} - \mathbf{q}), \tag{10}$$

and F is the state vector for the system which satisfies the Schrödinger equation,

$$HF = i\hbar\partial F/\partial t. \tag{11}$$

The sense in which χ_λ is to be interpreted as a density function is discussed in I, as well as the interesting question as to its statistical significance. It should be noted that the present introduction of a photon density in configuration and momentum space is somewhat in contrast to previous treatments of photon distributions (*6*, *7*). Analogous density functions for the particles were also introduced and discussed in some detail in (I), but we shall not be concerned with such densities in the present investigation.

To find an equation for the photon density, we introduce a method of temporal coarse graining which bears some formal resemblance to that employed by Mori and Ross (*8*) in their development of a transport equation for short-range-force gases. For convenience, we first rewrite the relation (9) as

$$\chi_\lambda(t) = (8/V) \sum_{\mathbf{q}} e^{-2i\mathbf{x}\cdot\mathbf{q}} TrD(t)\rho_{\lambda t}(\mathbf{k}, \mathbf{q}), \tag{12}$$

where $D(t)$ is the density matrix for the system in the photon interaction representation. Explicitly, if F is the state vector defined by Eq. (11), and if G is a new representation related to F by

$$F = UG, \tag{13}$$

where

$$U = \exp [-iT^{\gamma}t/\hbar], \tag{14}$$

then

$$D_{n\eta,n'\eta'}(t) = b^{*}_{n'\eta'}b_{n\eta}, \tag{15}$$

where the b's are given by

$$\begin{aligned} b_{n\eta}(t) &= <n\eta|G> \\ &= <n\eta|U^{-1}F>. \end{aligned} \tag{16}$$

Thus the b's are simply the coefficients of the expansion of the state vector G in terms of the set of base vectors $\{|n\eta>\}$. This set of base vectors is to be only partially specified at this point. The set is presumed complete and orthonormal and to diagonalize T^{γ} with eigenvalues ϵ_{η}, i.e.,

$$T^{\gamma}| n\eta> = \epsilon_{\eta} |n\eta>. \tag{17}$$

The explicit determination of the particle-space dependence of these eigenvectors will be accomplished variously in the subsequent development, depending upon the specific quantity to be computed. The matrix elements of $\rho_{\lambda t}$ are given by

$$\rho_{\lambda t}(\mathbf{k}, \mathbf{q})_{n\eta,n'\eta'} = <n\eta| U^{\gamma^{+}}(t)\rho_{\lambda}(\mathbf{k}, \mathbf{q})U^{\gamma}(t) |n'\eta'>. \tag{18}$$

Now consider[1]

$$\chi_{\lambda}(t + s) = (8/V)\sum_{\mathbf{q}} e^{-2i\mathbf{x}\cdot\mathbf{q}}TrD(t + s)\rho_{\lambda t+s}(\mathbf{k}, \mathbf{q}). \tag{19}$$

The time (s) dependence of $\rho_{\lambda t+s}$ may be approximated by

$$\rho_{\lambda t+s} = U^{\gamma^{+}}(s)\rho_{\lambda t}U^{\gamma}(s) \cong \rho_{\lambda t} + (is/\hbar)[T^{\gamma}, \rho_{\lambda t}], \tag{20}$$

if only a linear dependence upon the time displacement is retained. If now we rewrite

$$D(t + s) = D(t) + \bar{D}(t, s), \tag{21}$$

we obtain the equation,

$$\begin{aligned} \chi_{\lambda}(t + s) \cong \chi_{\lambda}(t) &+ (8/V)s \sum_{\mathbf{q}} e^{-2i\mathbf{x}\cdot\mathbf{q}}TrD(t)(i/\hbar)[T^{\gamma}, \rho_{\lambda t}] \\ &+ (8/V)\sum_{\mathbf{q}} e^{-2i\mathbf{x}\cdot\mathbf{q}}Tr\bar{D}(t, s)\rho_{\lambda t}(\mathbf{k}, \mathbf{q}), \end{aligned} \tag{22}$$

[1] A few of the succeeding steps in the derivation of an equation for χ_{λ} presented here in detail were erroneously summarized in I.

ignoring the term containing the factor, $s\bar{D}(t, s)$. A straightforward calculation (see Appendix A) leads to an evaluation of the second term on the right-hand side as

$$-2sc\chi_\lambda \left\{ \sin \frac{\overleftarrow{\nabla}_x \cdot \overrightarrow{\nabla}_k}{2} \right\} |\,\mathbf{k}\,|, \tag{23}$$

which, if we neglect terms of order $\hbar^2$ in the description of photon transport, becomes simply,

$$-sc\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\chi_\lambda\,, \tag{24}$$

where

$$\boldsymbol{\Omega} = \mathbf{k}/|\,k\,|. \tag{25}$$

Thus we now exhibit Eq. (22) as

$$\dot{\chi}_\lambda + c\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\chi_\lambda \cong \left(\frac{\delta\chi_\lambda}{\delta t}\right)_{Int}, \tag{26}$$

after identifying

$$s\dot{\chi}_\lambda = \chi_\lambda(t + s) - \chi_\lambda(t), \tag{27}$$

and

$$\left(\frac{\delta\chi_\lambda}{\delta t}\right)_{Int} = \frac{8}{Vs} \sum_{\mathbf{q}} e^{-2i\mathbf{x}\cdot\mathbf{q}} Tr\bar{D}(t, s)\rho_{\lambda t}(\mathbf{k}, \mathbf{q}). \tag{28}$$

Our task now is to calculate to some approximation the effect of photon-particle interactions on the time rate of change of χ_λ. To do this we choose our quantization volume V sufficiently small that we may assume that χ_λ is essentially constant throughout it. This assumption enables us to reinterpret χ_λ as a mean density which may still be usefully regarded as a continuously variable function of position such that

$$\int_V d^3x\,[\chi_\lambda + c\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\chi_\lambda] = V[\chi_\lambda + c\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\chi_\lambda]. \tag{29}$$

Since the volume of integration in (29) coincides with the volume of quantization, we find that

$$\int d^3x \frac{8}{Vs} \sum_{\mathbf{q}} e^{-2i\mathbf{x}\cdot\mathbf{q}} Tr\bar{D}(t, s)\rho_{\lambda t}(\mathbf{k}, \mathbf{q}) = \frac{1}{s} Tr\bar{D}(t, s)\rho_{\lambda t}(\mathbf{k}, 0). \tag{30}$$

Equation (29) implies an effective upper limit on the size of V if macroscopic spatial variation in a system of specified dimensions is to be meaningfully described. Conversely, Eq. (30) relates a lower limit on V to the maximum wavelength of the photons to be considered. This latter condition obtains because the minimum relative uncertainty $(\Delta k/k)$ allowable in our specification of the momenta described by χ_λ is essentially given by the ratio of the uncertainty in the momenta assigned to the initial and final states of the emitting particles to that of the emitted photon, i.e., $\Delta k/k \sim \Delta K_{f,i}/k$. But since the emitting particle is confined to the volume $V(V = L^3)$, it follows that $\Delta K_{f,i} > 1/L$, and thus further that $\Delta k/k > 1/kL = \lambda/L$. Hence if λ_{max} is the maximum wavelength of the radiation to be considered in a given case, it would seem that the quantization volume would have to be so chosen that $\lambda_{max}/L \ll 1$.

This spatial coarse-graining is somewhat reminiscent of Ono's method of quantization in cells—though far less formally executed. The present treatment is admittedly cavalier with respect to these approximations—particularly so with regard to the possibility of reconciling the opposing assumptions leading to Eqs. (29) and (30). But it was our stated intention here to merely make the assumptions and then explore the consequent implications.

We note that in the representation (17) the photon number operator is diagonal, and hence the matrix $\rho_{\lambda t}(\mathbf{k}, 0)$ has no off-diagonal elements, i.e.,

$$\rho_{\lambda t}(\mathbf{k}, 0)_{n\eta, n'\eta'} = \eta_{\lambda k}\delta_{nn'}\delta_{\eta\eta'}\,, \tag{31}$$

where $\eta_{\lambda k}$ is the eigenvalue of the number operator, $\alpha_\lambda{}^+(\mathbf{k})\alpha_\lambda(\mathbf{k})$. Our calculation of $\bar{D}$ is elementary but a little devious. Recalling Eqs. (15) and (16), we find that

$$\begin{aligned} b_{n\eta}(t + s) = {} & \exp\,[-iE_n s/\hbar][b_{n\eta}(t) \\ & + \sum_{n'\eta'} \exp\,[i(\epsilon_\eta - \epsilon_{\eta'})t/\hbar]Q_{n\eta n'\eta'}(s)b_{n'\eta'}(t)], \end{aligned} \tag{32}$$

where Q is the matrix whose elements are

$$\begin{aligned} Q_{n\eta n'\eta'}(s) = \sum_{j=1}^{\infty}\left(-\frac{i}{\hbar}\right)^j \int_{\tau_1=0}^{s} \cdots \int_{\tau_j=0}^{\tau_{j-1}} d\tau_1 \cdots d\tau_j \\ \cdot H^I_{n\eta n_1\eta_1}(\tau_1) \cdots H^I_{n_{j-1}\eta_{j-1}n'\eta'}(\tau_j). \end{aligned} \tag{33}$$

The time dependent matrix elements of the interaction, $H^I = H - T^\gamma - H^P$, are

$$H^I_{n\eta n'\eta'}(\tau) = (\exp\,[i(T^\gamma + H^P)\tau/\hbar]H^I \exp\,[-i(T^\gamma + H^P)\tau/\hbar])_{n\eta n'\eta'}\,. \tag{34}$$

The eigenstates with respect to which these matrix elements are to be calculated are formally defined by [as well as the eigenvalues E_n appearing in Eq. (32) above]

$$(T^\gamma + H^P)|\,n\eta\rangle = (\epsilon_\eta + E_n)\,|n\eta\rangle. \tag{35}$$

Since we require only the diagonal elements of $\bar{D}$, we find after inserting (32) into (15) that

$$\begin{aligned}\bar{D}_{n\eta n\eta}(t, s) &= D_{n\eta n\eta}(t + s) - D_{n\eta n\eta}(t) \\ &= \sum_{n'\eta'} \exp\,[i(\epsilon_\eta - \epsilon_{\eta'})t/\hbar]Q_{n\eta n'\eta'}(s)D_{n'\eta' n\eta}(t) \\ &+ \sum_{n'\eta'} \exp\,[i(\epsilon_{\eta'} - \epsilon_\eta)t/\hbar]Q^*_{n\eta n'\eta'}(s)D_{n\eta n'\eta'}(t) \\ &+ \sum_{n'\eta' n''\eta''} \exp\,[i(\epsilon_{\eta'} - \epsilon_{\eta''})t/h]Q^*_{n\eta n'\eta'}(s)Q_{n\eta n''\eta''}(s)D_{n''\eta'' n'\eta'}(t).\end{aligned} \tag{36}$$

Finally, if we assume that the off-diagonal elements of the matrices D make a contribution to the desired balance relation which is small compared to that provided by the diagonal elements, we find for Eq. (26),

$$\begin{aligned}\dot{\chi}_\lambda + c\mathbf{\Omega}\cdot\nabla\chi_\lambda \cong (1/Vs) \sum_{n\eta} \eta_{\lambda k}[Q^*_{n\eta,n\eta} + Q_{n\eta,n\eta}]\, D_{n\eta,n\eta} \\ + (1/Vs) \sum_{n\eta} \eta_{\lambda k} \sum_{n'\eta'} Q_{n\eta,n'\eta'}Q^*_{n\eta,n'\eta'}\, D_{n'\eta',n'\eta'}\, .\end{aligned} \tag{37}$$

An explicit calculation of the Q's through terms second order in the interactions (3) leads to the equation,

$$\dot{\chi}_\lambda + c\mathbf{\Omega}\cdot\nabla\chi_\lambda \cong \sum_{n\eta} \eta_{\lambda k}[\sum_{m\alpha} (W^{(1)}_{n\eta,m\alpha} + W^{(2)}_{n\eta,m\alpha})(D_{m\alpha,m\alpha} - D_{n\eta,n\eta})], \tag{38}$$

where

$$\begin{aligned}W^{(1)}_{n\eta,m\alpha} &= \frac{4}{V\hbar^2}\,|\,(H^I)_{n\eta,m\alpha}\,|^2\, \frac{\sin^2 \dfrac{\omega_{n\eta} - \omega_{m\alpha}}{2}\, s}{s(\omega_{n\eta} - \omega_{m\alpha})^2}\, , \\ W^{(2)}_{n\eta,m\alpha} &= \frac{4}{V\hbar^2}\left|\sum_{r\sigma}{}' \frac{(H^I)_{n\eta,r\sigma}(H^I)_{r\sigma,m\alpha}}{\hbar(\omega_{n\eta} - \omega_{r\sigma})}\right|^2 \frac{\sin^2 \dfrac{\omega_{n\eta} - \omega_{m\alpha}}{2}\, s}{s(\omega_{n\eta} - \omega_{m\alpha})^2}\, .\end{aligned} \tag{39}$$

As usual

$$\hbar\omega_{n\eta} = E_{n\eta}\, , \tag{40}$$

the energy of the system when in the state characterized by the occupation numbers $\{n\eta\}$.

The only second-order process which will be considered here is bremsstrahlung, and is considered only because it first enters at this order. Furthermore, since only those transitions in which the photon number changes can contribute to the rate of change of χ_λ, we see that W will be independent of $T^p + H^{pe}$. Hence, for subsequent purposes, we may explicitly exhibit $W^{(1)}$ and $W^{(2)}$ as,

$$W^{(1)} = \frac{4}{V\hbar^2} \frac{\sin^2 \frac{\omega_{n\eta} - \omega_{m\alpha}}{2} s}{s(\omega_{n\eta} - \omega_{m\alpha})^2} [| H^{p\gamma 1} + H^{p\gamma e})_{n\eta,m\alpha}|^2 + | H^{p\gamma 2}_{n\eta,m\alpha}|^2],$$
$$W^{(2)} = \frac{4}{V\hbar^2} \frac{\sin^2 \frac{\omega_{n\eta} - \omega_{m\alpha}}{2} s}{s(\omega_{n\eta} - \omega_{m\alpha})^2} \left| \sum_{r\sigma} \frac{V_{n\eta,r\sigma} H^{p\gamma 1}_{r\sigma,m\alpha}}{\hbar(\omega_{n\eta} - \omega_{r\sigma})} \right|^2. \tag{41}$$

The cross terms that have been ignored in $W^{(1)}$ vanish since $H^{p\gamma 1} + H^{p\gamma e}$ have nonvanishing elements only between states in which the photon number differs by one, whereas the matrix elements of $H^{p\gamma 2}$ are zero for such pairs of states—being nonzero only if the photon numbers of the pair differ by two.

The transitions described by $H^{p\gamma 2}$ (which is bilinear or quadratic in the photon creation and destruction operators) are essentially those which represent the scattering of photons, while those accomplished by $(H^{p\gamma 1} + H^{p\gamma e})$ (which is linear in the photon creation and destruction operators) are transitions in which either one or both of the initial and final particle states are bound states or both are the magnetic states of free (spinless) charged particles. Because of the dependence of the relevant matrix elements upon the mass of the particle interacting with the photons, it is clear that we may largely ignore the ions and neutrals except insofar as they provide electron scatterers for one stage in the bremsstrahlung process and centers of force in the context of which atomic bound states can be defined.

To proceed further it is necessary to specify in somewhat greater detail the nature of the particle-space dependence of the base vectors, $|n\eta>$. To do this we first note that, if we neglect terms in the Hamiltonian which describe particle-photon interactions, the wave operators satisfy the equation

$$(\mathbf{\Pi}^\sigma)^2\psi_\sigma + \left(\sum_{\sigma'} \frac{e_\sigma e_{\sigma'}}{1 + \delta_{\sigma\sigma'}} \int \frac{\psi^+_{\sigma'}(\mathbf{x}')\psi_{\sigma'}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'\right) \psi_\sigma = i\hbar \frac{\partial \psi_\sigma}{\partial t}, \tag{42}$$

where

$$\psi_\sigma(\mathbf{x}, t) = \exp [iH't/\hbar]\psi_\sigma(\mathbf{x}) \exp [-iH't/\hbar], \tag{43}$$

H' being the part of H (Eq. (2)) that survives after setting $\mathbf{A} = \mathbf{P} = 0$. Thus to find an appropriate set of base vectors in configuration space, we look for separable solutions of (42) with a time dependence of the form $e(-iEt/\hbar)$. Defining an effective potential experienced by the σth particle at $\mathbf{x}$ by

$$v_\sigma(\mathbf{x}) = \sum_{\sigma'} \frac{e_\sigma e_{\sigma'}}{1 + \delta_{\sigma\sigma'}} \int \frac{\psi^+_{\sigma'}(\mathbf{x}')\psi_{\sigma'}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x',$$

Eq. (42) then becomes[2]

$$(\mathbf{\Pi}^{\sigma})^2\psi_\sigma + v_\sigma\psi_\sigma = E\psi_\sigma . \tag{45}$$

For a quantization volume sufficiently large compared to the radius of the largest orbit of any bound state terminating transitions which are expected to contribute appreciably to our balance relation, and also large compared to the radii of gyration of the majority of the electrons in our system, we may anticipate that Eq. (45) defines a complete, nearly orthogonal set of states corresponding to both positive and negative eigenvalues. We are, of course, concerning ourselves only with electron eigenstates, and are working in the "binary collision" limit in which we assume that not more than one ion (or electron) is interacting with a given electron at any one time. Thus the potential v_σ is to be regarded for the purpose of computing contributions from transitions involving bound states, as simply the Coulomb potential of a single ion (which for the purpose of constructing our approximate representation may be taken to be infinitely massive and at rest) and consequently the states corresponding to negative eigenvalues will be bound Coulomb states with—to first order in the external magnetic field—no azimuthal degeneracy. The states corresponding to positive eigenvalues are expected to be quite well approximated by (for sufficiently large positive eigenvalues at least) the usual magnetic states (*10*) for electrons in a spatially uniform, temporally constant external magnetic field. Furthermore, for sufficiently high particle kinetic energies and sufficiently weak magnetic fields, the positive eigenvalue states should be further approximatable by plane waves. The states corresponding to positive and negative eigenvalues are not expected to be truly orthogonal for a finite volume of quantization. Furthermore the overlap between two such vectors will be the greater the smaller the absolute value of their respective eigenvalues. Nevertheless it will simply be asserted that Eq. (45) with periodic boundary conditions provides us with a sufficiently orthogonal set of base vectors to enable us to proceed to a calculation of the transition probabilities, Eq. (41).

In accordance with these remarks, we designate the eigenstates of Eq. (45) by $u_{\sigma\mathbf{K}}(x)$ and their corresponding eigenvalues by $E_{\mathbf{K}}$, where here $\mathbf{K}$ is simply a

[2] According to Eq. (41), the Coulomb energy of the particles was incorporated into the perturbing energy H^I, whereas Eq. (45) indicates explicitly that it is to be considered rather as part of H^P. This awkward treatment of the electrostatic interaction occurred because we wished to calculate the transition probabilities for bremsstrahlung employing free (plane wave) states for the particles. More accurately, the calculation of the transition probabilities for bremsstrahlung and inverse-bremsstrahlung should be viewed as a first-order process in Eq. (37)—the relevant matrix elements being defined with respect to positive energy Coulomb wave functions. Then the formula for the transition probability appearing in Eq. (47) arises as the consequence of approximating, by first-order perturbation theory, the positive energy Coulomb wave functions as plane waves plus a correction.

set of labels sufficient for complete specification of each state. We then expand

$$\psi_\sigma(\mathbf{x}) = (1/\sqrt{V}) \sum_K a_\sigma(\mathbf{K}) u_{\sigma\mathbf{K}}(\mathbf{x}), \tag{46}$$

where now $a_\sigma(\mathbf{K})$ is a destruction operator for a σth type particle in the $\mathbf{K}$th state. The factor $1/\sqrt{V}$ merely symbolizes that the eigenvectors have been normalized to unity.

In these terms it is a straightforward matter to compute the five "first"-order transition probabilities contained in Eq. (41). The physical processes they represent and the further assumptions employed in their calculation are:

(1) Photon scattering. Relevant matrix elements are those of $H^{p\gamma 2}$. For this calculation we approximate the initial and final particle states as plane waves.

(2) Emission and absorption of cyclotron radiation comes from the matrix elements of $H^{p\gamma 1} + H^{p\gamma e}$. Here we approximate the particle states as electron magnetic states—ignoring the perturbing influence of the Coulomb potential.

(3) The emission and absorption of photoradiation produced by electrons undergoing free-bound and bound-free transitions, respectively. In this case we approximate the free particle states by plane waves and the bound states as the usual Coulomb states in the absence of external fields.

(4) Emission and absorption of excitation radiation produced by electrons undergoing transitions between atomic bound states. Again we approximate these states by Coulomb wave functions appropriate to the instance when no external fields are present.

(5) Bremsstrahlung and inverse bremsstrahlung. Here, as in the scattering case, we approximate the initial and final electron states by plane waves. Performing the calculations of the quantities in Eq. (41) as indicated and substituting the results into Eq. (38), and carrying out the indicated summations (see Appendix B) we finally obtain (after replacing averages of products of particle and photon occupation numbers by products of averages) (*11*)[3]

$$\begin{aligned}
\dot{\chi}_\lambda + c\mathbf{\Omega}\cdot\nabla\chi_\lambda = &\sum_{\sigma K K_1 \lambda' k'} S^{\sigma K, \lambda k}_{\sigma K_1, \lambda' k'} [V\chi_{\lambda'}(\mathbf{k}')\{V\chi_\lambda(\mathbf{k}) + 1\} \\
&\times Vf_\sigma(\mathbf{K}_1)\{1 \pm Vf_\sigma(\mathbf{K})\} - V\chi_\lambda(\mathbf{k})\{V\chi_{\lambda'}(\mathbf{k}') + 1\} Vf_\sigma(\mathbf{K})\{1 \pm Vf_\sigma(\mathbf{K}_1)\}] \\
&+ \sum_{\sigma K K_1} [T^{K_1}_{c\sigma K}(\lambda\mathbf{k}) + T^{K_1}_{r\sigma K}(\lambda\mathbf{k}) + T^{K_1}_{e\sigma K}(\lambda\mathbf{k})][\{V\chi_\lambda(\mathbf{k}) + 1\} \\
&\times Vf_\sigma(\mathbf{K})\{1 \pm Vf_\sigma(\mathbf{K}_1)\} - V\chi_\lambda(\mathbf{k}) Vf_\sigma(\mathbf{K}_1)\{1 \pm Vf_\sigma(\mathbf{K})\}] \\
&+ \sum_{\sigma\sigma' K K_1 K_2 K_3} T_B(\lambda\mathbf{k})^{\sigma K_2, \sigma' K_3}_{\sigma K, \sigma' K_1} [\{V\chi_\lambda(\mathbf{k}) + 1\} Vf_\sigma(\mathbf{K}) Vf_{\sigma'}(\mathbf{K}_1) \\
&\times \{1 \pm Vf_\sigma(\mathbf{K}_2)\}\{1 \pm Vf_{\sigma'}(\mathbf{K}_3)\} - V\chi_\lambda(\mathbf{k}) Vf_\sigma(\mathbf{K}_2) Vf_{\sigma'}(\mathbf{K}_3) \\
&\times \{1 \pm Vf_\sigma(\mathbf{K})\}\{1 \pm Vf_{\sigma'}(\mathbf{K}_1)\}].
\end{aligned} \tag{47}$$

[3] In this reference, as in Ref. *9*, Ono develops a transport equation in which the influence of scattering is described in terms formally similar to those presented therein.

This is the balance relation sought. The quantities S, T_c, T_r, T_e, and T_B are transition probabilities per unit time for scattering, cyclotron emission, electron-ion recombination emission, de-excitation emission, and bremsstrahlung respectively. The scattering matrix is characterized by the symmetry property,

$$S_{\sigma K_1,\lambda' k'}^{\sigma K,\lambda k} = S_{\sigma K,\lambda k}^{\sigma K_1,\lambda' k'}, \tag{48}$$

whereas the emission probabilities transform to the corresponding absorption probabilities under interchange of particle coordinates. As will be seen explicitly, all transition probabilities guarantee appropriate conservation of energy and momentum. The quantities $f_\sigma(\mathbf{K})$ are the particle analogs of χ_λ, e.g., $f_\sigma(\mathbf{K})$ is the expected number of particles per cm^3 in the volume V having momentum $\hbar\mathbf{K}$ at time t. The plus or minus signs arise because the σth type particles may be either bosons or fermions. (In most of the succeeding discussion we shall assume Boltzmann statistics for the particles, i.e., assume that states to which transitions go are sufficiently improbably occupied that we may neglect $Vf_\sigma(\mathbf{K})$ compared to one. However, for the time being, and for most of the next section, we retain the quantum statistics as indicated.)

The specific formulas for the transition probabilities occurring in Eq. (47), computed to the level of approximation discussed above, are:

$$S_{\sigma K',\lambda' k'}^{\sigma K,\lambda k} = \left(\frac{e_\sigma^2}{m_\sigma c^2}\right)^2 \frac{8\pi^3}{V^3}\frac{c^2}{kk'} \,|\,\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\varepsilon}_{\lambda'}(\mathbf{k}')\,|^2 \times \delta(\mathbf{k}' + \mathbf{K}' - \mathbf{k} - \mathbf{K})\delta(\omega_{Kk} - \omega_{K'k'}); \tag{49a}$$

$$T_{c\sigma K}^{K_1}(\lambda\mathbf{k}) = \frac{4\pi^2 c}{\hbar V^2}\left(\frac{e_\sigma}{m_\sigma c}\right)^2 \frac{1}{k}\,|\,\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot \langle \mathbf{K}_1 \,|\, e^{-i\mathbf{k}\cdot\mathbf{x}}\boldsymbol{\Pi}^\sigma \,|\, \mathbf{K}\rangle\,|^2\delta(\omega_{K_1k} - \omega_{Kk}); \tag{49b}$$

$$T_{r\sigma K}^{K_1}(\lambda\mathbf{k}) = T_{e\sigma K}^{K_1}(\lambda\mathbf{k}) = \frac{4\pi^2}{V^2}\frac{e_\sigma^2}{m_\sigma c^2}\frac{\hbar c}{m_\sigma}\frac{1}{k} \times |\,\langle \mathbf{K}_1 \,|\, e^{-i\mathbf{k}\cdot\mathbf{x}}\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\nabla} \,|\, \mathbf{K}\rangle\,|^2\,\delta(\omega_{K_1k} - \omega_K); \tag{49c}$$

$$T_B(\lambda\mathbf{k})_{\sigma K,\sigma' K_1}^{\sigma K_2,\sigma' K_3} = \frac{\pi^2}{4V^4}\left(\frac{e_\sigma^2}{m_\sigma c^2}\right)^2\left(\frac{e_{\sigma'}^2}{\hbar c}\right)\frac{c^4}{k}\,|\,U(|\mathbf{K}_1 - \mathbf{K}_3|)\,|^2 \times \left|\frac{\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\mathbf{K}_2}{\omega_K + \omega_{K_1} - \omega_{K_3} - \omega_{\mathbf{K}_2+\mathbf{k}}} + \frac{\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\mathbf{K}}{\omega_{K_2} + \omega_{K_3} - \omega_{K_1} - \omega_{\mathbf{K}-\mathbf{k}}}\right|^2 \cdot\,\delta(\mathbf{K} + \mathbf{K}_1 - \mathbf{K}_2 - \mathbf{K}_3 - \mathbf{k})\delta(\omega_{K_2K_3k} - \omega_{KK_1}), \tag{49d}$$

where the momentum δ's are Kronecker δ's, but $\delta(\omega)$ is a Dirac delta arising from the identification

$$\frac{\sin^2(\omega s/2)}{s\omega^2} \longrightarrow \frac{\pi}{2}\,\delta(\omega). \tag{50}$$

In (49c) we have lumped the formulas for T_r (recombination emission) and T_e (de-excitation emission) together, since they differ only in the selection of the states $|K>$ and $|K_1>$ for the final step in the calculation of the emission and absorption coefficients conventionally employed in descriptions of radiative transfer. The Fourier transform of the Coulomb potential is defined by

$$U(|\mathbf{K}|) = \int \frac{d^3R}{R} e^{i\mathbf{K}\cdot\mathbf{R}}.$$

III. SOME ASPECTS OF EQUILIBRIUM

Before proceeding to the final reduction of Eq. (47) to the form commonly employed in the description of problems in radiative transfer, it is convenient to digress briefly for a discussion of some of the anticipated implications of the present analysis for equilibrium systems. Though most of these implications are perhaps obvious from the form of the equation itself (and in fact are generally well known), it nevertheless seems to us of some interest to point them out in the present context which is not quite the usual one. Actually not all of these implications are completely obvious from the form of Eq. (47) itself. The one that is obvious is the fact that this equation admits steady state solutions appropriate to the description of the equilibrium state. But the further fact that the nonequilibrium system is in some sense driven irreversibly to that state in which the densities assume their conventional form is not obvious from Eq. (47). One needs an H-theorem deducible from (47), but this is not possible since it describes only the behavior of the photon density in terms of the particle densities. In order to deduce an H-theorem in terms of the densities directly, it would be necessary to have at hand the equations for the particle densities (I) completed to account for all processes to the same order of approximation as they are in the photon Eq. (47) see reference (*12*). However, as we are not primarily concerned with the description of the particle densities in this paper, we shall base our discussion of an H-theorem in the present context upon an earlier phase of the analysis.

A cursory reappraisal of the argument leading to Eq. (38) reveals that it contains the more basic equation

$$\dot{D}_{n\eta,n\eta} = \sum_{m\alpha} W_{n\eta,m\alpha}[D_{m\alpha,m\alpha} - D_{n\eta,n\eta}], \tag{51}$$

where

$$W_{n\eta,m\alpha} = W_{m\alpha,n\eta}. \tag{52}$$

Recalling that the diagonal elements of the density matrix, D, have the interpretation of the probability of finding the system in a state characterized by a particular set of occupation numbers, it is convenient to introduce a notation which emphasises this interpretation; so we define

$$P(n\eta, t) = D_{n\eta,n\eta}(t), \tag{53}$$

and rewrite (51) as

$$P(n\eta, t) = \sum_{m\alpha} W_{n\eta,m\alpha}[P(m\alpha, t) - P(n\eta, t)]. \tag{54}$$

This equation and its implications have long been well known (*13*), so we merely sketch the succeeding argument. We first define a function H by

$$H = \sum_{n\eta} P(n\eta) \ln P(n\eta). \tag{55}$$

It is then readily shown that

$$dH/dt \leqq 0, \tag{56}$$

the equality holding only when

$$P(m\alpha) = P(n\eta), \tag{57}$$

all m, α, n, and η. The monotonicity of the time derivative of H suggests that we may tentatively interpret it as closely related to the entropy of the system, hence we identify

$$S = -\kappa H, \tag{58}$$

where κ is a constant of dimension ergs/°K.[4] It then follows that we should interpret the state for which

$$dS/dt = 0,\ S \text{ a maximum}, \tag{59}$$

as the equilibrium state. A solution of (57) which immediately suggests itself by virtue of the energy conservation condition contained in W is

$$P(n\eta) = P(E_{n\eta}). \tag{60}$$

A further condition on the solution if it is to describe the thermodynamic state of weakly interacting systems is;

$$\text{if} \quad E_{n\eta} \cong E_n + E_\eta, \qquad \text{then} \quad P(E_n + E_\eta) = P(E_n)P(E_\eta). \tag{61}$$

A solution to the functional Eq. (61) is, of course,

$$P(E) = Ce^{-\beta E}, \tag{62}$$

where C is to be determined by the requirement that P be a probability. Since β must be the same for both the photon and particle systems—and is the only macroscopic parameter they share—it follows that it must be related to the temperature, and is in fact $1/\kappa T$.

[4] The interpretation as an entropy of a functional of the distribution functions whose time derivative is always zero seems somewhat inappropriate (*14*).

One now readily establishes that the equilibrium photon and particle densities are:

$$\chi_\lambda(\mathbf{k}) = \sum_\eta \eta_{\lambda k} P(\eta) = \frac{1}{e^{\beta\hbar ck} - 1}, \quad \text{and} \quad f(\mathbf{K}) = \sum_n n_K P(n) = \frac{1}{Be^{\beta E}K \pm 1}. \tag{63}$$

It is also now readily established that the function identified above as the system entropy becomes, in the equilibrium state, the sum of the photon and particle entropies, respectively; and that the functional dependence of these partial entropies upon other thermodynamic variables is indeed that conventionally deduced by statistical mechanical arguments (*15*). Finally, employing the distribution functions (63), one easily shows that Eq. (47) is satisfied.

One of the principal reasons for presenting the relatively familiar detail of this section is to emphasize the fact that this detail and these results obtain naturally in the context of a description of systems with many degrees of freedom which introduces no specifically statistical considerations other than those inherent in the axioms of quantum mechanics themselves (*16*).

IV. SOME APPLICATIONS OF THE BALANCE RELATION

Although the processes influencing photon transport are well known and, to the order of approximation characterizing the present analysis, have been more or less thoroughly investigated, Eq. (47) is not in a form that is easily recognized. Thus in this section, the photon balance equation will be reduced to a more familiar form. The processes contributing to the scattering, emission, and absorption of radiation, enumerated in Section II, will be discussed in somewhat greater detail; and the corresponding transition probabilities will be reduced, when feasible, to forms that have already found useful application.

To initiate this reduction we now explicitly assume that the number of occupied particle states in any given energy range is small compared to the actual number of states in the same range. The effect of this assumption is to exclude from present consideration all systems characterized by particle degeneracy, and leads to a description of the particle densities in terms of Boltzmann statistics. We then go to the continuum in photon momentum space by defining

$$\sum_{\mathbf{k}\epsilon d^3k} \chi_\lambda = \bar{\chi}_\lambda \, dk \, d\Omega(k), \tag{64a}$$

and

$$\sum_{\mathbf{k}\epsilon d^3k} 1 = \rho_k \, dk \, d\Omega(k) = \frac{Vk^2 \, dk \, d\Omega(k)}{(2\pi)^3}. \tag{64b}$$

Furthermore, for the treatment of photon scattering—for which free-particle states are employed to describe the particles before and after collision—it is also convenient to go to the continuum in particle momentum space, i.e.,

$$\sum_{\mathbf{K}\epsilon d^3K} f_\sigma = \bar{f}_\sigma \, d^3K. \tag{64c}$$

Then Eq. (47) can be written as

$$\begin{aligned} \dot{\bar{\chi}}_\lambda + c\mathbf{\Omega}\cdot\nabla\bar{\chi}_\lambda = (s_i^\lambda + \epsilon_c^\lambda + \epsilon_r^\lambda + \epsilon_e^\lambda + \epsilon_B^\lambda)\left(\bar{\chi}_\lambda + \frac{\rho_k}{V}\right) \\ - (s_0^\lambda + \alpha_c^\lambda + \alpha_r^\lambda + \alpha_e^\lambda + \alpha_B^\lambda)\bar{\chi}_\lambda , \end{aligned} \tag{65}$$

where the reaction rates for scattering (s), emission (ϵ), and absorption (α) are now given by:

$$s_i^\lambda = \sum_{\sigma,\lambda'} \int_{\mathbf{K},\mathbf{K}_1,\mathbf{k}'} d^3K \, d^3K_1 \, dk' \, d\Omega' \; V^3 S_{\sigma K_1,\lambda' k'}^{\sigma K,\lambda k} \bar{\chi}_{\lambda'}(\mathbf{k}')\bar{f}_\sigma(\mathbf{K}_1), \tag{66a}$$

$$s_0^\lambda = \sum_{\sigma,\lambda'} \int_{\mathbf{K},\mathbf{K}_1,\mathbf{k}'} d^3K \, d^3K_1 \, dk' \, d\Omega' \; V^3 S_{\sigma K_1,\lambda' k'}^{\sigma K,\lambda k} \left[\bar{\chi}_{\lambda'}(\mathbf{k}') + \frac{\rho_{k'}}{V}\right]\bar{f}_\sigma(\mathbf{K}), \tag{66b}$$

$$\epsilon_{c,r,e}^\lambda = \sum_{KK_1} V^2 T_{c,r,eK}^{K_1}(\lambda\mathbf{k}) f(\mathbf{K}), \tag{66c}$$

$$\alpha_{c,r,e}^\lambda = \sum_{KK_1} V^2 T_{c,r,eK}^{K_1}(\lambda\mathbf{k}) f(\mathbf{K}_1), \tag{66d}$$

$$\epsilon_B^\lambda = \sum_{\sigma\sigma' KK_1K_2K_3} V^3 T_B(\lambda\mathbf{k})_{\sigma K,\sigma' K_1}^{\sigma K_2,\sigma' K_3} f_\sigma(\mathbf{K}) f_{\sigma'}(\mathbf{K}_1), \tag{66e}$$

$$\alpha_B^\lambda = \sum_{\sigma\sigma' KK_1K_2K_3} V^3 T_B(\lambda\mathbf{k})_{\sigma K,\sigma' K_1}^{\sigma K_2,\sigma' K_3} f_\sigma(\mathbf{K}_2) f_{\sigma'}(\mathbf{K}_3). \tag{66f}$$

These reaction rates are, in general, complicated functions of the photon wave vector, photon polarization, position, and time—the space and time dependence arising through the dependence of $\bar{\chi}_\lambda$ and f_σ on space and time. They represent total transition probabilities per unit time for transitions between all possible initial and final states such that a photon of momentum $\hbar\mathbf{k}$ and polarization λ is either gained or lost. The quantities $s_{i,o}$ are the "scattering in" and "scattering out" transition probabilities, whereas the ϵ's and α's are the corresponding probabilities for emission and absorption, respectively. The omission of the sum over the particle index for the cyclotron, recombination, and de-excitation radiation reaction rates is in accordance with the earlier discussion, in which it was indicated that essentially only electron transitions are important.

All of the emission and absorption processes enter the present analysis in the same fundamental way. In excitation and de-excitation for example, the photon field and the atom constitute two weakly coupled systems. The interaction between them causes an electronic transition from one atomic state to another accompanied by the emission or absorption of a photon. Electronic transitions leading to emission may proceed either spontaneously (at a rate independent of the presence or absence of photons) or at a rate proportional to the number of photons present (induced emission). The absorption rate is, of course, always proportional to the number of photons present. The other processes are described here in basically the same terms—cyclotron emission or absorption resulting from electronic transitions between unperturbed magnetic states, whereas bremsstrahlung and inverse bremsstrahlung are radiative free-free electronic transitions occurring in the field of another charged particle. Because of the present nonrelativistic treatment of the particles, Eq. (65) should probably be restricted to a description of systems in which the mean particle energies are not expected to much exceed 50 kev. Under such circumstances, pair creation and annihilation should not contribute appreciably to the photon balance, and hence the necessary absence of a description of such processes in Eq. (65) should lead to negligible error.

The scattering process is not of particular interest when dealing with nonrelativistic systems because the scattering rate (and corresponding cross section) is small in comparison with that of some of the absorption and emission processes. Since the cross section for nonrelativistic photon scattering is roughly proportional to the square of the radius of the electron, it is seen that, even for free electron densities of order 10^{18} per cm^3, the scattering mean-free-path is of order 10^6 cm. Such a process can hardly be expected to significantly influence photon distributions in laboratory-scale systems. Thus, in investigations of radiative transfer in this energy range, scattering rates which would be characteristically dependent upon the photon densities in both the initial and final collision states are not usually given any consideration.

Conversely, for the treatment of the problem of shielding high-energy gamma rays from a nuclear reactor, the scattering process does become a significant competitor with other relevant photon reactions. This is due in large part to the fact that the interactions of such high-energy photons with the electrons in the atoms in such systems may be satisfactorily treated as if the electrons were free, i.e., as if all such reactions are describable as Compton scattering. Consequently there is an enormous increase in the effective density of scatterers, leading to scattering mean-free-paths of the order of centimeters or less. However, in the description of scattering in these instances, the dependence of the scattering rates upon photon densities in post-collision states is always ignored (*2*). This is justified, of course, because in these far-from-photon-equilibrium systems, the photon densities are always very small when compared to the densities of available states.

In the absence of an external magnetic field, information about photon distributions in particular polarization states is no longer significant. In fact, in such instances, it is reasonable to assume random polarization for the photons—in which case, $\bar{\chi}_\lambda(\mathbf{k}) = \frac{1}{2}\bar{\chi}(\mathbf{k})$. Then Eq. (65) becomes

$$\dot{\bar{\chi}} + c\mathbf{\Omega}\cdot\nabla\bar{\chi} = (\bar{s}_i + \bar{\epsilon}_r + \bar{\epsilon}_e + \bar{\epsilon}_B)[\bar{\chi} + (2\rho_k/V)] - (\bar{s}_o + \bar{\alpha}_r + \bar{\alpha}_e + \bar{\alpha}_B)\bar{\chi}, \tag{67}$$

where we have defined

$$\bar{s}_{i,o} = \tfrac{1}{4}\sum_\lambda s_{i,o}^\lambda ,$$

$$\bar{\epsilon}_{r,e,B} = \tfrac{1}{2}\sum_\lambda \epsilon_{r,e,B}^\lambda ,$$

$$\bar{\alpha}_{r,e,B} = \tfrac{1}{2}\sum_\lambda \alpha_{r,e,B}^\lambda .$$

The scattering rates may now be rewritten somewhat more explicitly as,

$$\bar{s}_i = 4\pi^3 \sum_\sigma \int_{\mathbf{K}\mathbf{K}_1\mathbf{k}'} d^3K\, d^3K_1\, dk'\, d\Omega' \frac{c\sigma_T}{kk'} \{\hbar c\delta(\mathbf{k} + \mathbf{K} - \mathbf{k}' - \mathbf{K}_1\} \times \delta\left(\hbar ck + \frac{\hbar^2K^2}{2m_\sigma} - \hbar ck' - \frac{\hbar^2K_1^2}{2m_\sigma}\right) \bar{\chi}(\mathbf{k}')\bar{f}_\sigma(\mathbf{K}_1), \tag{68a}$$

$$\bar{s}_0 = 4\pi^3 \sum_\sigma \int_{\mathbf{K}\mathbf{K}_1\mathbf{k}'} d^3K\, d^3K_1\, dk'\, d\Omega' \frac{c\sigma_T}{kk'} \{\hbar c\delta(\mathbf{k} + \mathbf{K} - \mathbf{k}' - \mathbf{K}_1)\} \times \delta\left(\hbar ck + \frac{\hbar^2K^2}{2m_\sigma} - \hbar ck' - \frac{\hbar^2K_1^2}{2m_\sigma}\right)\left\{\bar{\chi}(\mathbf{k}') + \frac{2\rho_{k'}}{V}\right\} \bar{f}_\sigma(\mathbf{K}), \tag{68b}$$

where we have introduced the Thomson cross section,

$$\sigma_T = \tfrac{1}{2}(e_\sigma^2/m_\sigma c^2)[1 + (\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2].$$

Equation (67) may now be readily reduced further to a form familiar in reactor shielding studies (*2*). According to the above remarks about the ratio of photon densities to available state densities in these systems, we may drop $\bar{\chi}(\mathbf{k})$ whenever it is compared with the corresponding density of states, ρ_k/V. We then define a macroscopic linear absorption coefficient μ by

$$\mu = (1/c)[\bar{s}_o + \bar{\alpha}_r + \bar{\alpha}_e + \bar{\alpha}_B],$$

where "scattering out" is considered as an effective absorption. In this sense, μ represents a probability per unit path for small paths for the loss of a photon of momentum $\hbar\mathbf{k}$. Equation (67) may now be written as

$$\frac{1}{c}\dot{\bar{\chi}} + \boldsymbol{\Omega}\cdot\nabla\bar{\chi} + \mu\bar{\chi} = \frac{2\rho_k}{Vc}(\bar{\epsilon}_r + \bar{\epsilon}_e + \bar{\epsilon}_B) + \sum_\sigma \int dk'\, d\Omega'\, n_\sigma \bar{\chi}(\mathbf{k}')\sigma_\sigma(k', \boldsymbol{\Omega}'; k, \boldsymbol{\Omega}), \tag{69}$$

where

$$\sigma_\sigma(k', \boldsymbol{\Omega}'; k, \boldsymbol{\Omega}) = \int d^3K\, d^3K_1 \left[\frac{\bar{f}_\sigma(K)}{n_\sigma}\right]\frac{\hbar ck}{k'}\sigma_T \times \delta(\mathbf{k} + \mathbf{K} - \mathbf{k}' - \mathbf{K}_1)\delta\left(\hbar ck + \frac{\hbar^2K^2}{2m_\sigma} - \hbar ck' - \frac{\hbar^2K_1^2}{2m_\sigma}\right).$$

When finally cognizance is taken of the fact that atomic de-excitation radiation is considered to be of negligible importance in reactor shielding situations, and the high-energy photon transport equation is reduced to nonrelativistic form, it is seen that Eq. (69) is essentially the same as the one employed by Goldstein (*2*).

For the remainder of this section we shall be primarily concerned with low-energy plasma systems near kinetic equilibrium for which scattering can be neglected. Returning to Eq. (65) (without scattering), we accomplish a reduction to a form conventional in the discussion of low-energy radiative transfer by defining a source function, $j^\lambda(\mathbf{x}, \mathbf{k}, t)$; an "effective" absorption coefficient, $\alpha^{\lambda e}(\mathbf{x}, \mathbf{k}, t)$; and a radiation intensity, $I_\lambda(\mathbf{x}, \mathbf{k}, t)$ such that:

$$j^\lambda = \sum_{m=c,r,e,B} j_m^{\ \lambda} = \sum_{m=c,r,e,B} \frac{\hbar\omega\rho_k\epsilon_m^{\ \lambda}}{V}, \tag{70a}$$

$$\alpha^{\lambda e} = \sum_{m=c,r,e,B} \frac{\alpha_m^{\ \lambda} - \epsilon_m^{\ \lambda}}{c}, \tag{70b}$$

and

$$I_\lambda = \hbar\omega c\bar{\chi}_\lambda. \tag{70c}$$

Then radiation transport is described by the familiar expression (*1*),

$$(1/c)\dot{I}_\lambda + \boldsymbol{\Omega}\cdot\nabla I_\lambda = j^\lambda - \alpha^{\lambda e}I_\lambda. \tag{71}$$

The effective absorption coefficient $\alpha^{\lambda e}$ is a probability per unit path for small paths for energy loss by "net" absorption. The qualifications "effective" and/or "net" absorption imply a difference between the absorption and induced emission processes. The source function is essentially the rate of spontaneous emission of energy per cm^3 per unit k (frequency) per unit solid angle.

When a strong magnetic field is present, cyclotron radiation can cause a significant energy loss from a plasma system. For fully ionized plasmas it may even be the dominant mechanism for radiant energy loss. When this situation obtains, Eq. (71) becomes

$$(1/c)\dot{I}_\lambda + \mathbf{\Omega}\cdot\nabla I_\lambda = j_c^\lambda - \alpha_c^{\lambda e} I_\lambda . \tag{72}$$

We note that the source function j_c^λ is completely specified when ϵ_c^λ is specified, and that ϵ_c^λ is known (at least formally) when the electron distribution function is known and appropriate single-electron wave functions (previously discussed) are chosen.

To evaluate ϵ_c^λ, we choose [following Parzen (*17*)[5]] a coordinate system in which the external magnetic field is along the z-axis, and the photon propagation vector $\mathbf{k}$ lies in the y–z plane. We specify the polarization vectors by the usual spherical base vectors in the polar and azimuthal directions. The calculation of $T_{cK}^{K_1}(\lambda\mathbf{k})$ then follows directly from the work of Parzen, after replacing his K and β by k and $\gamma = v_\perp/c$, respectively—except that we have allowed arbitrary electron momenta in the z-direction, rather than restricting it to be zero. The results (Appendix C) are

$$\begin{aligned} V^2 T_{cK}^{K_1}(\bar{\phi}, \mathbf{k}) = {} & \frac{4\pi^2 e^2}{m^2 \hbar c k} (m v_\perp)^2 \delta(\omega_{K_1 k} - \omega_K) \\ & \times \delta(K_z - k_z - K_{1_z}) e\left[-\frac{2\hbar c k \hbar k_\perp}{\hbar\omega_0 m v_\perp} \right] [J_n'(n\gamma \sin\theta)]^2 \end{aligned} \tag{73a}$$

for the azimuthally polarized radiation, and

$$\begin{aligned} V^2 T_{cK}^{K_1}(\theta, k) = {} & \frac{4\pi^2 e^2}{m^2 \hbar c k} (m v_\perp)^2 \delta(\omega_{K_1 k} - \omega_K) \\ & \times \delta(K_z - k_z - K_{1_z}) e\left[-\frac{2\hbar c k \hbar k_\perp}{\hbar\omega_0 m v_\perp} \right] \cos^2\theta \left[1 - \frac{\hbar K_z \sin\theta}{mc} \tan\theta \right]^2 \\ & \times \left[\frac{J_n(n\gamma\sin\theta)}{\gamma \sin\theta} \right]^2 \end{aligned} \tag{73b}$$

for the photons polarized in the direction of the polar unit vector. We have introduced the notation ω_0 to represent the electron gyromagnetic frequency. The symbol n occurring in Eqs. (73a, b) is an integer equal to the difference between the radial energy quantum numbers characterizing the initial and final electron magnetic states appropriate to the transition under consideration. The quantity $\hbar k_\perp / m v_\perp$ occurring in the exponentials is the ratio of the perpendicular (to the magnetic field) component of the photon momentum to the corresponding component of the electron momentum. For frequencies of interest, this quantity is so small that for all practical purposes these exponentials may be replaced by unity. Energy and momentum conservation require that

[5] Note that although one of his approximations was not valid for the extreme relativistic case (*18*), Parzen's analysis is quite accurate for our problem. Also note that in his Eq. (26), k should be replaced by R.

$$n\omega_o = ck - v_z k_z \mp (\hbar k_z^2/2m). \tag{74}$$

Direct substitution of Eqs. (73a, b) into the definition (66c) of ϵ_c^λ gives,

$$\epsilon_c^\phi = \frac{4\pi^2 e^2}{m^2 \hbar ck} \sum_n \int d^3p\, f(\mathbf{p}) p_\perp^2 \left[J_n'\left(\frac{np_\perp \sin\theta}{mc}\right)\right]^2 \times \delta\left[n\omega_0 - ck\left(1 - \frac{p_z \cos\theta}{mc} + \frac{\hbar k_z \cos\theta}{2mc}\right)\right], \tag{75a}$$

$$\epsilon_c^\theta = \frac{4\pi^2 e^2}{m^2 \hbar ck} \sum_n \int d^3p\, f(\mathbf{p}) \left[J_n\left(\frac{np_\perp \sin\theta}{mc}\right)\right]^2 \times (p_z \sin\theta - mc\cot\theta)^2 \delta\left[n\omega_0 - ck\left(1 - \frac{p_z \cos\theta}{mc} + \frac{\hbar k_z \cos\theta}{2mc}\right)\right], \tag{75b}$$

where we have converted to the continuum in momentum space for the description of the pre-transition particle distributions, $f(\mathbf{p})$.

The evaluation of $\alpha_c^{\lambda e}$ is also of great importance. This parameter has been calculated in various ways by various authors (*4*, *19*). It is seen from Eqs. (66c, d) and (70b) that it can be written as

$$\alpha_c^{\lambda e} = (1/c) \sum_{K,K_1} V^2 T_{cK}^{K_1}(\lambda\mathbf{k})[f(\mathbf{K}_1) - f(\mathbf{K})]. \tag{76}$$

This expression for the effective absorption coefficient is quite general. Kirchoff's law—consisting of a relation between α_c^λ and ϵ_c^λ—can be developed at this point if we now assume local equilibrium for the particles, e.g., take

$$f(\mathbf{K}) = n(\beta\hbar^2/2m\pi)^{3/2} \exp[-\beta\hbar^2 K^2/2m], \tag{77}$$

where n is the (generally space- and time-dependent) particle density in configuration space, and $\beta = 1/\kappa T$ is also permitted an arbitrary space-time dependence.

Since $T_{cK}^{K_1}(\lambda\mathbf{k})$ conserves energy, $f(\mathbf{K}_1)$ may be expressed as

$$f(\mathbf{K}_1) = f(\mathbf{K}) \exp[\beta\hbar ck],$$

with the consequences that

$$\alpha_c^\lambda = \epsilon_c^\lambda \exp[\beta\hbar ck], \quad \text{and} \quad \alpha_c^{\lambda e} = (1/c)(\exp[\beta\hbar ck] - 1)\epsilon_c^\lambda. \tag{78}$$

Observing that j_c^λ is proportional to ϵ_c^λ, Eq. (71) assumes the form

$$(1/c)\dot{I}_\lambda + \mathbf{\Omega}\cdot\nabla I_\lambda = -\alpha_c^{\lambda e}(I_\lambda - \tfrac{1}{2}I_{BB}), \tag{79}$$

where

$$I_{BB}\, d\omega = \frac{[\hbar\omega^3/8\pi^3c^2]\, d\omega}{e^{\beta\hbar\omega} - 1}, \qquad \omega = ck.$$

Thus again (see Section III) we have arrived at an expression which provides us with an equilibrium solution for the photons, namely, I_λ independent of space and time and equal to $\frac{1}{2}I_{BB}$. The condition for the thermodynamic solution can also be rephrased as

$$j^\lambda/\alpha^{\lambda e} = \tfrac{1}{2}I_{BB},$$

which is a statement of Kirchoff's law for radiation of polarization λ.

When $\beta\hbar\omega \ll 1$ (which is the situation discussed in (*4*) and (*19*)), the Rayleigh-Jeans approximation to the black body distribution is valid. Then Eq. (79), for the steady state, takes the form used in these analyses, i.e.,

$$\mathbf{\Omega}\cdot\nabla I_\lambda = -\alpha_c^{\lambda e}(I_\lambda - \tfrac{1}{2}I_{RJ}). \tag{80}$$

Furthermore, in this instance, the effective absorption coefficients can be approximated from Eqs. (75a, b) and (78) as,

$$\begin{aligned}\alpha_c^{\Phi e} = \frac{4\pi^2 e^2\beta}{m^2 c}\sum_n \int d^3p\, f(\mathbf{p})p_\perp^2\left[J_n'\left(\frac{np_\perp \sin\theta}{mc}\right)\right]^2 \\ \times\, \delta\left[n\omega_0 - ck\left(1 - \frac{p_z\cos\theta}{mc} + \frac{\hbar k_z\cos\theta}{2mc}\right)\right],\end{aligned} \tag{81a}$$

$$\begin{aligned}\alpha_c^{\theta e} = \frac{4\pi^2 e^2\beta}{m^2 c}\sum_n \int d^3p\, f(\mathbf{p})\left[J_n\left(\frac{np_\perp \sin\theta}{mc}\right)\right]^2 \\ \times\, (p_z\sin\theta - mc\cot\theta)^2\delta\left[n\omega_0 - ck\left(1 - \frac{p_z\cos\theta}{mc} + \frac{\hbar k_z\cos\theta}{2mc}\right)\right].\end{aligned} \tag{81b}$$

Setting $m = c = 1$ and restricting attention to radiation proceeding nearly perpendicularly to the magnetic field ($\theta \sim \pi/2$), it is seen that (81a) becomes the nonrelativistic limit of the effective absorption coefficient obtained by Drummond and Rosenbluth (DR) (*4*). For $\theta = 0$, the absorption coefficients for the different polarizations vanish for all transitions except those between successive states, i.e., $n = 1$, and are equal for these transitions. However, the mean free path for absorption when $\theta = 0$ is much greater than for absorption at $\theta = \pi/2$. (Observe that $\lim_{\theta\to\pi/2}\alpha_c^{\theta e} \to 0$). Thus it is reasonable to expect that, for systems for which all the linear dimensions are of the same order of magnitude, the radiation loss parallel to the magnetic field will be only a small percentage of the total radiation loss. This seems to be a reasonable inference to be drawn from the calculations of DR for the infinite slab, which also indicate that the bulk of the radiation is emitted into an angular interval for which $\alpha_c^{\theta e} \ll \alpha_c^{\Phi e}$ so that radiation into the θ-polarization can probably be neglected entirely.

It is shown by Berman (*20*) that for a hydrogen plasma with no magnetic field and a kinetic temperature from 3 ev to 200 ev, radiative recombination is the dominant energy emission process. The calculation of the emission coefficient for this case in the present context can proceed in a rather general way by choosing for the electron and atomic wave functions,

$$| K> = (1/\sqrt{V})e^{i\mathbf{K}\cdot\mathbf{x}}, \quad | K_1> = \psi_n , \tag{82}$$

where n represents a sufficient set of labels to completely specify each atomic state. Since we have chosen a quantization volume such that only one ion is present, the ion density in the system is simply given by $n_I = 1/V$.

After converting to the continuum for the electron's initial momenta and summing over photon polarizations, we obtain for the recombination radiation emission coefficient

$$\bar{\epsilon}_r = 2\,\pi^2 n_I \left(\frac{e^2}{mc^2}\right) \frac{\hbar c}{m} \frac{1}{k} \sum_{\lambda n} \int d^3K\, f(\mathbf{K})\, (\boldsymbol{\varepsilon}_\lambda \cdot \mathbf{K})^2 \delta(\omega_{K_1 k} - \omega_K)\, | \psi_n(\boldsymbol{\kappa}) |^2, \tag{83}$$

where

$$\psi_n(\boldsymbol{\kappa}) = \int \psi_n^*(\mathbf{x}) e^{-i\boldsymbol{\kappa}_i\cdot\mathbf{x}}\, d^3x$$

and $\boldsymbol{\kappa} = \mathbf{k} - \mathbf{K}$. It is now a straightforward matter to obtain the results presented by Heitler (*21*) (p. 207), for transitions to the K-shell; or alternatively those presented by Bethe and Salpeter (*22*) for transitions into higher states (see Appendix D).

The remarks about the effective absorption coefficient for cyclotron radiation are also applicable to radiative recombination. Calculations of recombination radiation from a plasma have been performed by Berman (*20*) and Kogan (*23*).

For low-energy hydrogen plasmas (less than 3 ev), or for higher energy plasmas containing atoms of higher charge number (*24*), de-excitation radiation can be a serious energy loss mechanism. The transition probability $T^{K_1}_{e\sigma K}(\lambda\mathbf{k})$ can be put in a more convenient form for calculation in the dipole approximation by the elimination of the gradient operator from the matrix element. It is observed that (for the dipole approximation only),

$$<n\eta \mid H^{p\gamma 1} \mid m\alpha> = (e/\hbar c) < n\eta \,|[\psi^+\psi, H_p]|\, m\alpha>, \tag{84}$$

and thus, instead of (49b), we may write

$$T^{K_1}_{e\sigma K}(\lambda\mathbf{k}) = \frac{4\pi^2}{\hbar^3 V^2} \left(\frac{e_\sigma^{\,2}}{m_\sigma c^2}\right) \frac{m_\sigma c}{k} (\epsilon_{K_1 k} - \epsilon_K)^2 \mid \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \cdot < \mathbf{K}_1 \mid \mathbf{x} \mid \mathbf{K} > |^2. \tag{85}$$

Equation (85) is related to the corresponding expression for the rate of spontaneous de-excitation presented by Heitler (*21*) (p. 178, Eq. (10)) by

$$\omega\, d\Omega = V^2 T^{K_1}_{e\sigma K}(\lambda\mathbf{k}) \frac{\rho_k\, d\Omega}{cV} = \frac{e^2(ck)^3\, d\Omega}{2\pi\hbar c^3} \mid < \mathbf{K}_1 \mid x \mid \mathbf{K} > |^2 \cos^2\theta, \tag{86}$$

where θ is the angle between the direction of polarization and the vector, $\mathbf{x}$. If we now sum over polarization and integrate over angles, we obtain the transition probability A_{K_1K} given by Berman (*20*), Eq. (2-1). For further calculations applied to a hydrogen plasma, see reference *20*.

The last radiation mechanism which we will consider is bremsstrahlung. The calculation of $\bar{\epsilon}_B$ for electron-ion and electron-electron bremsstrahlung proceeds straightforwardly from Eqs. (49d) and (66e). However, nonrelativistic electron-electron bremsstrahlung contributes negligibly when compared with the electron-ion radiative collisions (*25*), and hence shall be given no explicit consideration here.

From the relations (49d), (66e), and (70a) we find that the source function in this instance may be written as

$$j_B\, d\omega = n_I\, \hbar\omega \int_{E,\Omega} v f_e(E, \Omega)\, dE\, d\Omega \int_{\Omega_2} \sigma_B\, d\omega\, d\Omega_2\,, \tag{87}$$

after assuming that the scattering ions are at rest before collision. The cross section $\sigma_B\, d\omega\, d\Omega_2$ is the one given by Heitler (*21*) (p. 245) for nonrelativistic electron-ion bremsstrahlung. Of course, Eq. (87) also effectively provides us with the emission coefficient $\bar{\epsilon}_B$ (recall (70a)) (see Appendix E). Hence if we again assume kinetic equilibrium for the particles, we may easily obtain the absorption coefficient α_B according to Eq. (78). However, for most laboratory situations the large bremsstrahlung mean-free-paths (*26*) imply that the photon densities will be exceedingly low (provided that bremsstrahlung is the principal emission mechanism). Hence

$$\bar{\alpha}_B{}^e I / j_B \ll 1, \tag{88}$$

and consequently the rate of loss of radiant energy from such systems is essentially given by the rate of emission, j_B. Extensive calculations of this emission rate have been carried out by Kvasnica (*27*),[6] and an investigation of the range of validity of the assumption in Eq. (88) is presented in reference *26*.

V. FIRST-ORDER COLLECTIVE EFFECTS ON PHOTON TRANSPORT IN THE FULLY IONIZED PLASMA

In the preceding sections we have developed a description of photon transport which implicitly assumes that the photons travel with speed c between successive events. (This assumption is realized explicitly in the form of the transport term, $c\mathbf{\Omega}\cdot\nabla\chi_\lambda$). The assumption slipped into the analysis through the choice of the transformation operator (14) which defined the interaction representation (13). However, a brief reappraisal of the discussion in Section II reveals that a certain

[6] Included among these calculations were rates of electron-electron bremsstrahlung as well.

amount of the information available about the system in the Hamiltonian (2) has simply been discarded. It is our purpose in this section to show that this information can be exploited with but trivial modification of the preceding analysis to enrich the treatment of radiative transfer in the fully ionized plasma.

The point is that in the term $H^{p\gamma 2}$ [Eq. (2g)] in the Hamiltonian for the system there is a part that describes simply an energy level shift for the photons in the medium, as well as other parts describing interactions between photons and particles leading to changes in the states of the photons. Only the latter parts of this interaction term were employed in the calculation of the influence of *transitions* upon the rate of change of the photon distribution function. To incorporate the effect of the former part, one need only add it to the energy of the "free" photon in the definition of the unitary transformation taking us to an appropriate interaction representation. Since it is this unitary transformation that describes how the photons propagate between events, we will then find that the phase velocity of photons of momentum $\hbar\mathbf{k}$ is modified to be

$$c(1 + \omega_p^2/2c^2k^2),$$

where ω_p^2 is the usual plasma frequency. The group velocity which enters into the transport term will be correspondingly modified to be $c(1 - \omega_p^2/2c^2k^2)$.

For an explicit realization of the content of these remarks, we rewrite $H^{p\gamma 2}$ in momentum space as

$$H^{p\gamma 2} = \frac{\pi\hbar}{cV}\sum_{\lambda k K\sigma}\frac{e_\sigma^2}{m_\sigma}\frac{a_\sigma^+(\mathbf{K})a_\sigma(\mathbf{K})\zeta_\lambda^+(\mathbf{k})\cdot\zeta_\lambda^+(-\mathbf{k})}{k} + \frac{\pi\hbar}{cV}\sum_{\substack{\lambda k K\sigma\\ \lambda' k' K'}}{}' \frac{e_\sigma^2}{m_\sigma}\frac{a_\sigma^+(\mathbf{K})a_\sigma(\mathbf{K}')\zeta_\lambda^+(\mathbf{k})\cdot\zeta_{\lambda'}^+(-\mathbf{k}')\delta(\mathbf{k}+\mathbf{K}-\mathbf{k}'-\mathbf{K}')}{\sqrt{kk'}}, \tag{90}$$

where the prime on the second summation implies that the terms for which $\lambda = \lambda'$, $k = k'$, and $K = K'$ are to be deleted. These latter terms are just the ones that have been employed in the discussion of photon scattering and hence shall be largely ignored in the following. Recalling Eq. (6), it is seen that the terms in the first summation in (90) include some that are proportional to the photon number operator. These we single out for special consideration and designate them as

$$H_0^{p\gamma 2} = \frac{2\pi\hbar}{cV}\sum_{\lambda k K\sigma}\frac{e_\sigma^2}{m_\sigma}\frac{a_\sigma^+(\mathbf{K})a_\sigma(\mathbf{K})\alpha_\lambda^+(\mathbf{k})\alpha_\lambda(\mathbf{k})}{k}. \tag{91}$$

Now define a transformation to an interaction representation by

$$F = UG, \tag{92}$$

where now

$$U = \exp\,[-i(T^{\gamma} + H_o^{p\gamma 2})t/\hbar]. \tag{93}$$

Proceeding as in Section II, we find that Eq. (47) is reproduced with the following two modifications:

(1) The transport term is altered from $c\mathbf{\Omega}\cdot\nabla\chi_\lambda$ to

$$\frac{\partial\chi_\lambda}{\partial x_j}\frac{\partial}{\partial k_j}\left\{ck + \frac{2\pi}{ckV}\sum_{K\sigma}\frac{e_\sigma^2}{m_\sigma}\,(F,\, a_\sigma^{+}(\mathbf{K})a_\sigma(\mathbf{K})F)\right\}, \tag{94}$$

where we have employed the approximation

$$(F,\, a_\sigma^{+}a_\sigma\rho_\lambda F) \cong (F,\, a_\sigma^{+}a_\sigma F)(F,\, \rho_\lambda F). \tag{95}$$

(2) The energies of the "free" photons—defined as the eigenvalues of $T^{\gamma} + H_o^{p\gamma 2}$ in the representation that diagonalizes the number operators for both particles and photons—are

$$\epsilon_\eta = \sum_{\lambda k}\hbar ck\left(1 + \frac{2\pi}{(ck)^2}\frac{1}{V}\sum_{K\sigma}\frac{e_\sigma^2}{m_\sigma}\,n_{\sigma K}\right)\eta_{\lambda k}\,. \tag{96}$$

This shift in the photon energies requires a corresponding modification in the energy conserving delta functions contained in the various transition probabilities in Eq. (47).

The transport term, Eq. (94), can be expressed in a more interpretable form. First note that

$$(1/V)\sum_{K}\,(F,\, a_\sigma^{+}(\mathbf{K})a_\sigma(\mathbf{K})F) = N_\sigma\,, \tag{97}$$

the expected density of particles of kind σ in the quantization volume V. Then, ignoring terms proportional to the ratio of the electron mass to the ion mass, one obtains

$$-\Omega_j(\partial\chi_\lambda/\partial x_j)(\partial/\partial k)(ck + \omega_p^2/2ck), \tag{98}$$

where we have introduced the notation $\omega_p^2 = 4\pi N_e e^2/m$. This suggests the assignment to photons of momentum $\hbar\mathbf{k}$ a frequency $\omega = ck + \omega_p^2/2ck$, a phase velocity $\omega/k = v_p = c(1 + \omega_p^2/2c^2k^2)$, and a group velocity

$$\partial\omega/\partial k = v_g = c(1 - \omega_p^2/2c^2k^2).$$

In these terms, transport is described by

$$-v_g\mathbf{\Omega}\cdot\nabla\chi_\lambda\,, \tag{99}$$

and we see that photons whose momenta are such that $\omega_p^2/2c^2k^2 \geqq 1$ do not propagate through the plasma. This is substantially the same conclusion with regard to "first-order" effects of collective particle behavior on electromagnetic wave propagation in plasmas as is drawn from conventional macroscopic electrodynamics (*28*). The restriction of the remarks in this section to the fully ionized gas is a consequence of the fact that in systems in which electron bound states are important, considerable modification of, say, the energies [Eq. (96)] is to be expected (*29*). Further investigation into the implications of the modifications (1 and 2) for transport processes will not be entered into here.

A consequence of the present description of "noninteracting" photons for the equilibrium state is of some interest. Recall that in Section III, the thermodynamic state was presumed characterized by the canonical distribution,

$$P = Ce^{-\beta(T^p + T^\gamma)} \tag{100}$$

In the present instance, however, this distribution should be generalized to

$$P = Ce^{-\beta(T^p + T^\gamma + H_o^{p\gamma 2})}. \tag{101}$$

It is readily shown that the density matrix (101) leads to particle densities the same as in Eq. (63). However, the photon density is altered to become

$$\chi_\lambda(\mathbf{k}) = [\exp(\beta\epsilon_\kappa) - 1]^{-1}, \tag{102}$$

where

$$\epsilon_k = \hbar ck\left(1 + \frac{2\pi}{c^2k^2V}\sum_\sigma \frac{e_\sigma^{\ 2}}{m_\sigma}\sum_K n_{\sigma K}\right). \tag{103}$$

Since in this instance we are considering a large, spatially uniform system, the quantization volume V may comprise the whole system and

$$(1/V)\sum_K n_{\sigma K} = N_\sigma, \tag{104}$$

the density of particles of the σth kind. Thus (102) may be written as

$$\chi_\lambda(\mathbf{k}) = \{\exp[\beta\hbar ck(1 + \omega_p^{\ 2}/2c^2k^2)] - 1\}^{-1}. \tag{105}$$

Perhaps the most significant aspect of this modified thermal radiation spectrum is the prediction of the rapid decrease in the expected number of photons with momenta such that $\omega_p^{\ 2}/2c^2k^2 \gg 1$.

APPENDIX A. A THEOREM ON TRANSPORT

A general theorem concerned with the transport term of the rate equation can be stated:
If

$$H_o = \sum_{\lambda k} O(\lambda k)\alpha_\lambda^{+}(\mathbf{k})\alpha_\lambda(\mathbf{k}),$$

then

$$\frac{8}{V}\sum_{\mathbf{q}} e^{-2i\mathbf{x}\cdot\mathbf{q}}\frac{i}{\hbar}(F,[H_o\,,\rho_\lambda(\mathbf{k},\mathbf{q})]F)$$
$$= -\frac{2}{\hbar}\frac{8}{V}\sum_{\mathbf{q}}\left(F, O(\lambda k)\sin\left\{\frac{\overleftarrow{\nabla}_k\cdot\overrightarrow{\nabla}_x}{2}\right\}\rho_\lambda(\mathbf{k},\mathbf{q})F\right)e^{-2i\mathbf{x}\cdot\mathbf{q}}, \tag{A.1}$$

where

$$\rho_\lambda(\mathbf{k},\mathbf{q}) = \alpha_\lambda^{+}(\mathbf{k}+\mathbf{q})\alpha_\lambda(\mathbf{k}-\mathbf{q}),$$

and $O(\lambda k)$ represents an operator containing factors which commute with the α operators.

Utilizing the commutation relation $[\alpha_\lambda(\mathbf{k}), \alpha_{\lambda'}^{+}(\mathbf{k}')] = \delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}')$ we obtain

$$[H_o\,,\rho\lambda\,(\mathbf{k},\mathbf{q})] = \sum_{k'\lambda'} O(\lambda' k')[\alpha_{\lambda'}^{+}(\mathbf{k}')\alpha_\lambda'(\mathbf{k}'),\ \alpha_\lambda^{+}(\mathbf{k}+\mathbf{q})\alpha_\lambda(\mathbf{k}-\mathbf{q})]$$
$$= [O(\lambda, |\,\mathbf{k}+\mathbf{q}\,|) - O(\lambda, |\,\mathbf{k}-\mathbf{q}\,|)]\rho_\lambda(\mathbf{k},\mathbf{q}). \tag{A.2}$$

The left-hand side of Eq. (A.1) thus becomes

$$\frac{8}{V}\sum_{q} e^{-2i\mathbf{x}\cdot\mathbf{q}}\frac{i}{\hbar}(F,[O(\lambda, |\,\mathbf{k}+\mathbf{q}\,|) - O(\lambda, |\,\mathbf{k}-\mathbf{q})\,|\,)]\rho_\lambda(\mathbf{k},\mathbf{q})F)$$
$$= -\frac{2}{\hbar}\left(\frac{8}{V}\right)\sum_{\mathbf{q}}\left(F, O(\lambda k)\left[\frac{e^{\overleftarrow{\nabla}_k\cdot\mathbf{q}} - e^{-\overleftarrow{\nabla}_k\cdot\mathbf{q}}}{2i}\right]\rho_\lambda(\mathbf{k},\mathbf{q})F\right)e^{-2i\mathbf{x}\cdot\mathbf{q}} \tag{A.3}$$
$$= -\frac{2}{\hbar}\left(\frac{8}{V}\right)\sum_{\mathbf{q}}\left(F, O(\lambda k)\sin\left\{\frac{\overleftarrow{\nabla}_{\mathbf{k}}\cdot\overrightarrow{\nabla}_{\mathbf{x}}}{2}\right\}\rho_\lambda(\mathbf{k},\mathbf{q})F\right)e^{-2i\mathbf{x}\cdot\mathbf{q}},$$

where the exponential operators are defined by their series expansion.

Two cases of interest are $H_o = T^\gamma$ and $H_o = T^\gamma + H_o^{p\gamma 2}$. For $H_o = T^\gamma$, $O(\lambda k) = \hbar c k$ and Eq. (A.3) reduces to Eq. (23). When we use $H_o = T^\gamma + H_o^{p\gamma 2}$ we obtain, in the classical limit, Eq. (94).

APPENDIX B. DERIVATION OF THE CYCLOTRON RADIATION TERM IN EQ. (47) FROM EQ. (38)

As pointed out in Section III, the cyclotron radiation results from that part of Eq. (38) which contains $H^{p\gamma 1} + H^{p\gamma e}$. Ignoring scattering and bremsstrahlung, Eq. (38) becomes

$$\dot{\chi}_\lambda + c\mathbf{\Omega}\cdot\nabla\chi_\lambda \cong \sum_{n\eta}\eta_{\lambda k}\sum_{m\alpha} W^{(1)}_{n\eta,m\alpha}(D_{m\alpha,m\alpha} - D_{n\eta,n\eta}), \tag{B.1}$$

where

$$W^{(1)}_{n\eta,m\alpha} = (2\pi/V\hbar^2)\delta(\omega_{n\eta} - \omega_{m\alpha})|(H^{p\gamma 1} + H^{p\gamma e})_{n\eta,m\alpha}|^2 \tag{B.2}$$

and we have replaced

$$\frac{\sin^2\left(\frac{\omega_{n\eta} - \omega_{m\alpha}}{2} s\right)}{s(\omega_{n\eta} - \omega_{m\alpha})^2} \text{ by } (\pi/2)\delta(\omega_{n\eta} - \omega_{m\alpha}).$$

It has been observed in Section II that a diagonal element of the density matrix $D_{n\eta,n\eta}$ is interpretable as the probability of finding the system in the state characterized by the occupation numbers $| n\eta >$ at time t. Consequently, we introduce $P(n\eta, t) = D_{n\eta,n\eta}{}^{(t)}$ and note that

$$V\chi_\lambda(\mathbf{k}) = \sum_{n\eta} \eta_{\lambda k} P(n\eta, t).$$

Thus,

$$V\dot{\chi}_\lambda(\mathbf{k}) = \sum_{n\eta} \eta_{\lambda k} \dot{P}(n\eta, t)$$

with

$$\dot{P}(n\eta, t) = \sum_{m\alpha} W^{(1)}_{n\eta,m\alpha}[P(m\alpha, t) - P(n\eta, t)]. \tag{B.3}$$

Instead of performing this sum algebraically, we appeal to the physical process, and recognize that the probability of the system being in the state characterized by occupation numbers $| n_{\sigma K} n_{\sigma K_1} \eta_{\lambda k} >$ is affected in the following way:

(1). $P(n_{\sigma K} n_{\sigma K_1} \eta_{\lambda k})$ is increased by:

(a) A photon absorption when the initial state has occupation numbers

$$| n_{\sigma K} - 1, n_{\sigma K_1} + 1, \eta_{\lambda k} + 1 >.$$

(b) A photon emission when the initial state has occupation numbers

$$| n_{\sigma K} - 1, n_{\sigma K_1} + 1, \eta_{\lambda k} - 1 >.$$

(2). $P(n_{\sigma K}, n_{\sigma K_1} \eta_{\lambda K})$ is decreased by the reverse processes.

Now $H^{(1)} = H^{p\gamma 1} + H^{p\gamma e}$ can easily be put in the form

$$\begin{aligned} H^{(1)} = - \sum_{\sigma\lambda k K K_1} \frac{e_\sigma}{m_\sigma c} \sqrt{\frac{2\pi\hbar c}{Vk}}\, a_\sigma{}^+(\mathbf{K}_1) a_\sigma(\mathbf{K}) \\ \times [\alpha^+(\mathbf{k})\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) + \alpha(-\mathbf{k})\boldsymbol{\varepsilon}_\lambda(-\mathbf{k})] \cdot \int d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}} u_{K_1}^{\sigma+} \mathbf{\Pi}^\sigma u_K^\sigma, \end{aligned} \tag{B.4}$$

where we have expanded

$$\psi_\sigma = \sum_K a_\sigma(\mathbf{K}) u_{\sigma K}(\mathbf{x}),$$

the $u_{\sigma K}(x)$ being the normalized magnetic states discussed in Appendix C. The sum over $\mathbf{k}$ is over both positive and negative values, so that when $(-\mathbf{k})$ appears it can be replaced everywhere by $(\mathbf{k})$. Equation (B.2) can now be written

$$W^{(1)}_{n\eta,m\alpha} = T^{\sigma K_1}_{c\sigma K}(\lambda'\mathbf{k}')\,|< n\eta \mid a^{+}(\mathbf{k}')a_\sigma{}^{+}(\mathbf{K}_1)a_\sigma(\mathbf{K}) \mid m\alpha >|^2 \tag{B.5}$$

or

$$T^{\sigma K_1}_{ca\sigma K}(\lambda'\mathbf{k}')\,|< n\eta \mid a(\mathbf{k}')a_\sigma{}^{+}(\mathbf{K}_1)a_\sigma(\mathbf{K}) \mid m\alpha >|^2,$$

where the first matrix element represents photon emission and the second matrix element represents absorption, and where

$$T^{\sigma K_1}_{c\sigma K}(\lambda'\mathbf{k}') = \frac{4\pi^2 c}{\hbar V^2}\left(\frac{e_\sigma}{m_\sigma c}\right)^2 \frac{1}{k'}\,\delta(\omega_{K_1 k'} - \omega_K) \times |\,\boldsymbol{\varepsilon}_{\lambda'}(\mathbf{k}')\cdot < \mathbf{K}_1 \mid e^{-i\mathbf{k}'\cdot\mathbf{x}}\boldsymbol{\Pi}^\sigma \mid \mathbf{K} >|^2 \tag{B.6a}$$

$$T^{\sigma K_1}_{ca\sigma K}(\lambda'\mathbf{k}') = \frac{4\pi^2 c}{\hbar V^2}\left(\frac{e_\sigma}{m_\sigma c}\right)^2 \frac{1}{k'}\,\delta(\omega_{K_1} - \omega_{Kk'}) \times |\,\boldsymbol{\varepsilon}_{\lambda}(\mathbf{k}')\cdot < \mathbf{K}_1 \mid e^{+i\mathbf{k}'\cdot\mathbf{x}}\boldsymbol{\Pi}^\sigma \mid \mathbf{K} >|^2. \tag{B.6b}$$

The states $|\mathbf{K}>$ and $<\mathbf{K}_1|$ characterize the initial and final electron states.

Equation (B.1) can now be reduced to

$$\begin{aligned}
(\partial\chi_\lambda/\partial t)_{\text{int}} &= \sum_{n\eta} \eta_{\lambda k} \sum_{m\alpha} W^{(1)}_{n\eta,m\alpha}[P(m\alpha, t) - P(n\eta, t)] \\
&= \sum_{n\eta} \eta_{\lambda k} \sum_{\sigma K K_1 \lambda' k'} T^{\sigma K_1}_{c\sigma K}(\lambda'\mathbf{k}')(1 \pm n_{\sigma K}) n_{\sigma K_1} \eta_{\lambda' k'} \\
&\qquad \times P(n_{\sigma K} + 1, n_{\sigma K_1} - 1, \eta_{\lambda' k'} - 1) \\
&+ \sum_{n\eta} \eta_{\lambda k} \sum_{\sigma K K_1 \lambda' k'} T^{\sigma K_1}_{ca\sigma K}(\lambda'\mathbf{k}')(1 \pm n_{\sigma K}) n_{\sigma K_1}(\eta_{\lambda' k'} + 1) \\
&\qquad \times P(n_{\sigma K} + 1, n_{\sigma K_1} - 1, \eta_{\lambda' k'} + 1) \\
&- \sum_{n\eta} \eta_{\lambda k} \sum_{\sigma K K_1 \lambda'_1 k'} T^{\sigma K_1}_{c\sigma K}(\lambda'\mathbf{k}') n_{\sigma K}(1 \pm n_{\sigma K_1}) \eta_{\lambda' k'} \\
&\qquad \times P(n_{\sigma K} n_{\sigma K_1} \eta_{\lambda' k'}) \\
&- \sum_{n\eta} \eta_{\lambda k} \sum_{\sigma K K_1 \lambda' k'} T^{\sigma K_1}_{ca\sigma K}(\lambda'\mathbf{k}') n_{\sigma K}(1 \pm n_{\sigma K_1})(\eta_{\lambda' k'} + 1) \\
&\qquad \times P(n_{\sigma K} n_{\sigma K_1} \eta_{\lambda' k'}),
\end{aligned} \tag{B.7}$$

where the positive sign is appropriate for bosons and the negative sign is for fermions. When the sums are broken into two parts, ($\lambda' k' = \lambda k$ and $\lambda' k' \neq \lambda k$) and the indices are appropriately shifted in the first two terms, we get

$$(\partial\chi_\lambda/\partial t)_{\text{int}} = \sum_{\sigma K K_1 n \eta} T_{c\sigma K}^{\sigma K_1}(\lambda\mathbf{k}) n_{\sigma K}(n_{\sigma K_1} + 1)(\eta_{\lambda k} + 1) P(n_{\sigma K}, n_{\sigma K_1}, \eta_{\lambda k})$$
$$- \sum_{\sigma K K_1 n \eta} T_{ca\sigma K}^{\sigma K_1}(\lambda\mathbf{k}) n_{\sigma K}(n_{\sigma K_1} + 1)\eta_{\lambda k} P(n_{\sigma K}, n_{\sigma K_1}, \eta_{\lambda k})$$
$$= \sum_{\sigma K K_1 n \eta} T_{c\sigma K}^{\sigma K_1}(\lambda\mathbf{k})[n_{\sigma K}(n_{\sigma K_1} + 1)(\eta_{\lambda k} + 1) - (n_{\sigma K} + 1) n_{\sigma K_1}\eta_{\lambda k}]$$
$$\times P(n_{\sigma K} n_{\sigma K_1} \eta_{\lambda k}). \quad \text{(B.8)}$$

If we approximate the average of the products by the product of averages, we obtain

$$(\partial\chi_\lambda/\partial t)_{\text{int}} = \sum_{\sigma K K_1} T_{c\sigma K}^{\sigma K_1}(\lambda\mathbf{k})[\{V\chi_\lambda(\mathbf{k}) + 1\} V f_\sigma(\mathbf{K})\{1 \pm V f_\sigma(\mathbf{K}_1)\} - V\chi_\lambda(\mathbf{k}) V f_\sigma(\mathbf{K}_1)\{1 \pm V f_\sigma(\mathbf{K})\}]. \quad \text{(B.9)}$$

The minus sign where $(\pm)$ appears would result from a rederivation with anticommutation rules.

APPENDIX C. REDUCTION OF EQ. (49b) TO EQS. (73a), (73b)

We first rewrite Eq. (49b) in the form

$$V^2 T_{c\sigma K}^{K_1}(\lambda\mathbf{k}) = \frac{4\pi^2 c}{\hbar}\left(\frac{e_\sigma}{m_\sigma c}\right)^2 \frac{1}{k}\,\delta(\omega_{K_1 k} - \omega_K)$$
$$\times\, |\langle j'l'm'|\, e^{-i\mathbf{k}\cdot\mathbf{x}}\{\tfrac{1}{2}[\epsilon_{\mathbf{k},\lambda}^{-}\Pi_+^{\sigma} + \epsilon_{\mathbf{k},\lambda}^{+}\Pi_-^{\sigma}] + \epsilon_{k,\lambda}^{z}\Pi_z^{\sigma}\}\,|jlm\rangle|^2 \quad \text{(C.1)}$$

where (*10*)

$$|K\rangle = |jlK_z\rangle = u_{jl}(\rho, \phi)\frac{e^{iK_z z}}{\sqrt{L}}$$
$$= \frac{1}{j!}\sqrt{\frac{l!}{\pi L j!}}\,(\alpha\rho^2)^{(j-l)/2} \exp\left(i(j - l)\phi + iK_z z - \frac{\alpha\rho^2}{2}\right) L_l^{j-l}(\alpha\rho^2),$$

with $\alpha = m\omega_0/2\hbar$, $L^{-1/2}$ being the box normalization length, and where $L_l^{j-l}(\alpha\rho^2)$ is the associated Laguerre polynomial. We have defined

$$\Pi_\pm^{\sigma} = \Pi_x^{\sigma} \pm i\Pi_y^{\sigma}$$

and $\boldsymbol{\epsilon}_{\mathbf{k},\lambda}^{\pm} = \epsilon_{\mathbf{k},\lambda}^{x} \pm i\epsilon_{\mathbf{k},\lambda}^{y}$. The operators $\Pi_\pm^{\sigma}$ are creation and annihilation operators with the property

$$\Pi_+^{\sigma} u_{jl} = i m_\sigma \omega_o b\sqrt{2(j + 1)}\, u_{j+1,l}\,, \qquad \Pi_-^{\sigma} u_{jl} = -i m_\sigma \omega_o b\sqrt{2j}\, u_{j-1,l}\,, \quad \text{(C.2)}$$

where $b^2 = \hbar c/eH$. When the $\mathbf{k}$ vector is oriented in the y–z plane with the magnetic field parallel to the z axis,

$$V^2 T_{c\sigma K}^{K_1}(\lambda\mathbf{k}) = \frac{4\pi^2 c}{\hbar}\left(\frac{e_\sigma}{m_\sigma c}\right)^2 \frac{1}{k}\,\delta(\omega_{K_1 k} - \omega_K)$$

$$\times \left| \frac{-i}{2} m_\sigma \omega_o b\sqrt{2j}\varepsilon^{+}_{\mathbf{k},\lambda}\,\delta(K_z - k_z - K_{1z})I(j'l' \mid j-1, l) \right.$$

$$+ \frac{i}{2} m_\sigma \omega_o \varepsilon^{-}_{\mathbf{k},\lambda} b\sqrt{(2j+1)}\,\delta(K_z - k_z - K_{1z})I(j'l' \mid j+1, l)$$

$$\left. + \varepsilon^{z}_{k,\lambda}\,\hbar K_z\,\delta(K_z - k_z - K_{1z})I(j'l' \mid jl) \right|^2,$$

where

$$I(j'l' \mid jl) = \langle j'l' \mid e^{-ik_1\rho\cos\theta} \mid jl\rangle \qquad \text{(C.3)}$$

and $\delta(K_z - k_z - K_{1z})$ results from the z integration. Taking $l = 0$ (the "well-centered orbit" approximation (*17*)) which is valid to first order (*30*) it is seen from Parzen that for cases of interest in plasmas only transitions between different j states are important. Thus, letting $n = j' - j$, we obtain (*17*)

$$\begin{aligned} I(j', 0 \mid j0) &= i^n e[-(\hbar ck\hbar k_\perp/\hbar\omega_o mv_\perp)]J_n(n\gamma \sin\theta) \\ I(j', 0 \mid j-1, 0) &= i^{n+1} e[-(\hbar ck\hbar k_\perp/\hbar\omega_o mv_\perp)]J_{n-1}[(n)\gamma \sin\theta] \qquad \text{(C.4)} \\ I(j', 0 \mid j+1, 0) &= i^{n-1} e[-(\hbar ck\hbar k_\perp/\hbar\omega_o mv_\perp)]J_{n+1}[(n)\gamma \sin\theta]. \end{aligned}$$

We specify the polarization vectors by the usual spherical base vectors in the polar and azimuthal directions, so $\varepsilon^{\pm}_{\mathbf{k},\theta} = \cos\theta$, $\varepsilon^{\pm}_{\mathbf{k},\phi} = \pm i$, $\varepsilon^{z}_{\mathbf{k},\theta} = -\sin\theta$, and $\varepsilon^{z}_{\mathbf{k},\phi} = 0$. Substituting Eq. (C4) into (C3) we obtain the desired result

$$\begin{aligned} V^2T^{K_1}_{c\sigma K}(\phi, \mathbf{k}) &= \frac{4\pi^2e^2}{m_\sigma^2\hbar ck}(m_\sigma v_\perp)^2\delta(\omega_{K_1k} - \omega_K) \\ &\qquad \times \delta(K_z - k_z - K_{1z})e\left[-\frac{2\hbar ck\hbar k_\perp}{\hbar\omega_0 m_\sigma v_\perp}\right][J_n'(n\gamma\sin\theta)]^2 \\ V^2T^{K_1}_{c\sigma K}(\theta, \mathbf{k}) &= \frac{4\pi^2e^2}{m_\sigma^2\hbar ck}(m_\sigma v_\perp)^2\delta(\omega_{K_1k} - \omega_K)\,\delta(K_z - k_z - K_{1z}) \qquad \text{(C.5)} \\ &\quad \times e\left[-\frac{2\hbar ck\hbar k_\perp}{\hbar\omega_0 m_\sigma v_\perp}\right]\cos^2\theta\left[1 - \frac{\hbar K_z\sin\theta}{m_\sigma c}\tan\theta\right]^2\left[\frac{J_n(n\gamma\sin\theta)}{\gamma\sin\theta}\right]^2, \end{aligned}$$

where we have made use of Bessel function recursion formulas and the relations $R \simeq b(2n)^{1/2} = v_\perp/\omega_o$, where R is the radius of the orbit.

APPENDIX D. REDUCTION OF $\bar{\alpha}_r$ TO A CROSS SECTION

The quantity $\bar{\alpha}_r$ is related to the cross section by

$$\bar{\alpha}_r = \sum_{\lambda K_1} c\int \sigma_p^{\lambda}(K_1, \mathbf{K}, \mathbf{k})f(K)\,d^3K, \qquad \text{(D.1)}$$

where

$$\sigma_p^{\lambda}(K_1, \mathbf{K}, \mathbf{k}) = \sum_{\mathbf{K}\in d^3K} \frac{V^2 T_{rK}^{K_1}(\lambda\mathbf{k})}{c}.$$

Then

$$\sigma_p^{\lambda}(K_1, E, \mathbf{\Omega k})\, dE\, d\Omega = \frac{V\, dE\, d\Omega}{(2\pi)^3} \left(\frac{m}{\hbar^2}\right)^{3/2} \sqrt{2\hbar ck}\, \frac{V^2 T_{rK}^{K_1}(\lambda\mathbf{k})}{c}, \quad \text{(D.2)}$$

where we have assumed

$$\hbar^2 K^2/2m \simeq \hbar ck.$$

We choose the plane wave state for the free electron as $|\,\mathbf{K}\!> \; = (1/\sqrt{V})e^{i\mathbf{K}\cdot\mathbf{x}}$, the factor $1/V$ indicating the number density of ions. Then

$$\sigma_p^{\lambda}(K_1, E, \mathbf{\Omega k})\, dE\, d\Omega = \frac{V\, dE\, d\Omega}{2\pi} \frac{e^2}{mc^2 k} \sqrt{\frac{2mck}{\hbar}} \times \delta(E_{K_1 k} - E_k) \,|<\mathbf{K}_1|\, e^{-i\mathbf{k}\cdot\mathbf{x}} \boldsymbol{\varepsilon}_\lambda \cdot \nabla \,|\,\mathbf{K}>|^2. \quad \text{(D.3)}$$

We can arrive at the result of Bethe and Salpeter (*22*) by defining

$$D_{\Omega K_1}^{k\lambda} \equiv \sqrt{mKV/(2\pi)^3\hbar^2} < \mathbf{K}_1 \,|\, e^{-i\mathbf{k}\cdot\mathbf{x}} \boldsymbol{\varepsilon}_\lambda \cdot \nabla \,|\, \mathbf{K}>, \quad \text{(D.4)}$$

where $D_{\Omega K_1}^{k\lambda}$ represents a matrix element with normalization different from that used above. The cross section can now be written

$$\sigma_p^{\lambda}(K_1, \mathbf{\Omega}, \mathbf{k})\, d\Omega = \int \sigma_p^{\lambda}(K_1, E, \mathbf{\Omega}, \mathbf{k})\, dE = (4\pi^2 e^2 \hbar^2/m^2 c^2 k)\, |\, D_{\Omega K_1}^{k\lambda}|^2\, d\Omega, \quad \text{(D.5)}$$

which is just the result given in reference (*22*), Eq. (69.5), when obvious notation changes are made.

A further reduction can be achieved by use of the hydrogen-like atom ground-state wave function

$$|\,\mathbf{K}_1 > \; = \sqrt{\frac{Z^3}{a_o^3 \pi}}\, e^{-r/a_0},$$

where $a_o = \hbar^2/me^2$. The integration over $\mathbf{x}$ yields

$$\sigma_p^{\lambda}(K_1, E, \mathbf{\Omega k})\, dE\, d\Omega = dE\, d\Omega \frac{32 Z^5}{a_o^5 k} \sqrt{\frac{2mck}{\hbar}} \left(\frac{e^2}{mc^2}\right) \times \delta(E_{K_1 k} - E_K) \left\{ \frac{\boldsymbol{\varepsilon}_\lambda \cdot \mathbf{K}}{\left[\frac{1}{a_o^2} + (\mathbf{k} - \mathbf{K})^2\right]^2} \right\}^2. \quad \text{(D.6)}$$

Assuming

$$\text{(a)}\ (\hbar^2 K/2m) \simeq \hbar ck, \qquad \text{(b)}\ a_o^{-2} \ll k^2 + K^2, \qquad \text{(c)}\ (\hbar ck/mc) \ll 1, \quad \text{(D.7)}$$

we obtain

$$\begin{aligned}\sigma_p(K_1, \mathbf{\Omega}, \mathbf{k})\, d\Omega &= \sum_\lambda \int dE\, \sigma_p^\lambda(K_1, E, \mathbf{\Omega}, \mathbf{K}) \\ &= 4\sqrt{2}Z^5 \left(\frac{mc}{\hbar k}\right)^{7/2} \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{e^2}{\hbar c}\right)^4 \left(1 - \frac{v}{c}\mu\right)^{-4} \sin^2\theta\, d\Omega,\end{aligned} \tag{D.8}$$

where $\mu \equiv \cos\theta = \mathbf{k}\cdot\mathbf{K}/kK$. Lastly

$$\sigma_p(K_1, \mathbf{k}) = \int d\Omega\, \sigma_p(K_1, \mathbf{\Omega}, \mathbf{k}) = \phi_0 \frac{Z^5}{(137)^4} 2\sqrt{2} \left(\frac{mc}{\hbar k}\right)^{7/2}, \tag{D.9}$$

where $\phi_o = 4\pi r_o^2/3$, $r_o = e^2/mc^2$, and $\alpha = e^2/\hbar c = 1/137$. If we multiply by a factor 2 to account for two electrons in the K shell, we obtain the result found in Heitler (*21*) [Eq. (14), p. 207].

APPENDIX E. REDUCTION OF ϵ_B

In Eq. (66e) we let σ denote electrons and σ' ions, then convert the sums to integrals to obtain

$$\begin{aligned}\epsilon_B^\lambda = \left(\frac{1}{2\pi}\right) r_0^2 Z^2 \alpha \frac{\hbar^2 c^4}{k} \int & d^3K\, d^3K_1\, d^3K_2\, d^3K_3\, \delta(\mathbf{K} + \mathbf{K}_1 - \mathbf{K}_2 - \mathbf{K}_3 - \mathbf{k}) \\ & \times \delta(\omega_f - \omega_i) f_\sigma(K) f_{\sigma'}(K_1) \,|\, U(|\mathbf{K}_1 - \mathbf{K}_3|)\,|^2 \\ & \times \left| \frac{\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\mathbf{K}_2}{\hbar ck - (\hbar^2 k^2/2m_\sigma) - (\hbar^2\mathbf{k}\cdot\mathbf{K}_2/m_\sigma)} \right. \\ & \left. - \frac{\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\mathbf{K}}{\hbar ck + (\hbar^2 k^2/2m_\sigma) - (\hbar^2\mathbf{k}\cdot\mathbf{K}/m_\sigma)} \right|^2,\end{aligned} \tag{E.1}$$

where $r_0 = e^2/mc_\sigma^2$, $\alpha = e^2/\hbar c$ and we have utilized energy conservation,

$$E_{\sigma K_2} + E_{\sigma' K_3} + \hbar ck = E_{\sigma K} - E_{\sigma' K_1}.$$

In a nonrelativistic approximation

$$\hbar ck \gg \frac{\hbar^2 k^2}{2m_\sigma} \quad \text{and} \quad \hbar ck \gg \frac{\hbar^2 \mathbf{k}\cdot\mathbf{K}_2}{m_\sigma} \simeq \frac{\hbar^2\mathbf{k}\cdot\mathbf{K}}{m_\sigma}.$$

If now we take the mass of the ions as infinite, then perform the K_3 and K_1 integrations, we find that

$$\begin{aligned}\epsilon_B^\lambda = 8\pi r_0^2 Z^2 \alpha \frac{c^2\hbar}{k^3} & \int f_\sigma(\mathbf{K})\, d^3K \\ & \times \int d^3K_2\, \delta(E_2 - E + \hbar ck) \frac{|\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\mathbf{K}_2 - \boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\mathbf{K}|^2}{|\mathbf{K}_2 - \mathbf{K} + \mathbf{k}|^4}.\end{aligned} \tag{E.2}$$

Note that $d^3K_2 = \sqrt{2}(m/\hbar^2)^{3/2}E^{1/2}\,dE\,d\Omega$. Now average over polarization, placing $\mathbf{k}$ along the z axis, and perform integrations over $\mathbf{K}_1$ and E_2 to get

$$\begin{aligned}\bar{\epsilon}_B &= 2\sqrt{2}\pi n_{\sigma'} r_0^2 Z^2 \alpha \frac{c^2 m_\sigma^{1/2}}{k^3} \int f_\sigma(E)\sqrt{E(E - \hbar ck)}\, dE\, d\Omega\, d\Omega_2 \\ &\times \frac{E\sin^2\theta + (E - \hbar ck)\sin^2\theta_2 - 2\sqrt{E(E - \hbar ck)}\sin\theta\sin\theta_2\cos(\phi - \phi_2)}{[E + (E - \hbar ck) - 2\mu_0\sqrt{E(E - \hbar ck)}]^2}\end{aligned} \tag{E.3}$$

where we have neglected $\mathbf{k}$ in comparison $\mathbf{K}_2 - \mathbf{K}$, and written

$$|\mathbf{K}_2 - \mathbf{K}|^4 = (2m_\sigma/\hbar^2)^2[E + (E - \hbar ck) - 2\mu_o\sqrt{E(E - \hbar ck)}]^2$$

with $\mu_o = \mu\mu_2 + \sin\theta\sin\theta_2\cos(\phi - \phi_2)$ and $\mu = \cos\theta$. We arrive at a more familiar form by observing that

$$j_B\, d\omega = d\omega n_e n_I \hbar\omega \int v\bar{f}_e\, dE\, d\Omega \int \sigma_B\, d\Omega_2, \tag{E.4}$$

where

$$\bar{f}_e = f_e/n_e\,.$$

But

$$j_B\, d\omega = \bar{\epsilon}_B(2k^2/(2\pi)^3)\hbar\omega\, d(ck)$$

so that

$$\begin{aligned}\sigma_B\, d\Omega_2 &= \frac{1}{2\pi^2}\frac{r_0^2 Z^2}{137}\frac{mc^2}{\omega}\sqrt{\frac{E - \hbar\omega}{E}} \\ &\times \frac{E\sin^2\theta + (E - \hbar ck)\sin^2\theta_2 - 2\sqrt{E(E - \hbar ck)}\sin\theta\sin\theta_2\cos(\phi - \phi_2)}{[E + (E - \hbar ck) - 2\mu_0\sqrt{E(E - \hbar ck)}]^2}\,.\end{aligned} \tag{E.5}$$

When appropriate variable change is accomplished and one ϕ integration is performed, this result is equivalent to that found in Heitler (*21*) [Eq. (17), p. 245].

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REFERENCES

1. S. CHANDRASEKHAAR, "Stellar Structure," Chapter 5. Dover, New York, 1957.
2. H. GOLDSTEIN, "The Attenuation of Gamma Rays in Reactor Shields," p. 139. Addison-Wesley, Reading, 1959.
3. R. K. OSBORN, University of Michigan Radiation Laboratory Report no. 2756-1-T, 1959.
4. W. E. DRUMMOND AND M. N. ROSENBLUTH, *Physics of Fluids* **3**, 45 (1960).
5. L. I. SCHIFF, "Quantum Mechanics," Chapter 9. McGraw-Hill, New York, 1949.
6. W. E. BRITTIN, *Phys. Rev.* **106**, 843 (1957).
7. A. SIMON AND E. G. HARRIS, *Physics of Fluids* **3**, 245 and 255 (1960).

8. H. Mori and J. Ross, *Phys. Rev.* **109**, 1877 (1958).
9. S. Ono, *Progr. Theoret. Phys. Japan* **12**, 113 (1954).
10. M. H. Johnson and B. A. Lippmann, *Phys. Rev.* **76**, 828 (1949).
11. S. Ono, *in* "Proceedings of the International Symposium on Transport Processes in Statistical Mechanics," p. 229. Interscience, New York, 1959.
12. S. Chapman and T. G. Cowling, "The Mathematical Theory of Non-Uniform Gases," Chapter 4. Cambridge Univ. Press, London and New York, 1958.
13. C. Kittel, "Elementary Statistical Physics," Section 37. Wiley, New York, 1958.
14. T. L. Hill, "Statistical Mechanics," p. 94. McGraw-Hill, New York, 1956.
15. J. E. Mayer and M. G. Mayer, "Statistical Mechanics," p. 363. Wiley, New York, 1940.
16. N. G. van Kampen, *in* "Proceedings of the International Symposium on Transport Processes in Statistical Mechanics." p. 239. Interscience, New York, 1959.
17. G. Parzen, *Phys. Rev.* **84**, 235 (1951).
18. D. L. Judd, J. V. Lepore, M. Ruderman, and P. A. Wolff, *Phys. Rev.* **86**, 123 (1952).
19. B. Trubnikov and V. S. Kudryavtsev, *in Proc. 2nd Intern. Conf. Peaceful Uses Atomic Energy*, Geneva, **31**, 93 (1958).
20. S. M. Berman, "Electromagnetic Radiation from an Ionized Hydrogen Plasma," Space Technology Laboratories, PRL 9-27 (1959).
21. V. Heitler, "The Quantum Theory of Radiation," 3rd ed., Oxford Univ. Press, London and New York, 1954.
22. H. Bethe and E. Salpeter, "Quantum Mechanics of One- and Two-Electron Atoms," p. 296. Academic Press, New York, 1957.
23. V. I. Kogan, "Plasma Physics and Controlled Thermonuclear Fusion," Vol. III, p. 116. Pergamon Press, London, 1959.
24. F. H. Clauser, "Plasma Dynamics," Chapter 2. Addison-Wesley, Reading, 1960.
25. R. F. Post, *in* "Annual Review of Nuclear Science," Vol. 9, p. 379. Annual Reviews, Stanford, 1959.
26. V. V. Babikov and V. I. Kogan, "Plasma Physics and Controlled Thermonuclear Reactions," Vol. III, p. 102. Pergamon Press, London, 1959.
27. J. Kvasnica, *Czech. J. Phys.* **B10**, 14 (1960).
28. L. Spitzer, Jr., "Physics of Fully Ionized Gases." Interscience, New York, 1956.
29. U. Fano, *Phys. Rev.* **103**, 1202 (1956).
30. H. Olsen and H. Wergeland, *Phys. Rev.* **86**, 123 (1952).