CLOSED-FORM SOLUTIONS FOR COMPUTING THE INTERSECTION BETWEEN TWO DISKS IN 3-D

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Abstract: This report presents an algorithm for computing the intersection between two circular disks in three-dimensional space. Rotation transformations and geometric properties of disks are used to simplify the intersection problem. The proposed algorithm enumerates the possible solutions of the simplified model, and uses deduction and/or induction techniques to find out the intersecting points. In all cases, closed-form solutions are obtained, and few tests are sufficient to compute the intersection, or to show that no such intersection exists.

Keywords: Computational geometry, circular disks, intersection detection and computation, algebraic computation.

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I. Introduction

The problems of minimum distance, and intersection detection and computation between two objects in three-dimensional space, have important applications in robotics, CAD systems, VLSI and other areas of information processing that deal with geometrical data. In particular, the intersection problem has been studied in the literature for a class of objects that can be represented by linear models. This class includes, line segments [1], rectangles [2, 3, and 4], boxes [5], planar polygons [6, 7, and 8], and convex polyhedrals [9, 10, and 11]. Other references on the intersection problem and its applications can be found in [12, 13, 14, and 15].

In this report, we are interested in solving the intersection problem between two circular disks defined in three-dimensional space with arbitrary position and orientation. The solutions to this problem depend on the representation used to describe a disk. For example, we can define a disk as a finite number of concentric circles, where each circle is defined by a finite number of points. In this case, for a given pair of disks, it might be possible to find a polynomial time algorithm for computing the intersection or the minimum distance between both disks. However, this method provides approximate solutions, depending on the number of circles and points used to define the disks. This approach may not be adequate if we are looking for exact solutions.

In this work, we consider a disk as a continuous area of points (an infinite number of points). A point of the disk may be represented by polar coordinates, Cartesian coordinates, or by some other equivalent parametric representations. Unlike the previous problems, where the objects are defined by half planes, the geometry of a disk is nonlinear by nature. In general, nonlinear models can be solved using nonlinear mathematical programming methods. However, for some problems, these methods may not be suitable for real-time applications because of their computational complexity. Besides, reasoning the geometry of disks may sensibly reduce the complexity of the problem, and provides fast computational solutions.

Our Approach for the intersection problem between two disks is based on using rotation transformations and geometric properties of disks. These operations produce a system of nonlinear equations that is more simple to solve than the original system. An algorithm that uses induction and/or deduction technique for solving the reduced system will be proposed. In all cases, closed-form solutions will be obtained for the intersection problem. The proposed algorithm (apparently the first in the literature) requires a small number of tests to compute the intersecting points, or to show that no such intersection exists.

II. Disks Intersection Formulation

Let F=(X,Y,Z) be the attached coordinates frame of a circular disk K centered at the origin O and lying in the X-Y plane of F. A disk is a collection of points V which can be represented as follows:

$$K = \{V=(x,y,z)^T \mid x = r\cos(q),\ y = r\sin(q),\ z = 0,\ 0 \le r \le D\} \qquad (1)$$

where D is the radius of the disk.

An arbitrary 3D disk is obtained by rotating K and translating its origin with respect to a fixed coordinates frame $F_0 = (X_0,Y_0,Z_0)$. Let R be a 3 x 3 rotation matrix that defines the disk plane, and let $P=(P_x,P_y,P_z)^T$ be its origin with respect to F_0. Let W be the absolute coordinates vector of a point V on the disk, then K can be defined as follows:

$$K = \{W = R\,V + P \mid V = (x,y,z)^T,\ x = r\cos(q),\ y = r\sin(q),\ z = 0,\ 0 \le r \le D\} \qquad (2)$$

Let K_1 and K_2 be two disks centered at $P_1 = (P_{1x}, P_{1y}, P_{1z})^T$ and $P_2 = (P_{2x}, P_{2y}, P_{2z})^T$ with radius D_1 and D_2 respectively. Let $R_1 = \{R_{1ij}\}$and $R_2 = \{R_{2ij}\}$, i, j = 1, 2, 3, be the rotation matrices that define the planes of K_1 and K_2 respectively. Determining the intersection between K_1 and K_2 is equivalent to solving the following problem:

$$\begin{array}{ll} & R_1 V_1 + P_1 = R_2 V_2 + P_2 \\ \text{subject to} & \|V_1\| \le D_1,\ \|V_2\| \le D_2, \end{array} \qquad (3)$$

where $V_1 = (V_{1x}, V_{1y}, 0)^T$, $V_2 = (V_{2y}, V_{2y}, 0)^T$, and $\|\cdot\|$ denotes the L_2-norm. Problem (3) represents a quadratic system having five equations in four unknowns, namely x_1, y_1, x_2, and y_2. In the following Lemma, we show a relationship between the locations of the optimal points V_1 and V_2 when the two disks have some intersection. This property is stated as follows:

Lemma 1: If K_1 and K_2 intersect, then there exists at least one intersecting point V for which:

$$r_1 = D_1 \quad \text{or} \quad r_2 = D_2$$

Proof: Assume that K_1 and K_2 have an intersection A, then three cases arise:

A is a point

A is a straight-line

A is an arbitrary area

It is clear in the first case that the intersecting point is on the edge of either K_1 and K_2, and hence either $r_1 = D_1$ or $r_2 = D_2$. If the intersection is a straight-line, then the two end points of the line lie either on the edge of K_1 or K_2, or both. This implies that the end points have $r_1 = D_1$ or $r_2 = D_2$. Case 3 implies that the two disks lie in the same plane which contains the intersection area.

Hence there exists at least one intersection point for which $r_1 = D_1$ and $r_2 = D_2$. This completes the proof.

Recall that a rotation matrix R is an orthogonal matrix satisfying $R\,R^T = R^T R = I$, where R^T denotes the transpose of R, and I is a 3 x 3 identity matrix. Multiplying both sides of (3) by $R_1{}^T$, one obtains after rearrangement:

$$V_1 = R_1{}^T R_2 V_2 + R_1{}^T (P_2 - P_1) \tag{4}$$

Let ${}^0P = R_1{}^T (P_2 - P_1)$, and ${}^0R = R_1{}^T R_2$, then problem (3) becomes:

$$\begin{aligned} & V_1 = {}^0R\, V_2 + {}^0P \\ \text{subject to} \quad & \|V_1\| \le D_1,\ \|V_2\| \le D_2, \end{aligned} \tag{5}$$

With the above transformations, the intersection problem is reduced to the problem of finding the intersection between a disk K_1 lying in the (X_0,Y_0) plane and centered at the origin O_0 of the fixed coordinates frame F_0, and a disk K_2 of center ${}^0P$ and rotation matrix ${}^0R$.

III. Geometric Transformations

Our approach for solving the intersection problem consists first on transforming the equality in (5) to a simpler form. This can be done by using rotation transformations that do not alternate the geometric structure of the problem. For this, we shall show that for any given matrix ${}^0R$ there exist three rotation matrices A, B, and R such that :

$$B\, {}^0R\, A^T = R \tag{6}$$

where A, B, and R are of the forms:

$$A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$

For this, let C be a 3 x 3 rotation matrix such that $C = {}^0R\, A^T$. Post multiply ${}^0R$ by A^T and set the entry C_{31} to zero, we obtain:

$$\begin{aligned} C_{31} &= {}^0R_{31} \cos(\alpha) - {}^0R_{31} \sin(\alpha) \\ &= 0 \end{aligned}$$

where α can be computed as follows:

$$\tan(\alpha) = \frac{{}^0R_{31}}{{}^0R_{32}} \quad \Rightarrow \quad \alpha = \operatorname{atan}\left(\frac{{}^0R_{31}}{{}^0R_{32}}\right) \tag{7}$$

The value of α is not unique, but since the disks are assumed to have two identical sides, the solutions $\alpha + \pi$ and $\alpha - \pi$ correspond to the same plane of the disk. Therefore, we can restrict α to be in $[0, \pi]$ by checking the signs of ${}^0R_{31}$ and ${}^0R_{31}$. Note that, if ${}^0R_{32} = 0$, then we can permute column 1 and 2 of ${}^0R$ so that the entry $C_{31} = 0$. In this case, the permutation matrix is obtained by setting $\alpha = \pi/2$ in A. Thus, the resulting matrix C is of the form:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix}$$

Let D be a 3 x 3 rotation matrix such that D = B C. Post multiply C by B and set the entry D_{21} to zero, we obtain:

$$\begin{aligned} D_{21} &= C_{11} \sin(\beta) + C_{21} \cos(\beta) \\ &= 0 \end{aligned}$$

which implies that:

$$\tan(\beta) = -\frac{C_{21}}{C_{11}} \quad \Rightarrow \quad \beta = \operatorname{atan}\left(\frac{C_{21}}{C_{11}}\right) \tag{8}$$

Similarly, the solutions $\beta + \pi$ and $\beta - \pi$ provide the same plane for the disk. Therefore, we can determine β in $[0, \pi]$ by checking the signs of C_{11} and C_{21}. If $C_{11} = 0$, then we can permute rows 1 and 2 of C so that the entry $D_{21} = 0$. The resulting matrix D is of the form:

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ 0 & D_{22} & D_{23} \\ 0 & D_{32} & D_{33} \end{bmatrix}$$

Since D is an orthonormal matrix, then D_{11} must be equal to 1 or -1, and $D_{12} = D_{13} = 0$. If $D_{11} = -1$, then we can pre multiply D by a matrix I_1 such that $R = I_1 D$, where I_1 is of the form:

$$I_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In this case, we have $R_{11} = 1$. Since D is a rotation matrix and $D_{11} = 1$, then $D_{22} = D_{33}$, and $D_{23} = -D_{32}$. Therefore, by setting $D_{22} = \cos(\gamma)$ and $D_{23} = -\sin(\gamma)$, the matrix R will be of the form:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix} \tag{9}$$

where $R = D$, if $D_{11}= 1$, or $R = I_1 D$, if $D_{11} = -1$, and γ is uniquely determined by D_{22} and D_{23}. The above transformations show that for a given matrix ${}^0R$, we can find a matrix R such that R is a rotation matrix about the X_0-axis of disk K_2.

Consider now the equality $V_1 = {}^0R\, V_2 + {}^0P$. Pre multiply this equality by B and post multiply ${}^0R$ by $A^T A$, we obtain:

$$B\, V_1 = B\, {}^0R\, A^T A\, V_2 + B\, {}^0P$$

Let $M_1 = B\, V_1$, $M_2 = A\, V_2$, and $P = B\, {}^0P$, and substitute R for $B\, {}^0R\, A^T$ in the above equation, we have:

$$M_1 = R\, M_2 + P$$

where: $M_1 = (r_1 \cos(q_1+ \beta), r_1 \sin(q_1+ \beta), 0)^T$, $M_2 = (r_2 \cos(q_2+ \alpha), r_2 \sin(q_2+ \alpha), 0)^T$, and $P = (P_x, P_y, P_z)^T$. Replacing $q_1+ \beta$ by θ_1 in M_1, and $q_2+ \alpha$ by θ_2 in M_2, we have:

$$\begin{aligned} M_1 &= (r_1 \cos(\theta_1), r_1 \sin(\theta_1), 0)^T \qquad , \theta_1 = q_1+ \beta \\ &= (x_1, y_1, 0)^T \end{aligned} \tag{10}$$

$$\begin{aligned} M_2 &= (r_2 \cos(\theta_2), r_2 \sin(\theta_2), 0)^T \qquad , \theta_2 = q_2+ \alpha \\ &= (x_2, y_2, 0)^T \end{aligned}$$

Finally, the disk intersection problem (5) is reduced to the following form:

$$\begin{aligned} & M_1 = R\, M_2 + P \\ \text{subject to} \quad & \|M_1\| \leq D_1, \; \|M_2\| \leq D_2, \end{aligned} \tag{11}$$

Expanding (11) in terms of the coordinates of M_1, M_2, and P, we have:

$$\begin{aligned} & x_1 = x_2 + P_x \\ & y_1 = \cos(\gamma)\, y_2 + P_y \\ & 0 = \sin(\gamma)\, y_2 + P_z \\ \text{subject to:} \quad & (x_1)^2 + (y_1)^2 \leq (D_1)^2 \\ & (x_2)^2 + (y_2)^2 \leq (D_2)^2 \end{aligned} \tag{12}$$

The above transformations, have reduced the problem of computing the intersection between two arbitrary disks in 3D, to a problem where disk K_1 is centered at the fixed origin, and disk K_2 is centered at a point P, and whose plane is rotated by an angle γ about the X_0-axis of the fixed coordinates frame F_0. Finally, it is clear that these transformations preserve the geometric structure of the intersection problem.

IV. Intersection Computation Algorithm

In this section, we give the algorithm for solving (12):

Case 1: If $\sin(\gamma) = 0$, which means that the planes of disks K_1 and K_2 are parallel. Then the intersection depends on the value of P_z.

Case 1.1: If $P_z \neq 0$, then there is no intersection between K_1 and K_2.

Case 1.2: If $P_z = 0$, then the intersection depends on the values of P_x and P_y.

Case 1.2.1: If $P_x = P_y = 0$. In this case the disks are centered at the origin. The following cases should be examined for the intersection:

Case 1.2.1.1: If $D_1 > D_2$, then K_2 is contained in K_1, and the intersection is the set of all points of K_2.

Case 1.2.1.2: If $D_1 = D_2$, then K_1 and K_2 are identical, and the intersection is the set of all points of K_1 or K_2.

Case 1.2.1.3: If $D_1 < D_2$, then K_1 is contained in K_2, and the intersection is the set of all points of K_1.

Case 1.2.2: If $P_x \neq 0$ or $P_y \neq 0$, then the intersection can be checked as follows:

Case 1.2.2.1: If $\sqrt{(P_x)^2 + (P_y)^2} > D_1 + D_2$, then there is no intersection between K_1 and K_2.

Case 1.2.2.2: If $\sqrt{(P_x)^2 + (P_y)^2} = D_1 + D_2$, then the intersection is reduced to a unique point given by:

$$x_1 = \frac{D_1 P_x}{D_1 + D_2} \qquad y_1 = \frac{D_1 P_y}{D_1 + D_2}$$

$$x_2 = \frac{- D_2 P_x}{D_1 + D_2} \qquad y_2 = \frac{-1}{\cos(\gamma)} \frac{D_2 P_y}{D_1 + D_2}$$

where $\cos(\gamma) = \pm 1$, depending on whether $\gamma = 0$ or π .

Case 1.2.2.3: If $\sqrt{(P_x)^2 + (P_y)^2} < D_1 + D_2$, then the following cases should be considered:

Case 1.2.2.3.1: If $\sqrt{(P_x)^2 + (P_y)^2} + D_2 \leq D_1$, then the set of intersecting points is disk K_2, since K_2 is contained in K_1.

Case 1.2.2.3.2: If $\sqrt{(P_x)^2 + (P_y)^2} + D_1 \leq D_2$, then the set of intersecting points is disk K_1, since K_1 is contained in K_2.

Case 1.2.2.3.3: If non of cases (1.2.2.3.1) and (1.2.2.3.2) is valid, then the disks have some overlapping area. In particular, the point M_1 on the border of K_1 satisfying the following coordinates:

$$x_1 = \frac{D_1 P_x}{\sqrt{(P_x)^2 + (P_y)^2}} \qquad y_1 = \frac{D_1 P_y}{\sqrt{(P_x)^2 + (P_y)^2}}$$

is an intersection point. Substituting x_1 and y_1 in (12) , we obtain the following coordinates for M_2:

$$x_2 = \frac{D_1 P_x}{\sqrt{(P_x)^2 + (P_y)^2}} - P_x \, , \, y_2 = \frac{1}{\cos(\gamma)} \left[\frac{D_1 P_y}{\sqrt{(P_x)^2 + (P_y)^2}} - P_y \right]$$

<u>Case 2</u>: If $\sin \gamma \neq 0$, then we should have for the intersection:

$$y_2 = - \frac{P_z}{\sin \gamma} \qquad \text{and} \qquad y_1 = - \frac{\cos \gamma}{\sin \gamma} P_z + P_y$$

We now use Lemma 1, which states that if K_1 and K_2 have a non-empty intersection, then one of the intersecting points must be on the border of K_1 or K_2. The following cases should be studied:

<u>Case 2.1</u>: Assume $(x_1)^2 + (y_1)^2 = D_1^2$.

<u>Case 2.1.1</u>: If $D_1^2 < (y_1)^2$, then the disks have no intersection, since $(x_1)^2 + (y_1)^2 = D_1^2$ has no real solutions.

<u>Case 2.1.2</u>: If $D_1^2 \geq (y_1)^2$, then the coordinates of M_1 are:

$$x_1 = \pm \sqrt{D_1^2 - \left[- \frac{\cos \gamma}{\sin \gamma} P_z + P_y \right]^2} \qquad y_1 = - \frac{\cos \gamma}{\sin \gamma} P_z + P_y$$

Using (12), the coordinates of M_2 are:

$$x_2 = x_1 - P_x \qquad y_2 = - \frac{P_z}{\sin \gamma}$$

<u>Case 2.1.2.1</u>: If $(x_2)^2 + (y_2)^2 \leq D_2^2$, then the above points verify the intersection.

<u>Case 2.1.2.2</u>: If $(x_2)^2 + (y_2)^2 > D_2^2$, then the disks have no intersection.

If Case 2.1 does not hold, then we have to check whether M_2 is on the border of K_2. This is done in the same way as in Case 2.1, that is:

<u>Case 2.2</u>: Assume $(x_2)^2 + (y_2)^2 = D_2^2$.

<u>Case 2.2.1</u>: If $D_2^2 < (y_2)^2$, then the disks have no intersection, since $(x_2)^2 + (y_2)^2 = D_2^2$ has no real solutions.

<u>Case 2.2.2</u>: If $D_2^2 \geq (y_2)^2$, then the coordinates of M_2 are:

$$x_2 = \pm \sqrt{D_2^2 - \left[\frac{P_z}{\sin \gamma} \right]^2} \qquad y_2 = - \frac{P_z}{\sin \gamma}$$

Using (12), the coordinates of M_1 are:

$$x_1 = x_2 + P_x \qquad y_1 = -\frac{\cos\gamma}{\sin\gamma} P_z + P_y$$

Case 2.2.2.1: If $(x_1)^2 + (y_1)^2 \leq D_1^2$, then the above points verify the intersection.

Case 2.2.2.2: If $(x_1)^2 + (y_1)^2 > D_1^2$, then the disks have no intersection.

Finally, if non of Cases 2.1 or 2.2 hold, then the disks have no intersection.

Note that the above algorithm provides closed-form solutions for all cases of intersection. The worst case is the case for which $\sin\gamma = 0$, and the disks have or not an overlapping area. In this case, at most 7 comparison tests are required to find the result. If the proposed algorithm terminates by finding the intersection, then one has to reapply the inverse transformations given in Section 3 to compute the exact solutions of the initial problem. In this case, if a solution $M_1 = (x_1, y_1, 0)^T$ and $M_2 = (x_2, y_2, 0)^T$ is found, then the relative coordinates of the original points $V_1 = (V_{1x}, V_{1y}, 0)^T$ and $V_2 = (V_{2x}, V_{2y}, 0)^T$ corresponding to M_1 and M_2 respectively are given by the following transformations:

$$V_1 = B^T M_1 \qquad \text{and} \qquad V_2 = A^T M_2$$

where B^T and A^T are defined by (6), (7) and (8).

V. Conclusion:

We presented in this report a geometric algorithm for computing the intersection between two circular disks in three-dimensional space. Several transformations were used to simplify the intersection problem and to provide closed-form solutions with simple expressions. The algorithm requires few tests to locate and compute the intersecting points. However, the implementation of the algorithm could be done without making these transformations. In fact, equations (5) will be enough for developing the algorithm. In this case, we may end up with an algorithm that requires more tests to find the intersection , than the proposed algorithm with transformations.

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