Classification and Evaluation of Upper Atmosphere Data

by H. A. Alperin and W. E. Burdick

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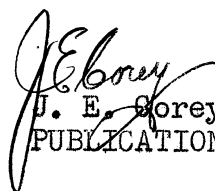
UNIVERSITY OF MICHIGAN

Willow Run Research Center
Engineering Research Institute

October 1, 1954

SUBJECT: Downgrading of Classification, University of Michigan
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1.   This is to advise that security classification of the
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     UMM-56 "Classification and Evaluation of Upper Atmosphere
             Data" by H. A. Alperin and W. E. Burdick , June 1950.
             Project MX-794   USAF Contract W33-038-ac-14222

                              J. E. Corey , Chief
                              PUBLICATIONS BRANCH

JEC/nc

Dist to:
ANAF-GM Mailing List No. 16 dated 15 July 1951,
to include Part A, Part B and Part C.

Errata on UMM-56, "Classification and Evaluation of Upper Atmosphere Data," by H. A. Alperin and W. E. Burdick.

Page 19, line 27 - Change "evergy" to "energy".

Delete the word "if".

J. E. Corey
Supervisor
Technical Data Services

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A. PREFACE

In January of 1949 the High Altitude Studies Section of Willow Run Research Center commenced an analysis of the parameters, coefficients, and constants pertaining to the atmosphere. The experimental literature was summarized (Ref. 1), and a new empirical temperature curve was formulated (Ref. 2).

This report explains the processes which determined this new curve. It also discusses the methodology of using this curve to obtain the parameters and variables that are used by the other groups of WRRC to predict and compute missile trajectories.

This report fulfills both a need and a responsibility. In the Wizard program the Willow Run Research Center needs the best available information governing all the ramifications of offensive and defensive missile flight. It then is the responsibility of the High Altitude Studies Section to learn what information exists for the atmosphere, how this information can be improved, and how it can be utilized for the greatest overall efficiency in reaching the ultimate goal of WRRC.

Keeve M. Siegel

B. INTRODUCTION

The atmospheric variables and parameters which most affect trajectory analysis are temperature, density, mean free path, pressure, speed of sound and coefficient of viscosity. It is the purpose of this report to provide values of each of these quantities or methods of calculating them.

Underlying all calculations of pressure and density is an equation of state. This report presents an explanation of why the perfect gas law may be used in all calculations.

After having established the correct equation of state, the functional relationship between temperature and altitude must be ascertained. This has been done empirically by considering all the important experimental determinations of temperature. A new temperature curve results.

Using the equation of state, the hydrostatic equation, and the new temperature curve, values of pressure and density may now be calculated.

Because of the extreme altitude covered, it is well to re-examine our basic concepts of temperature, the speed of sound, and the coefficient of viscosity. This has been done in this report, and the applicable relations for the calculation of these quantities is presented.

C. LIST OF SYMBOLS

R = Reynolds number

R_c = Reynolds number associated with the speed of sound $= \dfrac{\rho_o c_a}{\eta} \dfrac{L}{2\pi}$

W = Molecular weight

S = Entropy

R_m = Universal gas constant $= 82.06 \dfrac{\text{cc.atm.}}{\text{deg.}}$

v = Most probable molecular velocity

c = Local speed of sound

c_a = Adiabatic speed of sound

$\gamma = \dfrac{C_P}{C_V} = \dfrac{\text{specific heat at constant pressure}}{\text{specific heat at constant volume}}$

η = Coefficient of viscosity

L = Wavelength

λ = Mean free path

$p = P$ = Pressure

ρ = Density

h = Altitude

M = Mach number

V = Volume per mole

r = Distance between two molecules

$F = \rho_o g_o T_o / P_o$

P_t = Tangential stress

σ = Collision diameter for molecules considered as hard elastic spheres

ζ = Coefficient of slip

D. ANALYSIS OF THE EQUATIONS OF STATE AS APPLICABLE TO THE PHYSICAL
PROPERTIES OF THE ATMOSPHERE

In kinetic theory an ideal gas is defined as one in which molecules are assumed to be of infinitesimal size and exert no forces on each other.

We would expect that at low pressures and densities the mean free path of real gases would be large, the interaction of the molecules would be small, they would appear as points to one another; thus, in this situation, the gas would approach an ideal one.

That this conclusion is correct has been noted by many experimenters. For example, Knudsen noticed when performing his famous rates of effusion experiments that the experimental values checked with those values derived from kinetic theory on the basis of a perfect gas only if the pressure was sufficiently low (in his case, about .8 mm Hg) (Ref. 3).

Therefore, for the problem at hand, we can conclude without any further discussion that the air will behave as a perfect gas to a higher and higher degree of approximation as the altitude increases, that is as the density decreases - for truly, a gas approaches a perfect gas as the density approaches zero. It only remains to determine the percent of deviation of the air from that of a perfect gas as the altitude decreases, and to select an appropriate equation of state if at any point this deviation should become appreciable.

A quick calculation to determine the deviation of the true equation of state for the air from that of the perfect gas law equation of state may be carried out, using the fact that this deviation will be of the same order of magnitude as the deviation of the specific heats.

The specific heats of an ideal gas follow the equation:

$$C_p - C_V = T\left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_p$$

From the perfect gas law we have the well-known relation, $C_p - C_V = R_m$.

Using the Van der Waals equation of state $p = \dfrac{R_m T}{V - b} - \dfrac{a}{V^2}$ as the true equation of state for the air*, we arrive at

$$C_p - C_V = \frac{R_m}{1 - \dfrac{2a\rho}{R_m TW}\left(1 - \dfrac{b\rho}{W}\right)^2}$$

The values of the constants a and b may be evaluated from the critical data (Ref. 4), using the relations $b = \dfrac{V_c}{3}$ and $a = P_c V_c^2$; where V_c is the critical volume for air, expressed in liters/mole, and P_c is the critical pressure expressed in atmospheres.

At the surface of the earth the per cent deviation of $C_p - C_V$ from that of a perfect gas is 0.27%, and this value will decrease with increasing altitude.

Using the Beattie-Bridgeman equation of state,

$$p = \frac{\rho^2 R_m T}{W^2}\left[W/\rho + B_0\left(1 - \frac{b\rho}{W}\right)\right]\left[1 - \frac{c\rho}{WT^3}\right] - \frac{A_0\rho^2}{W^2}\left(1 - \frac{a\rho}{W}\right),$$

which has an accuracy of 0.5%, we can actually determine the pressure at any altitude and compare it with that pressure as determined by the perfect gas law, $p = \dfrac{\rho R_m T}{W}$.

For our calculations the values of density as a function of altitude were taken from the compilation of 231 summer sounding balloon flights in Germany (Ref. 5). The molecular weight of the air was taken as 28.9;

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*The assumptions made in deriving Van der Waals' equation (namely, that the molecules are of finite size, spherical in shape, and exert weak forces of cohesion on one another) would apply very well to our low density gas.

i.e., having a composition of 21% O_2, 78% N_2, and .93% A.* The temperature as a function of altitude was expressed as $T = T_0 - .0065(h)$ in accordance with NACA Report No. 218, where h is given in meters and T in degrees Kelvin. The five adjustable constants were those originally determined by Beattie and Bridgeman (Ref. 6) for air; namely, $A_0 = 1.3012$, a = .01931, $B_0 = .04611$, b = - .001101, and c = 4.34 × 10^4.

At an altitude of 0.0 meters the per cent deviation in the pressures, as calculated by the Beattie-Bridgeman equation and the perfect gas law, is 0.19%; and at 5000 meters (16,404 feet) it is only 0.11%. Any other representative set of data could have been used, and the results would yield the same order of magnitude.

If one uses the virial equation of state (Ref. 7), $\dfrac{pV}{R_m T} = 1 + \dfrac{B}{V} + \dfrac{C}{V^2} + ..$

which is perhaps more accurate at low pressures than the Beattie-Bridgeman equation, one finds that computations at sea level, using the perfect gas law, agree with the virial equation to within 0.08%.

Thus, in all cases, for our purposes the deviations from the perfect gas law are entirely negligible.

The fact that one can use the equation of state of a perfect gas becomes very important in that it enables us to develop simple expressions for the pressure in terms of the temperature as a function of altitude by using the barometric equation $dp = -\rho g dh$.

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*The inclusion of water vapor and CO_2 in the composition of the air, the effect of oxygen decomposition, or the pressures of the ionized layers will not affect the conclusions of this section; since it is the low density which is the principal factor.

E. TEMPERATURE

1. DEFINITION OF TEMPERATURE

The ordinary thermodynamic definition of temperature is based on the following macroscopic experiment: a thermometer is put into contact with the medium to be measured, and by conduction comes into thermal equilibrium; the reading of the thermometer is now "the temperature" of the medium. If such an experiment were carried out in an attempt to measure the temperature of the gas in interstellar space, the thermometer would, in fact, lose heat by radiation far faster than it could gain heat by conduction from the gas. Thus, although the "true temperature" of the gas is several thousand degrees, the reading of the thermometer would be close to absolute zero. "True temperature" is the kinetic temperature, based on the relation $1/2 \ m\overline{v^2} = 3/2 \ kT$, where $\overline{v^2}$ is the mean square particle velocity.

This hypothetical experiment indicates that the thermodynamic definition of temperature breaks down at very low pressures. However, inconsistencies may be avoided by using only the kinetic definition of temperature. The equation of state can be derived from this definition, and from the equation $P = 1/3 \ nm\overline{v^2} = 1/3 \ \rho\overline{v^2}$, which can be derived solely from kinetic theory. Pressure and density can also be defined by these equations, even in an extremely diffuse gas.

The end product of this study, the value of temperature at various altitudes, will be used in situations where the thermodynamic definition is involved (for example, the extent to which the skin temperature of a missile will rise). However, if there are sufficient molecules to heat the missile, there will be sufficient molecules to heat a hypothetical thermometer, and hence, the thermodynamic definition of temperature will be identical with the kinetic definition. Looked at another way, if the gas is very sparse, the flow will not be viscous, and the kinetic definition of temperature will be adequate to determine the heating effect. If the density is higher, the flow will be viscous (there will be a boundary layer of stagnant molecules), and thermodynamic temperature will be involved; but if there are enough molecules to form such a boundary layer, then there are enough molecules to make the thermodynamic temperature identical with the true temperature. Looked at still another way, the

molecular density at the highest altitude of interest (120 km) is 1.4×10^{13} molecules per cc. The expected value of the fluctuation of the number of molecules per cc is $\sqrt{N}$, which is extremely small compared to N; and the expected value of the fluctuation of thermodynamic temperature in one cc of gas is of the same order of magnitude as $\sqrt{N}/N$. Now the quantity of gas involved in any physical measurement cannot be very much smaller than one cc, and so the measured values cannot fluctuate appreciably. This is another way of saying that if there are sufficient molecules to effect a reading of a physical measuring device such as those used in direct macroscopic measurements of such quantities as density or temperature, there are sufficient molecules to render the thermodynamic temperature meaningful, and hence, identical with the true temperature.

It may therefore be assumed that our usual concepts of temperature are applicable over all of the areas of interest in this report.

2. TEMPERATURE DISTRIBUTION FOR THE UPPER ATMOSPHERE

The computation of Wizard trajectories requires that the temperature versus altitude distribution for the upper atmosphere be known for the region of the missile flight. It also is desirable for computing purposes that this distribution curve be smooth and representable by a simple functional relationship. A survey of the literature on the upper atmosphere has shown that there is a general agreement as to the form of this distribution within rather wide and loose limits (Ref. 1). Since there is no measure of error associated with most of the data in the literature, it is impossible to find a true mean distribution curve. The best temperature distribution that can be determined from the present literature must be one representing "probable" mean values for middle latitudes; these values are not statistical averages but are estimates based on the rather limited quantity of data available.

Several "average" or estimated temperature distributions have appeared in the literature. The latest, and probably the best, are those of Warfield (Ref. 9), Grimminger (Ref. 8), and Penndorf (Ref. 10). It is of some interest that Warfield and Grimminger arrived at essentially the same average temperature distribution, working independently and at about the same time. Penndorf arrived at a distribution of the same general form as those of Warfield and Grimminger but with somewhat lower values up to about 80 km.

These distributions are plotted in Figure 1, along with other summary and experimental curves.

A study of the original literature shows that the Warfield-Grimminger temperature curve is an adequate mean curve based on the data available in 1947. Since this curve was published, however, several sets of data have been added to the literature which indicate that, at altitudes above 45 km, these temperatures may be too high to represent mean values.

Measurements made on the sound waves from the Helgoland explosion of April 18, 1947, and reported by Cox (Ref. 11) give temperature values considerably below the average values of the Warfield-Grimminger curve. Jacchia (Ref. 12) has extended the work of Whipple (Ref. 13); and while he does not give his results in terms of a temperature curve, his corrected pressure curve is very close to the curve derived from the Warfield-Grimminger temperature values. The V-2 (Ref. 14) rocket flights in the United States have given direct pressure measurements up to about 100 km. Converting these pressures to temperatures gives a temperature distribution much lower than the Warfield-Grimminger curves.

The following table compares this new data with the Warfield-Grimminger temperature curve.

Source	45 km	50	55	60	70	80	100
Warfield	315° K	350	350	350	290	240	300
Grimminger	315° K	355	355	355	290	240	300
Cox	285	290	294			180	320
Jacchia	About the same as Warfield-Grimminger						
V-2, 3-7-47	310	320 ±25	315 ±25	300 ±25	190 ±15		230 ±40
V-2, 1-23-48	325	310		255	180		
V-2, U of M 12-8-47		220	235	255	260	220	280

Up to an altitude of 45 km the Warfield-Grimminger curve was accepted as agreeing well with the majority of the data in this region. In the 50

to 60 km region, the maximum Warfield-Grimminger values are probably a little too high, as indicated by the V-2 data and by the later sound data of Cox. In the 80 km region the minimum Warfield-Grimminger values are probably considerably too high, inasmuch as there is less data here to support these higher values. With these changes in mind, the smooth curve marked "ARC June 1949" was drawn; and it is believed that this curve gives a truer representation of the probable average temperature distribution for the middle latitudes than is given by previous "averages".

F. SPEED OF SOUND

SPEED OF SOUND IN AIR

The velocity of sound for a wave of infinitesimal amplitude is de-
fined as $c = \left[\left(\dfrac{\partial p}{\partial \rho}\right)_S\right]^{\frac{1}{2}}$ (Ref. 15). Using the thermodynamic relations
$\left(\dfrac{\partial p}{\partial v}\right)_S = \gamma\left(\dfrac{\partial p}{\partial v}\right)_T$ (Ref. 15) and $v \propto 1/\rho$, we get $\left(\dfrac{\partial p}{\partial \rho}\right)_S = \gamma\left(\dfrac{\partial p}{\partial \rho}\right)_T$. Using the
perfect gas law, $p = \dfrac{\rho R_m T}{W}$, then the velocity becomes $c = \sqrt{\dfrac{\gamma R_m T}{W}}$.

In actual practice this relation is not strictly correct, since the
air does not behave exactly as a perfect gas. Furthermore, it is found
experimentally that γ, which for a particular gas would be a constant
(the usual values are $\gamma = 1.667$ for monatomic gases and $\gamma = 1.400$ for
diatomic gases), actually varies with temperature and pressure. This
is due to the fact that, as the temperature of a gas is raised, the
vibrational and rotational energy states begin to contribute to the
internal energy of the molecules; and thus γ will decrease. Further-
more, as the density of a gas is lowered to the point where molecular
encounters become very infrequent, vibrational and rotational states
will be less easily excited; and so a polyatomic gas will tend to be-
have more like a monatomic gas with a corresponding increase in γ.

Tsien and Schamberg (Ref. 16) have calculated $\alpha = c/c_a$, the ratio
of the actual speed of sound to the adiabatic speed of sound; and b,
the damping (of the amplitude) coefficient as functions of a Reynolds
number defined as $R_C = \dfrac{\rho_o c_a}{\eta}\dfrac{L}{2\pi}$, where L = wavelength of the sound.
Unfortunately, Tsien and Schamberg's analysis cannot be applied for
low values of R_C; and their work merely shows that α increases very
slowly as R_C decreases. At $R_C = 6$, $b = .15$ and $\alpha = 1.02$, which would
represent an increase of 2% in the true velocity of sound over the
adiabatic velocity.

Assuming we have a body which would propagate a disturbance of wavelength 14 cm (the effective length), a Reynolds number $R_C = 6$ would occur at a height of 113 km.

Although it is a matter of conjecture just how far the α versus R_C curve as it stands now can be extended to lower values of R_C, it can be said that at about 120 km $(\lambda / \ell = 41/14 \simeq 3)$ (see Sec. G, below) we are somewhere in the transition region between slip flow and free molecule flow (Ref. 17); and, therefore, at approximately this altitude (depending upon the body in question) our point of view must change from a macroscopic to a microscopic one. One of the properties of a macroscopic quantity is that it may be measured directly. We know, however, that when the mean free paths of the air molecules are comparable with the wavelength of the sound wave, the damping action due to the viscosity and heat conduction becomes great. Thus, even if the velocity of propagation should increase with decreasing density, because of the damping a point will be reached where a disturbance will be physically undetectable at any appreciable distance; and the velocity of propagation of this disturbance can no longer have macroscopic meaning.

For the very same reasons as above (micro-macro argument), we know that we must change our equations of motion from a gas dynamical set to a gas kinetic set as soon as we enter the region of free molecule flow. In this case a convenient parameter to be used will be the most probable molecular speed v, instead of the speed of sound c_a. They are related by $v/c_a = \sqrt{2/\gamma}$.

CONCLUSION

Since the true values of the speed of sound differ little from the adiabatic values, it is permissible to use the adiabatic formula $c_a = \sqrt{\dfrac{\gamma R_m T}{W}}$ in calculating the speed of sound up to an altitude of approximately 120 km, at which point one must start using the most probable speed in the new gas kinetic equations, $v = \sqrt{2 R_m T / W}$.

G. COEFFICIENT OF VISCOSITY OF AIR

The problem is to determine which of the laws (Sutherland's, Rayleigh's, or the $T^{\frac{1}{2}}$ law) governing the dependence of the coefficient of viscosity upon the temperature is a suitable one to be applied to atmospheric air in the temperature range 200° K to 375° K, which represents the variation of temperature with altitude from sea level to 120 km.

The criterion to be used will be, in all cases, the closeness of agreement of the particular theoretical law with the experimentally determined values.

Because of the nature of the particular problem at hand, however, many difficulties present themselves. First of all, atmospheric air as we know it differs in composition from the air used in laboratory determinations of the coefficient of viscosity, in that the air used in the laboratory has been dried (water vapor eliminated) and rid of all carbon dioxide. The correction for the water vapor and CO_2 in the air was calculated, and turns out to be very small. This result might have been anticipated from the small amounts of these substances which are present.

The next difficulty encountered is the change in composition of the air; namely, the gradual linear dissociation of molecular oxygen with altitude commencing at 80 km and ending at 120 km, at which height essentially all oxygen is in the atomic form.

Although no experimental data are available for the coefficient of viscosity of atomic oxygen, it was found possible to estimate approximately what this value should be. Thus, one can calculate the viscosity for air, using the new compositions. The actual calculations have not been carried out in this report. These calculations will be of some value in future analyses.

VISCOSITY IN THE THREE REGIONS OF FLUID MECHANICS

Fluid mechanics is divided into three regions: continuum analysis, slip flow, and free molecule flow. The usual criteria denoting these boundaries of fluid mechanics are as follows (Ref. 17):

For the boundary between continuum analysis and slip flow, $M\lambda/\ell \simeq .02$, where ℓ is the effective length of the body in question. For the lower bound of free molecule flow, $M/R = 10$.

Aside from the inevitable change in the equations of motion which must be taken into account when a body passes from one region to another, we must also realize the differences in the definitions of viscosity for the three regions.

1. Continuum Analysis

The phenomenon of viscosity in this region occurs in a fluid undergoing shearing motion. Assume that a gas is in streamline motion with a velocity which is in the same direction (the y direction) at every point, but which has a velocity gradient dv_o/dx in a direction perpendicular to the direction of the velocity. It is experimentally found that the mass motion of the gas exerts not only a normal stress across any plane perpendicular to the velocity gradient, but also a shearing or tangential component in a direction tending to oppose the mass motion of the gas.

This force per unit area is proportional to the velocity gradient, $P_t = -\eta \; dv_o/dx$. The factor of proportionality, η, is called the coefficient of viscosity.

2. Slip Flow

Assume the gas is contained between two parallel plates. As the pressure of a gas is decreased, a situation is reached where the gas in mass motion will slip over the walls past which it is flowing. The velocity of slip u relative to the wall will have the direction of v_o and can be written $u = \zeta \; dv_o/dx$, where ζ is the coefficient of slip. To connect this with the ordinary coefficient of viscosity η, we note that for steady motion, the viscous force/unit area in the gas, $\eta \; dv_o/dx$, must be equal to the force/unit area exerted by the gas on the wall; and for this we can write ϵu, where ϵ is the coefficient of external friction of the gas. Thus, $\epsilon u = \eta \; dv_o/dx$; and $\zeta = \eta/\epsilon$. The important point to be noted in slip flow is that the concept of the viscosity (as due to the velocity gradient and arising from the internal shearing motion of the gas) still holds. The slip due to the external shear simply adds a correction term to the ordinary viscous shear term.

3. Free Molecule Flow

In this case the density of the gas is so low; i.e., the mean free path is so large, that the collisions of molecules with each other may be entirely neglected in comparison with their impacts upon any containing boundary walls. If, for instance, the gas is flowing through a tube, a molecule striking the lower part of the tube will move with constant velocity until it strikes the upper half of this tube, because the mean free path for this free molecule flow case is much greater than the diameter of the tube. Thus, there can be no velocity gradient; and, consequently, viscosity as we know it does not exist in this situation. There is surely a retardation of the motion of the gas through the tube; but this arises only from the axial component of the transfer of momentum from the gas to the walls of the tube and not from any internal shearing motion of the gas itself.

It also should be noted that if a boundary layer is established, it is possible for the free stream gas to be in free molecule flow while the boundary layer is in slip flow. Then we may have viscosity within the boundary layer but none in the surrounding gas. However, in any drag analysis of a body in free molecule flow, the drag arising from this phenomenon will be small compared to the other forces acting, and hence, is usually neglected.

The most important point to be noted in this discussion is that our tabular values for the coefficient of viscosity must be terminated as soon as we enter the free molecule flow region, since in this region viscosity no longer has any meaning.

THE STANDARD VALUE OF η

The value of $\eta_{23^\circ C} = 1822.6 \times 10^{-7}$, based upon Harrington's (Ref. 18) determination of the coefficient of viscosity of dry air, is definitely outmoded. Although Van Dyke (Ref. 19) obtained a value of $\eta_{23} = 1822.1 \times 10^{-7}$, it is to be remembered that this value was obtained by using Harrington's old apparatus which was not even dismantled before the determination was begun. Although Harrington's value was never checked independently, it has nevertheless persisted as the standard value in the United States and Canada. This is the value found in the "International Critical Tables", Volume V, page 1. Even as late as 1947 this value was still being used in NACA reports (Ref. 9).

Listed below are more recent determinations of the coefficient of viscosity of pure dry air at 23° C.

1829.2×10^{-7}	Houston, "Phys. Rev.", Vol. 52, p. 751 (1937)
1834.9	Kellstrom, "Phil. Mag.", Vol. 23, p. 313 (1937)
1834.7	Bond, "Nature", Vol. 137, p. 1031 (1936)
1834.4	Majudmar and Vajifdar, "Proc. Ind. Acad. Sci.", Vol. 8A, p. 171 (1938)
1833.3	Banerjea and Plattanaik, "Zeit. Phys.", Vol. 110, p. 676 (1938)
1830.3	Rigden, "Phil. Mag.", Vol. 25, p. 961 (1938)
1834.1	Fortier, "Comptes Rendus", Vol. 208, p. 506 (1939)
1833.0	Vasilesco, "Ann. de Physique", 11 série 20, p.292(1945)

Johnston and McCloskey (Ref. 20) use the average ($\eta_{23} = 1833.0 \times 10^{-7}$) of the above values, and note that this value agrees well with the long accepted European standard 1832×10^{-7} by Vogel (Ref. 21). We will use the value $\eta_{23} = 1833.0 \times 10^{-7}$ as our standard.

At other temperatures, the experimental data of four experimenters, Fortier (Ref. 22), Vasilesco (Ref. 23), Sutherland-Maass (Ref. 24), and Johnston and McCloskey (Ref. 20), has been plotted with all data corrected to our standard value of η. It is seen that the results are in good agreement (Fig. 2 and 3). Sutherland and Maass claim an accuracy of their results of 0.5%. Johnston and McCloskey claim an accuracy of about 0.4%, and Vasilesco claims an accuracy of better than 0.2%.

VARIATION OF THE COEFFICIENT OF VISCOSITY WITH PRESSURE

According to kinetic theory the coefficient of viscosity is shown to be $\eta = 0.499\, \rho \bar{v} \lambda$. Now, for a period of time which is great enough for a mean free path and an average collision rate to have meaning, the distance a molecule moves in a time t is $\bar{v}t = \textcircled{H}t\lambda$; where $\bar{v}$ is the average molecular speed, and $\textcircled{H}$ is the collision rate. Thus, $\bar{v} = \textcircled{H}\lambda$. Increasing the temperature is equivalent to multiplying the velocity of each molecule by a certain constant factor. The collision probability per unit time of each molecule thereby becomes multiplied by the same factor as the velocities and so does the general collision rate, $\textcircled{H}$. Thus, $\textcircled{H} \propto \sqrt{T}$, $\bar{v} \propto \sqrt{T}$ and λ is independent of temperature.
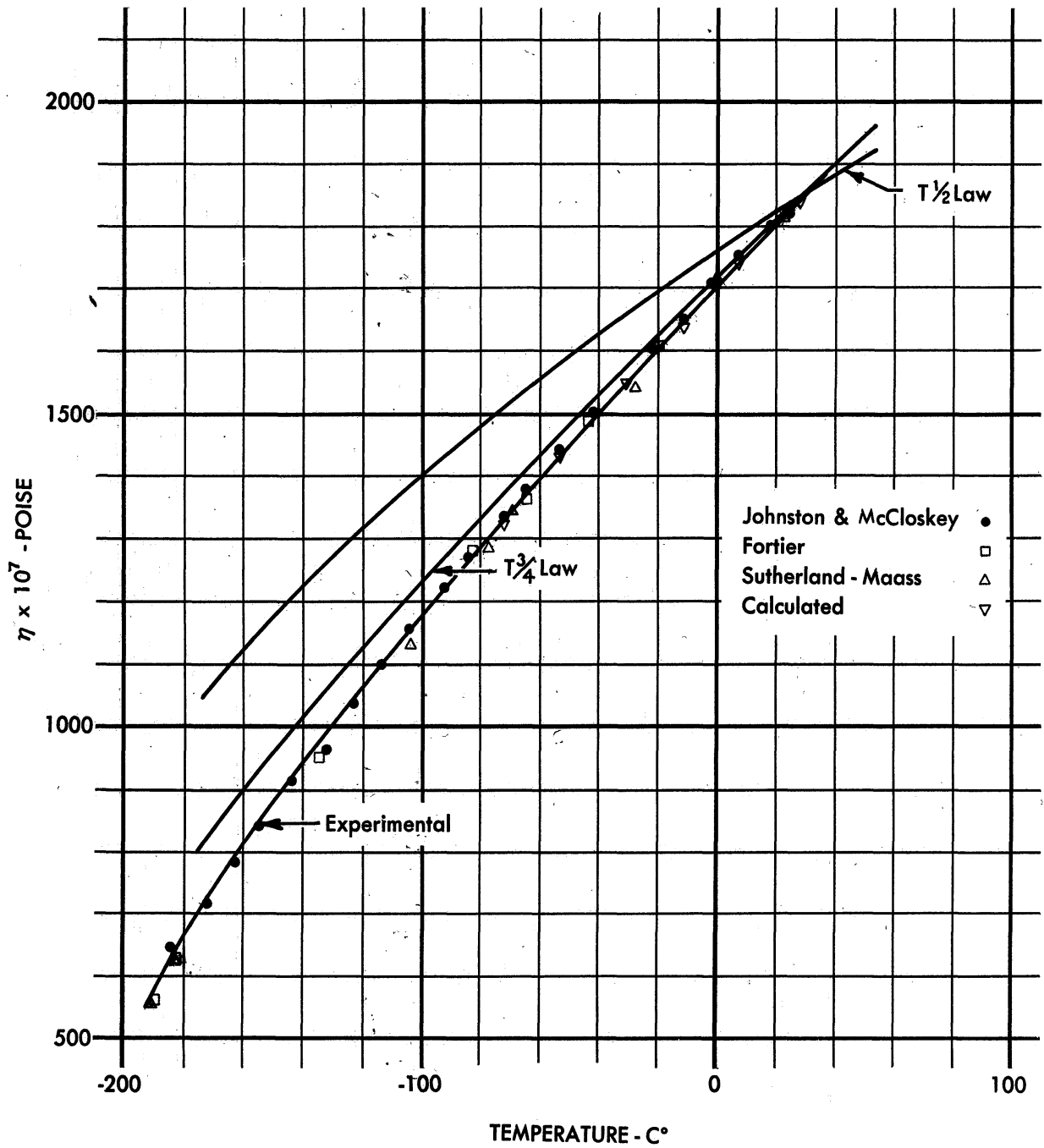
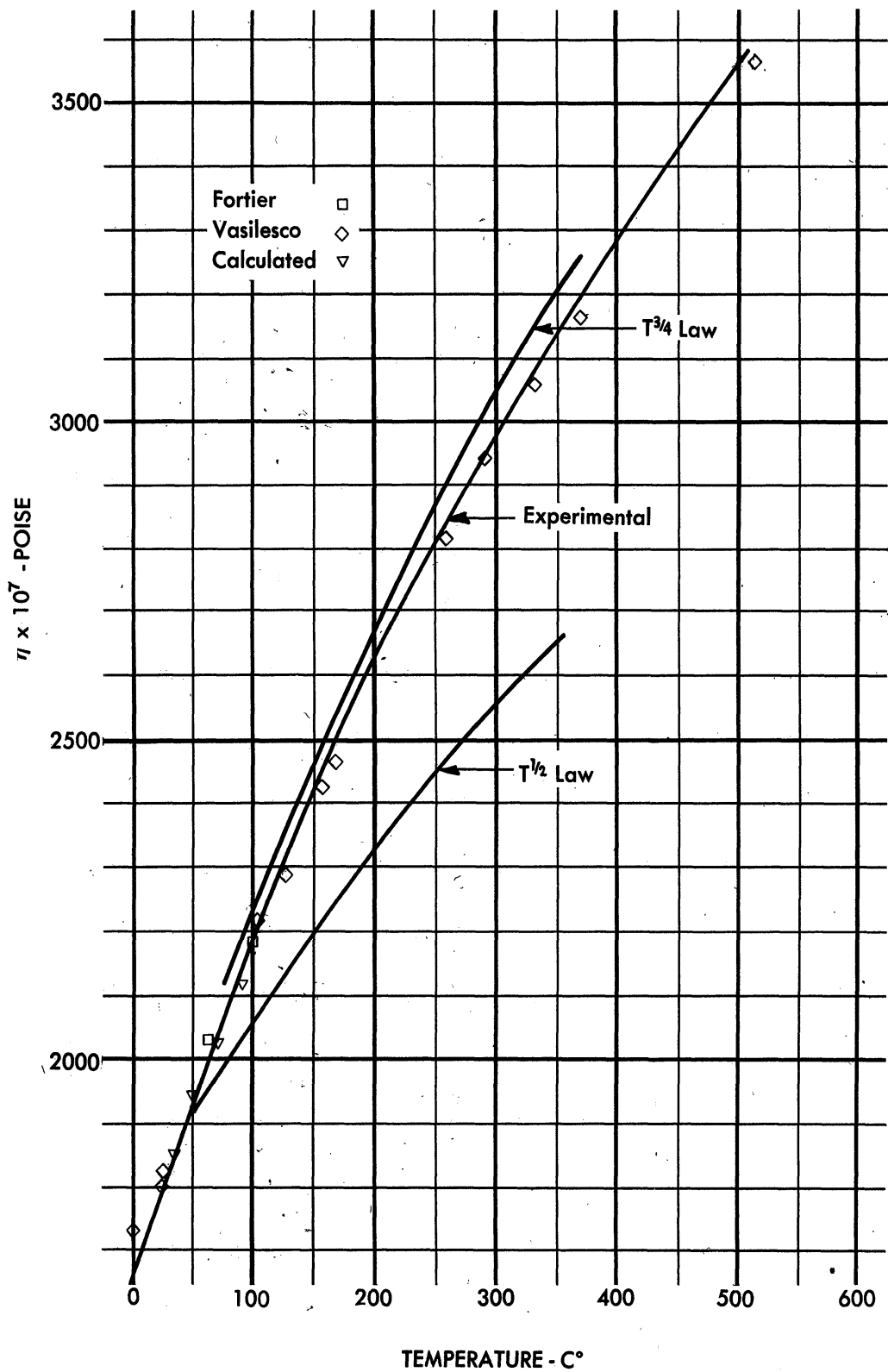FIG. 2   COEFFICIENT OF VISCOSITY VERSUS TEMPERATURE FOR DRY CO_2 - LESS AIR

FIG. 3 COEFFICIENT OF VISCOSITY VERSUS TEMPERATURE FOR DRY CO_2-LESS AIR

If now the temperature is held constant and the density is increased, the mean free path will decrease, and the collision rate will increase. At a given temperature the mean free path is inversely proportional to the density and the pressure.

Therefore, at a constant temperature the coefficient of viscosity will be a constant and thus independent of the pressure.

When the mean free path becomes comparable with the physical distances in which we are interested, the derivation of the formula for the coefficient of viscosity no longer holds. In particular, slip flow sets in; and one must consider the coefficient of slip (ζ) which, for the case of the gas bounded by a fixed plane surface, is given by $\zeta = 0.998 \left(\dfrac{2 - f}{f} \right) \lambda$; where f is the fraction of the tangential momentum of the molecules which strikes the surface of the body in question (i.e., the fraction of molecules striking diffusely). Thus, ζ is inversely proportional to the pressure.

In using the rotating cylinder method for the determination of the coefficient of viscosity, Stacy (Ref. 25) found evidence of the fact that if correction for slip is made, the viscosity of a gas is indeed independent of the pressure. The expression he used was the one derived for this method by Millikan (Ref. 26); namely, $\zeta = 1/K(\eta/\eta_a - 1)$, where η is the true viscosity independent of pressure, η_a is the apparent viscosity (i.e., the viscosity which the instrument would measure directly), and where K is a constant of the apparatus. Thus the prediction that the coefficient of viscosity is independent of the pressure is correct and has been experimentally verified -- certainly if we consider 1 atmosphere as being an upper limit to the pressure. Hence, whenever the coefficient of viscosity is to be used, it will mean the true viscosity (independent of the pressure) as distinguished from an apparent viscosity uncorrected for slip.

DEPENDENCE OF THE COEFFICIENT OF VISCOSITY UPON TEMPERATURE

If the density is kept constant, and the temperature is varied, then λ will be a constant. Also, $\bar{v} \propto \sqrt{T}$; and thus, $\eta \propto \sqrt{T}$.

This law, unfortunately, does not agree with experimental data (see graph). The reason for this is quite clear. Inherent in our theory is the assumption that the molecules behave as hard spheres, repelling each other with a repulsive force $\propto 1/r^n$, where $n = \infty$. This is certainly not true; n must be a finite number, varying for different molecules.

Various attempts have been made to derive or deduce more general force laws that would be applicable to all gases over a wide range of temperature. Lord Rayleigh, in 1900, derived a force law (Ref. 27, p. 153) by using a method of similitude which gives $\eta \propto T^{\frac{1}{2} + \frac{2}{s-1}}$. An expression sometimes used in aerodynamics (Ref. 28) takes s = 9; and thus,

$$\eta = \eta_0 \left(\frac{T}{T_0}\right)^{\frac{3}{4}} .$$

This function is plotted in Figures 2 and 3; and while it is better than the $T^{\frac{1}{2}}$ law, it still does not represent the experimental values accurately.

The temperature law proposed by Sutherland (Ref. 29) assumes that the molecules are hard spheres having, in addition to the infinite repulsive force, a weak attractive force falling off rapidly with distance. The formula is

$$\eta = \eta_0 \left(\frac{T}{T_0}\right)^{\frac{1}{2}} \frac{1 + C/T_0}{1 + C/T} ,$$ where C is supposedly a constant (the Sutherland constant) for a given gas. This formula holds for gases which are above their critical temperature and at pressures for which the deviations from Boyles law are not great (i.e., at not too high pressures). Unfortunately, the theoretical basis of this law is rather weak; as evidenced by the fact that, experimentally (Ref. 23), it neither holds equally well for all gases nor over a wide range of temperature for a particular gas. Theoretical considerations (Ref. 30) agree with experiment in showing that the Sutherland constant varies with temperature. However, it is possible to regard this law in somewhat of an empirical light and thereby determine just where it may be applied.

Following a method proposed by Vasilesco (Ref. 23), we can write the

Sutherland law as $\eta = \eta_0 \left(\dfrac{T}{T_0}\right)^{\frac{1}{2}} \dfrac{1 + C_0/T_0}{1 + C_T/T}$; where C_0 is the value of the

Sutherland constant at temperature $T_0 = 296.1°K$, and C_T is the value at the temperature T.

Then, $C_T = \dfrac{KT^{\frac{3}{2}}}{\eta} - T$, where $K = \eta_0 \dfrac{\left(T_0 + C_0\right)}{T_0^{\frac{3}{2}}}$. Vasilesco calculated

C_T as a function of temperature, using data by Fortier and himself. The results over part of the temperature range are plotted in Figure 4. Although the points are quite scattered, a consistent increase of C_T with temperature is apparent, and becomes more evident at higher temperatures. However, we are interested mainly in the region $200°$ K to $375°$ K. (See temperature curve, Part E.)

In this region, C_T can be given a constant value of 113.5; and the

equation becomes $\eta = 1833 \times 10^{-7} \left(\dfrac{T}{296.1}\right)^{\frac{3}{2}} \left(\dfrac{296.1 + 113.5}{T + 113.5}\right)$. The success of this assumption can be ascertained by comparing the close agreement between the experimental curve and the values calculated between $-75°$ and $+100°$ C using the above value for the Sutherland constant (Fig. 2 and 3).

One of the most useful representations of transport phenomena in general, and viscosity in particular, has been given by Hirschfelder, Bird and Spotz (Ref. 30). Their method assumes the energy of interaction between two simple molecules to be expressed in the form

$\epsilon(r) = 4\epsilon_m \left[\left(\dfrac{r_0}{r}\right)^{12} - \left(\dfrac{r_0}{r}\right)^6\right]$; where ϵ_m is the maximum repulsion energy,

and r_0 is the distance at which the net interaction is zero. Using the Chapman and Cowling (Ref. 31) expressions for the transport phenomena in terms of a set of collision integrals which are in turn functions of the cross section of the molecules, the viscosity for a single gas is given by the relation $\eta \times 10^7 = 266.93 \, (WT)^{\frac{1}{2}}[r_0]^{-2} \, V/[W^{(2)}(2)]$; where W is the molecular weight, T the temperature in degrees K, r_0 the low evergy of collision diameter expressed in Angstroms, and V and $W^{(2)}(2)$ are slowly varying functions of kT/ϵ (k = Boltzmann's constant). Although the theory,
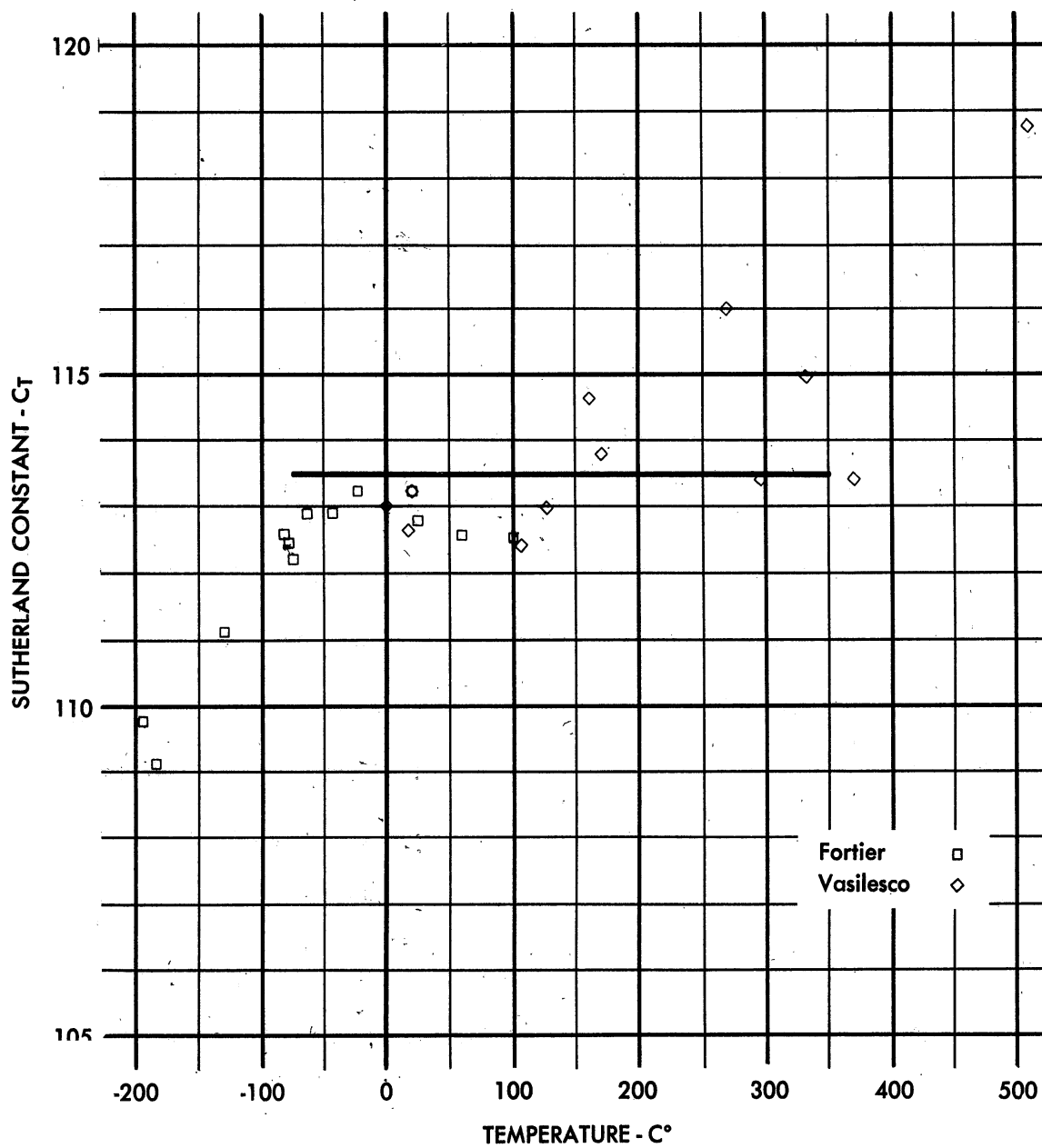
FIG. 4   SUTHERLAND CONSTANT VERSUS TEMPERATURE

for the present, has been worked out only for air and various non-polar gases, the agreement with experiment is quite good. The method is particularly worthwhile in that it enables one to calculate the force constants for a gas by knowing experimental values for only one of the transport phenomena. Knowledge of the force constants r_0 and ϵ_m enables us to calculate the other two transport coefficients. Another particular success of the theory is the ability to calculate the transport' coefficients for a mixture of gases. The application of this to atmospheric air will be discussed below.

VISCOSITY OF ATMOSPHERIC AIR

All experimental values of viscosity are for dry air without CO_2, and everything said previously has pertained to this type of air. Clearly, our atmosphere does not consist of dry air without CO_2. To further complicate matters, we know that atmospheric oxygen begins dissociating at 80 km, dissociation assumed as being complete at 120 km. Since the air consists of 20% oxygen, the change in composition of this great a constituent from molecular to atomic form will surely produce quite an effect upon the viscosity coefficient.

The first point to be considered is the effect of water vapor upon the viscosity of dry air.

Hirschfelder, Bird, and Spotz have developed an extension of their theory, whereby they may calculate the transport phenomenon of multi-component systems. They have calculated the viscosity of moist air, assuming that moist air may be treated as a mixture of two components, air and water vapor. For the temperature range in which we are interested, the values of the coefficient of viscosity ($\times 10^7$) for varying percentages of moisture is given in the table below.

Temp. °K	Percent Moisture				.03% CO_2	2% H_2O and .03% CO_2
	0.0	0.5	1.0	5.0		
200	1337	1335	1333	1315	1335	1328
273.16	1724	1722	1720	1704		
296.1	1833				1827	1823
300	1851	1849	1847	1832	1854	
400	2290	2289	2288	2278	2278	

Using the Hirschfelder method, the correction for CO_2 in dry air has been calculated. The generally accepted value for the percentage of CO_2 in the atmosphere is .03%. Using this value we have the coefficient of viscosity as given in the sixth column of the above table.

By means of the quite complicated Hirschfelder formulas (see Section H5b) for multicomponent systems, the coefficient of viscosity for air containing .03% CO_2 and 2% water vapor has also been calculated. This water vapor value was chosen arbitrarily as being an average between the upper and lower limits of 4% and .2% (Ref. 32) usually given for the amount of water vapor present in the air. The coefficient of viscosity in this case is given in the last column of the above table for the temperatures $200°$ K and $296.1°$ K.

As can be seen, the change (due to the addition of CO_2 and H_2O in air at $296.1°$ K) is very small (only 0.5%) and could be neglected. However, we will use the corrected value as the standard. Sutherland's Law may again be applied to the variation of the coefficient of viscosity with temperature; now taking as the standard at $23°$ C the value $\eta_{23} = 1823 \times 10^{-7}$, which represents the value of the coefficient for air containing 2% water vapor and .03% CO_2.

We must take care of the correction to the viscosity for the dissociation of molecular oxygen. First, however, one must see whether this phenomenon takes place in the region in which viscosity is defined; i.e., we must make sure that we are not in free molecule flow, for here the concept of viscosity does not exist.

Using $\lambda/\ell = 10$ as the boundary of free molecule flow and taking λ as the mean free path to be 31 cm (120 km) and ℓ the effective length of a typical missile body as 14 m; then $\lambda/\ell \approx .02$, which shows that 120 km is still not in free molecule flow. Thus, the region of oxygen dissociation (80 to 120 km) is one in which the viscosity exists; and therefore the correction for this change in composition of the air becomes important.

Since no experimental data is available for atomic oxygen, we must estimate its force coefficients from what is known about other monatomic gases. The constants to be determined are the force constants, ϵ and r_0. In Figure 5 these constants are plotted versus the atomic weight of monatomic gases, for which the constants have been determined from viscosity
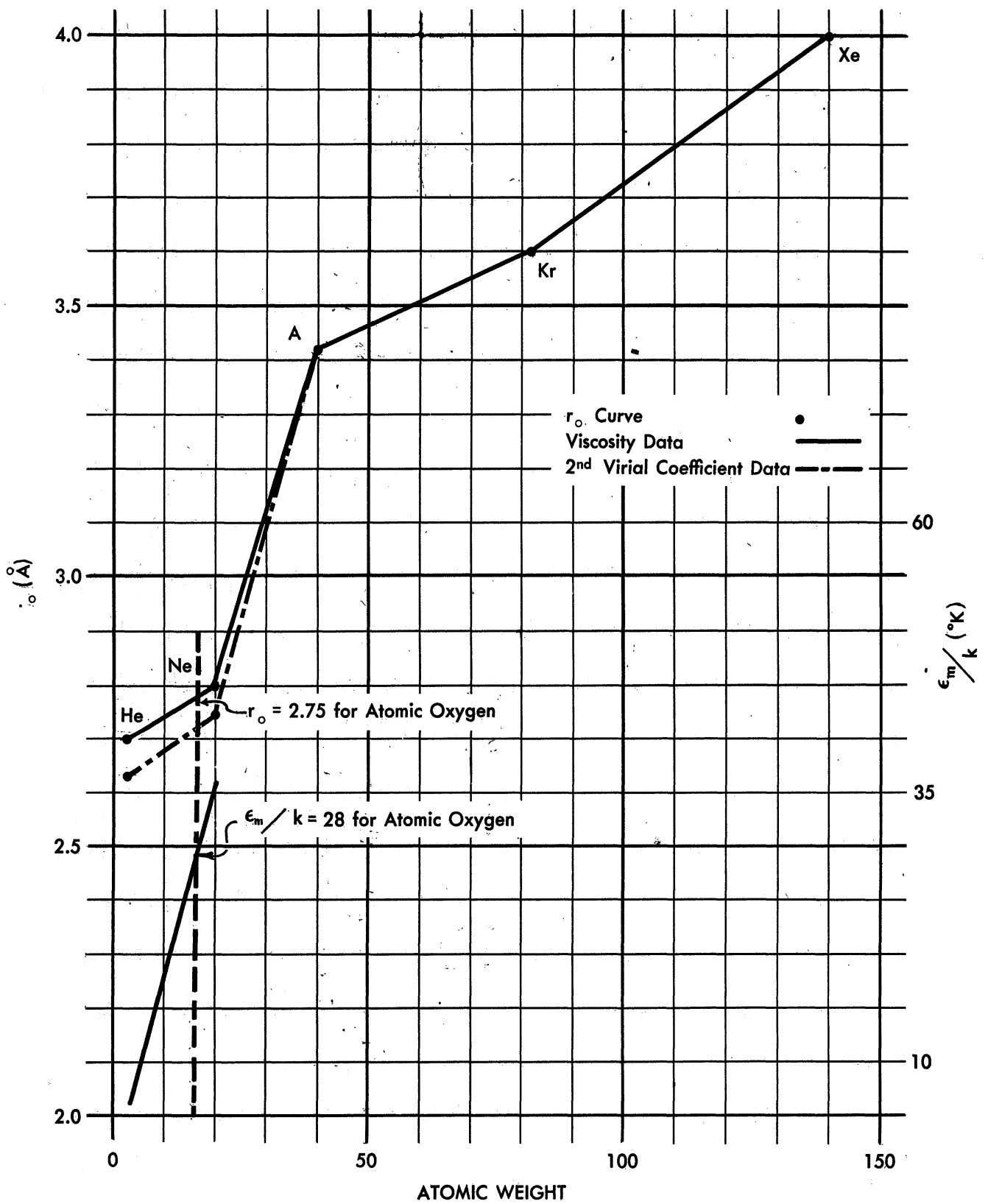
FIG. 5 FORCE CONSTANTS FOR MONATOMIC GASES

data and the second virial coefficients of the equation of state. On the basis of this graph, $r_0 = 2.75$ and $\epsilon/k = 28$ have been chosen as the values for atomic oxygen (At. Wt. = 16). It is noted that in the expression being used for the viscosity; namely,

$$\eta \times 10^7 = 266.93 (WT)^{\frac{1}{2}} \left(\frac{1}{r_0^2}\right) \left(\frac{V}{W^{(2)}(2)}\right)$$

the most important of the two constants is r_0, $W^{(2)}(2)$ being only a slowly varying function of kT/ϵ. It can be assumed safely that the r_0 for O (At.Wt. = 16) will be a little less than that for Ne (At.Wt. = 20). Assuming, therefore, that at most, the value of r_0 for O can vary between the values for He and Ne; the maximum error introduced by this method of estimating r_0 is less than 4%.

By means of the above viscosity formula at $T = 280°$ K, $\eta_0 \times 10^7 = 2857$, while $\eta_{O_2} \times 10^7 = 1958$.

Thus, it is now possible for us to compute the viscosity of atmospheric air in the region of oxygen dissociation. Here the air will be assumed to have the composition given by Warfield (Ref. 9) and a temperature distribution as given by the ARC temperature curve for the upper atmosphere, Section E. The formulae and constants to be used are shown in Section H5.

A theoretical remark should be made concerning the above method. The oxygen in the atomic state will be subject to recombination mainly by two processes: $O + O \rightarrow O_2 + h\nu$ (radiative recombination) and $O + O^- \rightarrow O_2 + e$ (due to the formation of negative atomic oxygen ions because of the considerable electron affinity of atomic oxygen). Because of this tendency to recombine, the oxygen atoms are actually subject to an additional force; and strictly speaking, the force law that we have

assumed throughout; namely, $\epsilon(r) = 4 \epsilon_m \left[\left(\frac{r_0}{r}\right)^{12} - \left(\frac{r_0}{r}\right)^6\right]$, should be modi-

fied. In addition the density of atomic oxygen will not be constant, but will possess a higher value during the daytime than at night.

However, if we assume, as Warfield (Ref. 9) has done, that the density of atomic oxygen possesses two distinct equilibrium values for night

and day, then we may consider that recombination takes place only for a very short length of time, and hence, no modification of the force law need be made.

CONCLUSIONS

For the region up to 80 km, the coefficient of viscosity as a function of temperature is given by the formula

$$\eta = 1827 \times 10^{-7} \left(\frac{T}{296.1}\right)^{\frac{3}{2}} \left(\frac{409.6}{T + 113.5}\right)$$

The use of this formula should not introduce an error greater than 1%.

For the region from 80 to 120 km, the formulas given in H5(b) are to be used in conjunction with the Warfield values of composition and the ARC temperature curve. The error, using this method, should be less than 5%.

For a body in motion through the atmosphere, any computed values of η for which $M/R \geq 10$ will cease to have any meaning.

H. CALCULATIONS

1. TEMPERATURE CALCULATIONS

The "ARC June 1949" temperature curve introduced in Section E has been fitted by Brown (Ref. 33) with a set of simple functions consisting of straight lines and parabolas. These are reproduced here for use in the remainder of the calculations. Altitude is expressed in feet and temperature in degrees Rankine, since all trajectory computations are carried out in this system.

Equation	Altitude Range (Feet)	Temperature Range (Degrees Rankine)
(a) $T = \dfrac{-1404}{393700} y + 518.4$	0 (and below) to $\dfrac{2755900}{78}$	518.4 to 392.4
(b) $T = 392.4$	$\dfrac{2755900}{78}$ to $\dfrac{314960}{3}$	392.4
(c) $T = \dfrac{1584}{393700} y - 30$	$\dfrac{314960}{3}$ to $\dfrac{433070}{3}$	392.4 to 550.8
(d) $T = \dfrac{-9504 \times 10^{-4}}{15499969}(y-177165)^2 + 616.8$	$\dfrac{433070}{3}$ to 177,165 177,165 to $\dfrac{629920}{3}$	550.8 to 616.8 616.8 to 550.8
(e) $T = \dfrac{-1584}{393700} y + 1395.6$	$\dfrac{629920}{3}$ to $\dfrac{728345}{3}$	550.8 to 418.8
(f) $T = \dfrac{9504 \times 10^{-4}}{15499969}(y-275590)^2 + 352.8$	$\dfrac{728345}{3}$ to 275,590 275,590 to 295,275	418.8 to 352.8 352.8 to 376.56

Equation	Altitude Range (Feet)	Temperature Range (Degrees Rankine)
(g) $T = \dfrac{9504}{393700} y - 336.24$	295,275 to 393,700 (and above)	376.56 to 614.16

2. CALCULATION OF ρ/ρ_o

For these calculations the composition of the atmosphere used by Warfield (Ref. 9) is accepted. This gives a constant composition to an altitude of 262,467 feet (80 km) in the daytime and to 344,487 feet (105 km) at night with a linear gradient of dissociation of oxygen giving all atomic oxygen at 328,083 feet (100 km) in the daytime and at 393,700 feet (120 km) at night.

The equations on which the calculations are based are the perfect gas law, which has been shown in (D) to hold in the upper atmosphere, and the hydrostatic law; i.e.,

$$dp = - g\rho dh$$

$$\rho/\rho_o = \frac{p}{p_o} \frac{T_o}{T} \frac{W}{W_o} \qquad (2.1)$$

In the region of dissociation of oxygen $\dfrac{W}{W_o} = \dfrac{1}{1 - K(h - h_m)}$; and $\dfrac{\gamma}{\gamma_o} = 1 - \dfrac{128K(h - h_m)}{21W_o}$, where $K = - \dfrac{V_o}{h_a - h_m}$, and V_o is the volume of all molecular oxygen at height h_m (Ref. 13).

The acceleration due to gravity (g) is considered constant in these calculations. If g were considered as varying according to an inverse square law $\left[g = g_o \left(\dfrac{r}{r + h} \right)^2 \right]$, the reduction in g at 393,700 feet would be less than 4%; and this would change the derived values of pressure and density by about 20%. These differences, however, are probably small compared to those due to the unknown errors in the temperature distribution.

The equations (2.1) are always integrated over the region for which
the function $T = T(h)$, as given by Brown (Ref. 33), and the equation for
the composition hold simultaneously. This necessitates a further subdi-
vision of the regions of the temperature distribution as follows.

Equations			Range of h
(a) $T = a + bh$	$\dfrac{W}{W_0} = 1$	$\dfrac{\gamma}{\gamma_0} = 1$	0 to $\dfrac{2755900}{78}$ ft.
(b$_1$) $T = 392.400$	$\dfrac{W}{W_0} = 1$	$\dfrac{\gamma}{\gamma_0} = 1$	$\dfrac{2755900}{78}$ to $\dfrac{314960}{3}$ ft.
(c) $T = a + bh$	$\dfrac{W}{W_0} = 1$	$\dfrac{\gamma}{\gamma_0} = 1$	$\dfrac{314960}{3}$ to $\dfrac{433070}{3}$ ft.
(d) $T = r(h-s)^2 + u$	$\dfrac{W}{W_0} = 1$	$\dfrac{\gamma}{\gamma_0} = 1$	$\dfrac{433070}{3}$ to 177,165 ft. 177,165 to $\dfrac{629920}{3}$ ft.
(e) $T = a + bh$	$\dfrac{W}{W_0} = 1$	$\dfrac{\gamma}{\gamma_0} = 1$	$\dfrac{629920}{3}$ to $\dfrac{728345}{3}$ ft.
(f$_1$) $T = r(h-s)^2 + u$	$\dfrac{W}{W_0} = 1$	$\dfrac{\gamma}{\gamma_0} = 1$	$\dfrac{728345}{3}$ to 262,467 ft. (day) $\dfrac{728345}{3}$ to 295,275 ft. (night)
(f$_2$) $T = r(h-s)^2 + u$	$\dfrac{W}{W_0} = \dfrac{1}{1-K(h-h_m)}$	$\dfrac{\gamma}{\gamma_0} = 1-\dfrac{128K(h-h_m)}{21W_0}$	262,467 to 295,275 ft. (day)
(g$_1$) $T = a + bh$	$\dfrac{W}{W_0} = 1$	$\dfrac{\gamma}{\gamma_0} = 1$	295,275 to 344,275 ft. (night)
(g$_2$) $T = a + bh$	$\dfrac{W}{W_0} = \dfrac{1}{1-K(h-h_m)}$	$\dfrac{\gamma}{\gamma_0} = 1-\dfrac{128K(h-h_m)}{21W_0}$	295,275 to 328,083 ft. (day) 344,487 to 393,700 ft. (night)
(g$_3$) $T = a + bh$	$\dfrac{W}{W_0} = .8267879$	$\dfrac{\gamma}{\gamma_0} = 1.0440845$	328,083 to 393,700 ft. (day)

From (2.1) we have

$$\frac{dp}{p} = - \frac{\rho_o g_o T_o W}{p_o T W_o} \, dh;$$

and if we write $F = \dfrac{\rho_o g_o T_o}{p_o}$, $\dfrac{dp}{p} = - F \dfrac{W}{W_o} \dfrac{dh}{T} = - F \dfrac{W}{W_o} \dfrac{dh}{T(h)}$.

This equation then must integrated for each of the above regions:

(a) $\dfrac{dp}{p} = - F \dfrac{dh}{a + bh}$

$$p = p_o \left(\frac{T}{a}\right)^{-\frac{F}{b}}$$

and

$$\frac{\rho}{\rho_o} = \frac{T_o}{T} \left(\frac{T}{a}\right)^{-\frac{F}{b}} = \left(\frac{T}{T_o}\right)^{-\left(\frac{F}{b} + 1\right)} \quad \text{since here } a = T_o$$

(b_1) $\dfrac{dp}{p} = - F \dfrac{dh}{392.4}$

$$p = p_{b_1} e^{-\frac{F}{392.4}(h - h_{b_1})}$$

$$\frac{\rho}{\rho_o} = \frac{T_o}{p_o} \frac{p_{b_1}}{392.4} e^{-\frac{F}{392.4}(h - h_{b_1})}$$

(c) $\dfrac{dp}{p} = - F \dfrac{dh}{a + bh}$

$$p = p_c \left(\frac{T}{T_c}\right)^{-\frac{F}{b}}$$

$$\frac{\rho}{\rho_o} = \frac{T_o p_c}{p_o} \left(T_c\right)^{\frac{F}{b}} (T)^{-\left(\frac{F}{b} + 1\right)}$$

(d) $\quad \dfrac{dp}{p} = -F \dfrac{dh}{r(h - s)^2 + u}$

$$\log \frac{p}{p_d} = \frac{2F}{\sqrt{-q}} \left\{ \tanh^{-1} \frac{2r(h - s)}{\sqrt{-q}} - \tanh^{-1} \frac{2r(h_d - s)}{\sqrt{-q}} \right\} \quad \text{where } q = 4ru.$$

$$p = p_d \, e^{\frac{2F}{\sqrt{-q}} \left\{ \tanh^{-1} \frac{2r(h - s)}{\sqrt{-q}} - \tanh^{-1} \frac{2r(h_d - s)}{\sqrt{-q}} \right\}}$$

$$\frac{\rho}{\rho_o} = \frac{T_o p_d}{p_o} \frac{1}{T} e^{\frac{2F}{\sqrt{-q}} \left\{ \tanh^{-1} \frac{2r(h - s)}{\sqrt{-q}} - \tanh^{-1} \frac{2r(h_d - s)}{\sqrt{-q}} \right\}}$$

(e) $\quad \dfrac{dp}{p} = -F \dfrac{dh}{a + bh}$

$$p = p_e \left(\frac{T}{T_e}\right)^{-\frac{F}{b}}$$

$$\frac{\rho}{\rho_o} = \frac{T_o p_e}{p_o T} \left(\frac{T}{T_e}\right)^{-\frac{F}{b}} = \frac{T_o p_e}{p_o} \left(T_e\right)^{\frac{F}{b}} (T)^{-\left(\frac{F}{b} + 1\right)}$$

(f_1) Same as in (d)

$$\frac{\rho}{\rho_o} = \frac{T_o p_{f_1}}{p_o T} e^{-\frac{2F}{\sqrt{q}} \left\{ \tan^{-1} \frac{2r(h - s)}{\sqrt{q}} - \tan^{-1} \frac{2r\left(h_{f_1} - s\right)}{\sqrt{q}} \right\}}$$

(f_2) $\quad \dfrac{dp}{p} = -F \dfrac{1}{[1 - K(h - h_m)]} \dfrac{dh}{[r(h - s)^2 + u]}$

To integrate this we let $y = 1 - K(h - h_m)$. Then, $dh = -\dfrac{dy}{K}$; and the right hand member becomes

$$\frac{F}{K} \frac{dy}{y(\omega y^2 + \beta y + \alpha)} \quad \text{which integrates as}$$

$$\frac{F}{k} \left\{ \frac{1}{2\alpha} \log \frac{y^2}{(\omega y^2 + \beta y + \alpha)} - \frac{\beta}{\alpha \sqrt{q}} \tan^{-1} \frac{2\omega y + \beta}{\sqrt{q}} \right\}$$

where

$$\alpha = rs^2 + u + \frac{r(1 + Kh_m)^2}{K^2} - \frac{2rs(1 + Kh_m)}{K}$$

$$\beta = -\left[\frac{2r(1 + Kh_m)}{K^2} - \frac{2rs}{K} \right]$$

$$\omega = \frac{r}{K^2}$$

$$q = 4\alpha\omega - \beta^2 = \frac{4ru}{K^2}$$

then

$$\log \frac{p}{p_{f_2}} = \frac{F}{K} \left\{ \frac{1}{2\alpha} \log \frac{y^2}{\omega y^2 + \beta y + \alpha} - \frac{\beta}{\alpha \sqrt{q}} \tan^{-1} \frac{2\omega y + \beta}{\sqrt{q}} \right\}_1^{1 - K(h - h_m)} ,$$

or

$$p = p_{f_2} \left\{ \frac{[1 - K(h - h_m)]^2 (\alpha + \beta + \omega)}{r(h - s)^2 + u} \right\}^{\frac{F}{2K\alpha}}$$

$$e^{-\frac{F\beta}{K\alpha \sqrt{q}} \left[\tan^{-1} \frac{2\omega[1 - k(h - h_m)] + \beta}{\sqrt{q}} - \tan^{-1} \frac{2\omega + \beta}{\sqrt{q}} \right]}$$

and

$$\frac{\rho}{\rho_o} = \frac{T_o \, p}{p_o \, T} , \quad \text{where } p \text{ is taken as above.}$$

(g_1) Same integration as in (e)

$$\frac{\rho}{\rho_0} = \frac{T_0 p_{g_1}}{p_0 T}\left(\frac{T}{T'_{g_1}}\right)^{-\frac{F}{b}}$$

(g_2) $\frac{dp}{p} = -F\frac{dh}{\lceil 1 - K(h - h_m)\rceil(a + bh}$

$$p = p_{g_2}\left\{\frac{(a + bh)\left[1 - K\left(h_{g_2} - h_m\right)\right]}{\left(a + bh_{g_2}\right)[1 - K(h - h_m)]}\right\}^{-\frac{F}{[(1 + Kh_m)b + Ka]}}$$

$$\frac{\rho}{\rho_0} = \frac{T_0 Wp}{p_0 W_0 T}\quad\text{where } p \text{ is as above.}$$

It is noted that at night $h_{g_2} = h_m$,

and $\quad p = p_{g_2}\left\{\frac{a + bh}{(a + bh_m)[1 - K(h - h_m)]}\right\}^{-\frac{F}{[(1 + Kh_m)b + Ka]}}$

(g_3) The integration is the same as in (e).

$$\frac{\rho}{\rho_0} = \frac{T_0 Wp_{g_3}}{p_0 W_0 T}\left(\frac{T}{T_{g_3}}\right)^{-\frac{FW}{bW_0}}$$

3. CALCULATION OF $\frac{\lambda}{\lambda_0}$

The mean free path of the particles (considered as hard elastic spheres) is shown by kinetic theory (Ref. 27, p. 113) to be

$$\lambda = \frac{1}{\pi\sqrt{2}N\sigma^2}\quad\text{or}\quad\frac{\lambda}{\lambda_0} = \frac{N_0}{N}\left(\frac{\sigma_0}{\sigma}\right)^2.\quad\text{If }\frac{N}{N_0}\text{ is now replaced by its value in terms}$$

of (p,T,g), then $\frac{\lambda}{\lambda_0} = \frac{p_0 Tg}{pT_0 g_0}\left(\frac{\sigma_0}{\sigma}\right)^2.$

It is assumed that $g = g_o$ and $\sigma = \sigma_o$ at all altitudes. This equation then becomes $\dfrac{\lambda}{\lambda_o} = \dfrac{p_o T}{p T_o}$ where $T = f(h)$ and $\dfrac{dp}{p} = -\dfrac{g_o \rho_o T_o W}{p_o W_o} \dfrac{dh}{T}$. Then, for altitudes below the dissociation of oxygen; i.e., $\dfrac{W}{W_o} = 1$, the value of $\dfrac{\lambda}{\lambda_o}$ is merely the reciprocal of the $\dfrac{\rho}{\rho_o}$ value or

$$\frac{\lambda}{\lambda_o} = \frac{\rho_o}{\rho} \qquad \text{for} \qquad \begin{array}{l} h = 0 - 262,467 \text{ ft. (day)} \\ h = 0 - 344,487 \text{ ft. (night)} \end{array}$$

For the regions where $\dfrac{W}{W_o} = g(h) \neq 1$, this factor still appears; and now,

$$\frac{\rho}{\rho_o} = \frac{p T_o W}{p_o T W_o}$$

or

$$\frac{\lambda}{\lambda_o} = \frac{W \rho_o}{W_o \rho} \, .$$

4. CALCULATIONS OF $\dfrac{c}{c_o}$

It was shown in (F) that using the perfect gas law the speed of sound $c = \sqrt{\dfrac{\gamma R_m T}{W}}$. For the calculation of $\dfrac{c}{c_o}$ this is

$$\frac{c}{c_o} = \left(\frac{\gamma T W_o}{\gamma_o T_o W} \right)^{\frac{1}{2}} .$$

For all regions below the dissociation of oxygen $\dfrac{W_o}{W} = 1$, $\dfrac{\gamma}{\gamma_o} = 1$; and this becomes $\dfrac{c}{c_o} = \sqrt{T/T_o}$.

For the regions of the dissociation of oxygen

$$\frac{\gamma}{\gamma_o} = 1 - \frac{128K(h - h_m)}{21W_o} \qquad K = -\frac{V_o}{h_a - h_m}$$

$$\frac{W}{W_o} = \frac{1}{1 - K(h - h_m)}$$

and the relation becomes

$$\frac{c}{c_o} = \left[1 + \frac{128\left(\dfrac{V_o}{h_a - h_m}\right)(h - h_m)}{21W_o} \right]^{\frac{1}{2}} \left[\frac{T}{T_o}\left\{ 1 + \frac{V_o}{h_a - h_m}(h - h_m) \right\} \right]^{\frac{1}{2}}$$

5. CALCULATIONS OF VISCOSITY

a. The formula for coefficient of viscosity below the region of oxygen dissociation has been developed in Section G and is merely stated here.

$$\eta = \eta_o \left(\frac{T}{296.1}\right)^{\frac{3}{2}} \left(\frac{409.6}{T + 113.5}\right)$$

where

$$\eta_o = 1827 \times 10^{-7} \text{ and } T \text{ is in } °K.$$

b. The calculation of the coefficient of viscosity in the region of oxygen dissociation follows[1].

For the viscosity of a three component gas

$$\eta \times 10^7 = -\frac{\begin{vmatrix} J_{11} & J_{12} & J_{13} & 1 \\ J_{21} & J_{22} & J_{23} & 1 \\ J_{31} & J_{32} & J_{33} & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} J_{ij} \end{vmatrix}}$$

---

[1] Taken from Hirschfelder, Bird, Spotz (Ref. 30).

where

$$J_{ij} = -A_{ij} + B_{ij} \ , \ J_{ii} = -A_{ii} + B_{ii} + \sum_{K=1}^{3} \frac{X_K}{X_i} \left(A_{iK} + B_{iK} \frac{W_K}{W_i} \right)$$

$$A_{ij} = (0.017663)T^{-\frac{1}{2}} \left[(r_o)_{ij} \right]^2 \left[W^{(1)}(1;kT/\epsilon_{ij}) \right] \left[\frac{W_i W_j}{(W_i + W_j)^3} \right]^{\frac{1}{2}}$$

$$B_{ij} = (0.0052988)T^{-\frac{1}{2}} \left[(r_o)_{ij} \right]^2 \left[W^{(2)}(1;kT/\epsilon_{ij}) \right] \left[\frac{W_i W_j}{(W_i + W_j)^3} \right]^{\frac{1}{2}}$$

$$\epsilon_{ij} = \sqrt{\epsilon_{ii} \epsilon_{jj}}$$

$$(r_o)_{ij} = \frac{(r_o)_{ii} + (r_o)_{jj}}{2}$$

$(\epsilon)_{ii}$ = maximum energy of attraction between molecules (or atoms) of the ith component

$(\epsilon)_{ij}$ = maximum energy of attraction between molecules (or atoms) of the ith and jth components

X_i = mole fraction of the ith component

W_i = molecular weight of the ith component

$(r_o)_{ii}$ = low velocity collision diameter of molecules (or atoms) of the ith component colliding with each other

$(r_o)_{ij}$ = interaction collision diameter for molecules (or atoms) of the ith component colliding with molecules (or atoms) of the jth component

For the case of the atmosphere from 80 km upwards, the air can be considered as a mixture of N_2, O_2, and O.

	$(r_o)_{ii}$ A	ϵ_{ii}/k $°K$	W_i
N_2	3.681	91.46	28.016
O_2	3.433	113.2	32.000
O	2.75	28	16.000

$\frac{kT}{\epsilon}$	$W^{(1)}(1)$	$W^{(2)}(2)$	$\frac{kT}{\epsilon}$	$W^{(1)}(1)$	$W^{(2)}(2)$
.3	1.331	2.785	2.1	.5284	1.156
.35	1.238	2.628	2.2	.5203	1.138
.4	1.159	2.492	2.3	.5129	1.122
.45	1.092	2.368	2.4	.5061	1.107
.5	1.033	2.257	2.5	.4998	1.093
.55	.9829	2.156	2.6	.4939	1.081
.6	.9383	2.065	2.7	.4885	1.069
.65	.8991	1.982	2.8	.4836	1.058
.7	.8644	1.908	2.9	.4788	1.048
.75	.8335	1.841	3.0	.4745	1.039
.8	.8058	1.780	3.1	.4703	1.030
.85	.7809	1.725	3.2	.4664	1.022
.9	.7585	1.675	3.3	.4628	1.014
.95	.7382	1.629	3.4	.4593	1.007
1.0	.7197	1.587	3.5	.4560	.9999
1.05	.7028	1.549	3.6	.4529	.9932
1.1	.6873	1.514	3.7	.4499	.9870
1.15	.6731	1.482	3.8	.4471	.9811
1.2	.6601	1.452	3.9	.4444	.9755
1.25	.6479	1.424	4.0	.4418	.9700
1.3	.6367	1.399	4.1	.4394	.9649
1.35	.6263	1.375	4.2	.4370	.9600
1.4	.6166	1.353	4.3	.4347	.9553
1.45	.6075	1.333	4.4	.4326	.9507
1.5	.5991	1.314	4.5	.4305	.9464
1.55	.5912	1.296	4.6	.4284	.9422
1.6	.5837	1.279	4.7	.4265	.9382
1.65	.5767	1.264	4.8	.4246	.9343
1.7	.5701	1.248	4.9	.4228	.9305
1.75	.5639	1.234	5.0	.4211	.9269
1.8	.5580	1.221	6	.4062	.8963
1.85	.5523	1.209	7	.3943	.8727
1.9	.5471	1.197	8	.3856	.8538
1.95	.5421	1.186	9	.3778	.8379
2.0	.5373	1.175	10	.3712	.8242

$\dfrac{kT}{\epsilon}$	$W^{(1)}(1)$	$W^{(2)}(2)$	$\dfrac{kT}{\epsilon}$	$W^{(1)}(1)$	$W^{(2)}(2)$
20	.3320	.7432	70	.2732	.6194
30	.3116	.7005	80	.2676	.6076
40	.2980	.6718	90	.2628	.5973
50	.2878	.6504	100	.2585	.5882
60	.2798	.6335	200	.2322	.5320
			300	.2180	.5016
			400	.2085	.4811

6. COMPUTATIONS

In the previous parts (1 through 5) of this section, formulas have been derived for computing changes in temperature, density, speed of sound, mean free path, and viscosity as a function of altitude. To use these formulas in the computation of missile trajectories, the quantities $\frac{\rho}{\rho_0}$, $\frac{c}{c_0}$, $\frac{\lambda}{\lambda_0}$, and $\frac{\eta}{\eta_0}$ must be tabulated for the altitude range of these trajectories. This tabulation was begun under the supervision of Dr. D. M. Brown, but it was little more than started when a change in emphasis in the Wizard contract caused it to be stopped.

At present the values of $\frac{c}{c_0}$ are in a form which can be easily tabulated. The term $\left(\frac{T}{T_0}\right)^{\frac{1}{2}}$ is expressed as a polynomial in h (altitude), and $\left(\frac{\gamma W_0}{\gamma_0 W}\right)^{\frac{1}{2}}$ is tabulated for the altitudes at which its value is not unity.

Work on the density ratio, $\left(\frac{\rho}{\rho_0}\right)$ has not progressed very far. No work has been started on the computation of $\frac{\lambda}{\lambda_0}$ or $\frac{\eta}{\eta_0}$.

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