

On Mixed Boundary-Value Problems for Axially-Symmetric Potentials

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This paper concerns the determination of an axially symmetric harmonic function that is a function $u(r, y)$ satisfying the equation

$$u_{rr} + r^{-1}u_r + u_{yy} = 0 \quad \text{in} \quad y < 0 \quad 0 \leq r, \quad (E)$$

when u is subject to the mixed boundary conditions,

$$u(r, 0) = g(r) \quad 0 \leq r < 1, \quad (A)$$

$$u_y(r, 0) = 0 \quad r > 1. \quad (B)$$

This problem, which physically corresponds to the potential of a disk in an external field, has a long history dating back to Kelvin. There are several known methods of solution all quite complicated. Of these we mention the dual integral equation method of Titchmarsh [1] and a later method of Copson [2]. It is the purpose of this note to point out that these methods are not necessary but that the problem is easily reduced to an elementary boundary-value problem for a harmonic function of two variables.

We suppose,

$$u(r, y) \rightarrow 0 \quad \text{as} \quad r^2 + y^2 \rightarrow \infty.$$

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Then $u(r, y)$ can be represented in the integral form,

$$u(r, y) = (2\pi)^{-1} \int_0^1 \int_0^{2\pi} [r^2 + \rho^2 - 2r\rho \cos \theta + y^2]^{-1/2} \rho u_y(\rho, 0) d\theta d\rho.$$

From this we have

$$u(0, y) = \int_0^1 (\rho^2 + y^2)^{-1/2} \rho u_y(\rho, 0) d\rho. \tag{1}$$

While Eq. (1) is derived initially for y real we observe that the integral on the right side is an analytic function for complex $y = \eta + i\tau$. Accordingly $u(0, y)$ is extended analytically to complex y . Let us denote the extension by

$$u(0, y) = U(y) = \varphi(\eta, \tau) + i\psi(\eta, \tau)$$

From Eq. (1) we find immediately then that

$$\text{Re } U(y) = \varphi(\eta, \tau) = 0 \quad \text{for} \quad \eta = 0 \quad |\tau| > 1. \tag{2}$$

Note also that $U(y)$ equals $u(0, \eta)$ on $\tau = 0$ hence is real there and we have,

$$U(\bar{y}) = \overline{U(y)} \quad \text{or} \quad \varphi(\eta, -\tau) = \varphi(\eta, \tau), \psi(\eta, -\tau) = -\psi(\eta, \tau). \tag{3}$$

Finally, it follows from Eq. (1) that

$$U(y) \rightarrow 0 \quad \text{as} \quad |y| \rightarrow \infty. \tag{4}$$

Now it is known that solutions of equation (E) are determined by their values on $r = 0$, the axis of symmetry, provided those values are analytic. In fact the answer may be written down [3] in the integral form,

$$u(r, y) = \pi^{-1} \int_{-r}^{+r} (r^2 - \tau^2)^{-1/2} U(y + i\tau) d\tau,$$

or, using (3),

$$u(r, y) = 2(\pi)^{-1} \int_0^r (r^2 - \tau^2)^{-1/2} \varphi(y, \tau) d\tau. \tag{5}$$

Let us now enter the boundary condition (A). We find,

$$g(r) = 2(\pi)^{-1} \int_0^r (r^2 - \tau^2)^{-1/2} \varphi(0, \tau) d\tau \quad 0 \leq r < 1.$$

This Volterra integral equation is easily inverted to yield (c.f. [4])

$$\varphi(0, \tau) = \frac{d}{d\tau} \int_0^\tau \rho g(\rho) (\tau^2 - \rho^2)^{-1/2} d\rho \equiv h(\tau) \quad 0 \leq \tau < 1. \quad (6)$$

By Eq. (3) we have also

$$\varphi(0, \tau) = h(-\tau) \quad -1 < \tau \leq 0. \quad (7)$$

The harmonic function of two variables $\varphi(\eta, \tau)$ is then known from Eqs. (2), (6), and (7) on the entire τ axis. Further, according to (4) it must vanish as $\eta^2 + \tau^2 \rightarrow \infty$. The required function is then given by the Poisson integral,

$$\varphi(\eta, \tau) = (\pi^{-1})\eta \int_{-1}^{+1} h(t) [(\eta^2 + (\tau - t)^2)^{-1}] dt.$$

Having determined $\varphi(\eta, \tau)$ we then find $u(r, y)$ from Eq. (5).

The use of the integral representation (5) for an axially-symmetric function in terms of its values on the axis of symmetry seems to be a rather powerful tool. We remark that the case of a circular disk in a non-symmetric field can be handled similarly. One separates the angular dependence in a Fourier series. The coefficients of $\cos n\theta$ or $\sin n\theta$ are then functions $u^n(r, \theta)$ which satisfy instead of (E) equations of the form,

$$u_{rr}^n + r^{-1} u_r^n + u_{yy}^n - n^2 r^{-2} u^n = 0.$$

Analogous of formula (5) exist for these equations (see [3]) and the whole technique can be duplicated. Quite similar although somewhat more complicated methods have been successfully employed by the authors in the study of acoustic wave diffraction by a circular disk [5].

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