WORK SESSION IN LYAPUNOV'S SECOND METHOD

Sponsored by the Nonlinear Control Subcommittee
of the AIEE Feedback Control Systems Committee

Edited by

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PREFACE

This pamphlet contains papers and problems of a Workshop Session on the Second Method of Lyapunov, sponsored by the Nonlinear Control Theory Subcommittee of the American Institute of Electrical Engineers held on September 6, preceding the Joint Automatic Control Conference for 1960, at the Massachusetts Institute of Technology, Cambridge, Massachusetts.

In 1892 A. M. Lyapunov, a Russian Mathematician, postulated in his book, The General Problem of Motion Stability a number of sufficient conditions for stability or instability of undisturbed systems. By reducing the problem of stability on an undisturbed system to the problem of the stability of the equilibrium position, Lyapunov connected the fact of stability or instability with the presence of a "v" function, the time derivative of which has certain properties. For a long time it was not clear whether the conditions postulated by him were necessary. This question remained unanswered for a long time and only much later were the necessary conditions established which would insure the existence of a Lyapunov Function.

In 1949 the work was translated into English and since then it has received some notoriety in this country, but only within the past five years has it been given serious consideration by those interested in feedback control systems. At the present time it is considered to be the most general method of studying the stability of nonlinear systems.

The purpose of the Workshop Session as planned by the Nonlinear Control Theory Subcommittee was to organize a group of papers which would serve to introduce this subject to the uninformed by starting with introductory concepts and culminating in a group of home problems designed to enhance the reader's understanding of the subject.

The workshop committee directly responsible for the success of this Session were Professor I. Flugge-Lotz, Stanford University Chairman; Dr. Kan Chen, Westinghouse Electric Corporation; Professor John E. Gibson, Purdue University; and Professor T. J. Higgins, University of Wisconsin; although all the members of the Nonlinear

Control Theory Subcommittee should be congratulated for their willingness to assist in every way possible.

Thanks go to The University of Michigan Industry Program, who made possible this publication.

Louis F. Kazda
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AN INTRODUCTION TO LYAPUNOV'S SECOND METHOD

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1. INTRODUCTION

A fundamental problem associated with the study of dynamic systems is the determination of their stability. While various techniques are available for investigating stability these techniques typically become difficult and tedious to apply if the system is of high order, or is nonlinear or time-varying. An approach to this problem, developed seventy years ago in Russia but almost unrecognized in this country until quite recently, is the so-called second method of Lyapunov. This is a method which has been much exploited in the Soviet Union, and which appears to have great power and flexibility. It does not provide a purely mechanical procedure applicable to all situations; it does require ingenuity to apply to other than standard situations. On the other hand, it may give information about systems that cannot be analyzed in other ways.

An introduction to this second method of Lyapunov is given in the following discussion. The purpose here is to describe the basic idea of the method, to show something of the range of problems to which it applies, and to provide simple illustrative examples. The discussion is presented with a minimum of mathematical niceties, and no attempt is made to justify the procedures which are employed. Most of the published Western literature about the method has been written by mathematicians for a mathematical audience. Those readers interested in this aspect of the subject are referred to the literature[1,2]. The recent paper by Kalman and Bertram[3], in particular, provides an excellent survey of the method, together with appropriate mathematical proofs, and gives many bibliographic references.

2. MATHEMATICAL DESCRIPTION OF SYSTEM

Throughout the following discussion it is assumed that the description of the dynamic system under study has been reduced to a set of simultaneous first-order differential equations

$$
\begin{aligned}
dx_1/dt &= \dot{x}_1 = f_1(x_1,\ldots,x_n) \\
dx_n/dt &= \dot{x}_n = f_n(x_1,\ldots,x_n)
\end{aligned}
\tag{1}
$$

The independent variable in these equations is time t, and the various dependent variables are $x_1,\ldots x_n$. The functions $f_1(x_1,\ldots,x_n)$, $\ldots,f_n(x_1,\ldots,x_n)$ may be nonlinear but are assumed to be differentiable. The order of the system is the integer n. The system described by Equation (1) is said to be stationary in the sense that functions $f_1,\ldots,f_n$ do not depend upon time t. The system is said to be free in the sense that no explicit functions of time appear as forcing functions. A free stationary system is sometimes said to be autonomous.

An equilibrium condition exists if the variables have such values $x_{1e},\ldots,x_{ne}$ that all the derivatives $dx_1/dt,\ldots,dx_n/dt$ are simultaneously zero. In a general way, an equilibrium condition is described as stable if the system tends to remain at that condition following any small disturbance away from the condition.

The dependent variables $x_1,\ldots x_n$ must be chosed in such a way and in sufficient number to describe completely the system under study. For a physical system, these variables will generally be quantities which have certain physical dimensions. It is often possible to make more than a single choice for the variables used in describing a particular system. Sometimes one choice has advantages over some other choice. It should be noted here that in many of the operations which appear in the following discussion it is essential that all the dependent variables be chosen so as to have the same physical dimensions. Either the choice must be made intentionally with this criterion in mind, or else the dimensions must be made the same by suitable conversion factors. Sometimes it is desirable to normalize all quantities into pure numerics having no dimensions. Such normalization can always be done. This necessity for having a common dimension comes about because of the way in which coefficients arising at several points in the equations must be combined in subsequent work.

It is evident that if the order of the system is other than quite small, the set of equations forming Equation (1) is going to be complicated and difficult to manipulate. For this reason it is essential to use matrix notation in the analysis. Rewritten in this notation, Equation (1) becomes

$$d\underset{\sim}{x}/dt = \dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}) \tag{2}$$

where $\underset{\sim}{x}$ is the column matrix, or vector, made up of the n dependent variables, and $\underset{\sim}{f}(\underset{\sim}{x})$ is a similar column matrix of the functions. An equilibrium condition for Equation (2) is $\underset{\sim}{x} = \underset{\sim}{x}_e$ for which $d\underset{\sim}{x}/dt = \dot{\underset{\sim}{x}} = \underset{\sim}{0}$, where $\underset{\sim}{0}$ is the zero column matrix. If functions $\underset{\sim}{f}(\underset{\sim}{x})$ are linear, and $\underset{\sim}{f}(\underset{\sim}{0}) = \underset{\sim}{0}$, the one equilibrium condition is $\underset{\sim}{x}_e = \underset{\sim}{0}$. If functions $\underset{\sim}{f}(\underset{\sim}{x})$ are nonlinear, there may be more than one equilibrium condition.

The values of the n variables in the column matrix x at any instant describe the state of the system at that instant. It is convenient to have a single number to represent, at least in part, the state of the system. Such a single number may be a norm, which can be defined in any of several ways. The norm is sometimes taken as the sum of the magnitudes of all the state variables. For the present discussion, it is more convenient to take the square of the Euclidean norm, written as follows,

$$||\underset{\sim}{x}||^2 = \underset{\sim}{x}'\,\underset{\sim}{x} = (x_1, \dots x_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1^{\,2} + \cdots + x_n^{\,2}$$

The primed matrix $\underset{\sim}{x}'$ is the transpose of the unprimed matrix $\underset{\sim}{x}$. This quantity $||x||^2$ can readily be interpreted geometrically, at least for the cases of two or three variables. It is the square of the distance from the origin to the point representing the particular state $\underset{\sim}{x}$ of the system, as plotted in rectangular coordinates. If the origin is an equilibrium point $\underset{\sim}{x}_e = \underset{\sim}{0}$ the norm $||x||$ provides a simple measure of the departure from this equilibrium point.

3. STABILITY OF SYSTEM

The precise definition of stability, particularly for a nonlinear or time-varying system, is not simple. This question will not be explored here. Rather, for the present discussion, only the concept of asymptotic stability will be employed. An equilibrium condition is asymptotically stable if the system ultimately returns to this condition following any slight disturbance away from it. Stated in another way, if an initial disturbance $||\underset{\sim}{x} - \underset{\sim}{x}_e||$ is small, asymptotic stability implies that ultimately $\underset{\sim}{x} \rightarrow \underset{\sim}{x}_e$ as $t \rightarrow \infty$. For a linear system the disturbance need not be limited in magnitude. Since a nonlinear system may have more than just one equalibrium condition, the disturbance used to test its stability must be small enough so that the system remains near the point being investigated. The exact nature of the nonlinearity governs the required smallness here.

For a linear system with an equilibrium condition at $\underset{\sim}{x} = \underset{\sim}{x}_e = \underset{\sim}{0}$, Equation (2) may be written

$$d\underset{\sim}{x}/dt = \dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} \tag{3}$$

where $\underset{\sim}{A}$ is a square matrix of constant coefficients. The solution for Equation (3) is known to be of the general form

$$x_i = \sum_{j=1}^{n} C_{ij} \exp(\lambda_j t) \tag{4}$$

where the n characteristic exponents, or eigenvalues, are λj, and the constants C_{ij} depend upon initial conditions at $t = 0$. Eigenvalues are determined by coefficients $\underset{\sim}{A}$ in Equation (3), and are roots of the characteristic equation

$$\lfloor \underset{\sim}{A} - \lambda \underset{\sim}{I} \rfloor = 0 \tag{5}$$

where $\underset{\sim}{I}$ is the unit matrix. For a real physical system, matrix $\underset{\sim}{A}$ is real, and the eigenvalues must either be real or occur in complex conjugate pairs. The system is asymptotically stable only if every eigenvalue has a negative real part.

4. TESTS FOR STABILITY

a. Routh-Hurwitz Method

The determination of the characteristic equation, and its factorization to find the various eigenvalues, is a tedious process, particularly if the order of the system is large. If a complete solution is not needed, but information about stability is all that is required, it is sufficient only to test whether every eigenvalue has a negative real part. Such a test is provided by the well known Routh-Hurwitz criterion. This criterion requires expansion of the characteristic equation, Equation (5), into the polynomial form

$$a_o \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0 \tag{6}$$

where coefficient a_o has been made positive. The following matrix is formed from the coefficients of Equation (6)

$$\underset{\sim}{M} = \begin{bmatrix} a_1 & a_o & 0 & 0 & \cdots & & 0 \\ a_3 & a_2 & a_1 & a_o & \cdots & & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & a_n & a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} \\ 0 & \cdots & 0 & 0 & a_n & a_{n-1} & a_{n-2} \\ 0 & \cdots & 0 & 0 & 0 & 0 & a_n \end{bmatrix} \tag{7}$$

All the eigenvalues, found as roots of Equation (6), can have only negative real parts if matrix $\underset{\sim}{M}$ is positive definite. This is the case if each principal minor of $\underset{\sim}{M}$ is positive. These minors are the following determinants

$$\begin{vmatrix} a_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \quad \cdots$$

If all these determinants are positive, and $a_0 > 0$ as already assumed, the system leading to Equation (6) is asymptotically stable. The Routh-Hurwitz criterion for stability requires the expansion of the determinant of Equation (5) into the form of Equation (6), and the subsequent evaluationof the determinants derived from the matrix of Equation (7). While this is all perfectly straightforward, it is tedious to carry out if the system is large.

b. First Method of Lyapunov

If the system described by Equation (2) is nonlinear, the process just described is not immediately applicable. An approach commonly used with a nonlinear system is based on what is known as the first method of Lyapunov. In this method each equilibrium point must be investigated in turn. The nonlinear functions $\underset{\sim}{f}(\underset{\sim}{x})$ of Equation (2) are expanded in Taylor series about the equilibrium point. It is convenient to introduce the new variable $\underset{\sim}{y} = \underset{\sim}{x} - \underset{\sim}{x}_e$, and to write Equation (2) as

$$d\underset{\sim}{y}/dt = \underset{\sim}{F}\,\underset{\sim}{y} + \underset{\sim}{G}(\underset{\sim}{y})\,\underset{\sim}{y} \tag{8}$$

where

$$\underset{\sim}{F} = \begin{matrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots \\ \vdots & & \end{matrix}$$

and $\underset{\sim}{F}$ is the so-called Jacobian matrix, with all its partial derivatives evaluated at the equilibrium point, $\underset{\sim}{x} = \underset{\sim}{x}_e$. The second matrix, $\underset{\sim}{G}(\underset{\sim}{y})$ of Equation (8), contains terms arising from the higher-order derivatives in the Taylor series expansions. This matrix must have its elements vanish at the equilibrium point, that is

$$||\underset{\sim}{G}(\underset{\sim}{y})\ \underset{\sim}{y}|| \ / \ ||\ \underset{\sim}{y}\ || \rightarrow 0 \quad \text{as} \quad ||\ \underset{\sim}{y}\ || \rightarrow 0.$$

The linearized equation

$$d\underset{\sim}{y}/dt = \underset{\sim}{F}\ \underset{\sim}{y} \tag{9}$$

is the first approximation to the nonlinear equation, Equation (8). Lyapunov showed that if the real parts of the eigenvalues corresponding to the linearized equation are not zero, the stability of the nonlinear equation near the equilibrium point is the same as that of the linearized equation. Thus, the stability of a nonlinear system under some conditions can be investigated using the same techniques as are used with linear systems. The procedure here is similar to that which would be used in attempting to find an explicit solution for the system.

5. SECOND METHOD OF LYAPUNOV

The second method of Lyapunov is based on a somewhat different idea, and one that is closely related to the concept of energy. The energy stored in any physical system is, of course, a scalar quantity represented by a single number, even though a complete description of the system may require many variables. In an asymptotically stable system, the stored energy decays with increasing time. Thus a stable system may be characterized by stored energy, which is itself a positive quantity, but which has a time derivative which is negative.

A simple electric circuit, consisting of a capacitance C and a conductance G in series, is described by the equation

$$C\, de/dt + Ge = 0 \tag{10}$$

where e is the voltage across the capacitor. Voltage e is given by the solution $e = E \exp(-Gt/C)$, where $e = E$ at $t = 0$. The system is obviously stable. The instantaneous stored energy is $W = \frac{1}{2} Ce^2 = \frac{1}{2} CE^2 \exp(-2Gt/C)$ which is positive. The time derivative is $dW/dt = \dot{W} = -GE^2 \exp(-2Gt/C)$ which is negative. The ratio $-W/\dot{W} = C/2G$ can be interpreted as a time constant for energy change. Its value is half the more usual time constant, C/G, applying to voltage change.

This concept of energy and its rate of change is extended in the second method. In this method, however, a more general "Lyapunov function" is used, rather than energy itself. If a system is asymptotically stable, a Lyapunov function can be determined for the system. This is a scalar function of time and of the state variables. It is positive, itself, and it has a negative time derivative. Conversely the existence of such a function for a given system implies that the system is asymptotically stable. The Lyapunov function is given the symbol $V(\underset{\sim}{x})$, and the requirements for asymptotic stability are

$$\begin{aligned} V(\underset{\sim}{x}) &> 0 \quad \text{for } \underset{\sim}{x} \neq \underset{\sim}{x}_e \\ dV/dt = \dot{V}(\underset{\sim}{x}) &< 0 \quad \text{for } \underset{\sim}{x} \neq \underset{\sim}{x}_e \\ V(\underset{\sim}{x}) &= 0 \quad \text{for } \underset{\sim}{x} = \underset{\sim}{x}_e \\ V(\underset{\sim}{x}) &\rightarrow \infty \quad \text{for } \|\underset{\sim}{x}\| \rightarrow \infty \end{aligned} \tag{11}$$

For simple systems, $V(\underset{\sim}{x})$ may be taken directly as the energy of the system. For more complicated systems, usually $V(\underset{\sim}{x})$ is better chosen to be something other than the energy. In fact there may be great flexibility in the choice of $V(\underset{\sim}{x})$, and this is one of the features of this method of analysis. At the same time, this flexibility requires ingenuity and experience on the part of the analyst.

The intent in applying the method to test stability is to determine a Lyapunov function for the system directly from the differential equations. It is hoped to avoid many of the steps needed in attempting to find an explicit solution for the equations. A Lyapunov function is known for a few simple sorts of equations. There are some indications of how such a function might be sought for more complicated equations. There is opportunity for further work in this area.

In addition to providing a test for stability, the Lyapunov function may also give information about the transient response of the system. This possibility is based on the observation that the function gives a simple measure of the state of the system at any instant. A parameter η may be defined as

$$\eta = (-\dot{V}/V)_{min} \tag{12}$$

in which case $1/\eta$ is the largest time constant relating to changes in the Lyapunov function $V(\underset{\sim}{x})$. Since $V(\underset{\sim}{x})$ is somewhat similar to energy which generally depends upon the squares of the state variables $\underset{\sim}{x}$, this time constant $1/\eta$ is half the more conventional time constant defined for the state variables themselves. Usually rapid response is desirable so that parameter η can be considered as a kind of figure of merit. Larger values of η correspond to more rapid response.

The point of interest here is that it may be possible to determine a Lyapunov function $V(\underset{\sim}{x})$ for a system without going through the usual steps of finding a solution in a conventional manner. It may be possible to find a $V(\underset{\sim}{x})$ for a nonlinear system that could not be solved at all in the usual way. In either case, figure of merit η can be found. For a nonlinear system, η will change as the state of the system changes. The nature of such changes may represent useful information itself. It should be noted, however, that the actual value of η will depend upon the particular Lyapunov function that is used. Since there may be several such functions for a given system, several alternate values of η may result. Presumably, the choice of $V(\underset{\sim}{x})$ should be made so as to minimize the resulting value of η. Just how to make this choice is usually not known in advance.

6. LYAPUNOV'S THEOREM FOR LINEAR SYSTEM

The Lyapunov function $V(\underset{\sim}{x})$ is known for a free, linear, stationary system as described by Equation (3)

$$d\underset{\sim}{x}/dt = \dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} \qquad (3)$$

Here, $\underset{\sim}{A}$ is a square matrix of constants and equilibrium exists for $\underset{\sim}{x} = \underset{\sim}{x}_e = \underset{\sim}{0}$. For this system, Lyapunov himself showed that one suitable function is

$$V(\underset{\sim}{x}) = ||\underset{\sim}{x}||^2_{\underset{\sim}{P}} \qquad (13)$$

and the notation of the right side of the equation has the meaning $||\underset{\sim}{x}||^2_{\underset{\sim}{P}} = \underset{\sim}{x}'\,\underset{\sim}{P}\,\underset{\sim}{x}$. Matrix $\underset{\sim}{P}$ is a symmetric positive definite matrix satisfying the equation

$$\underset{\sim}{A}'\,\underset{\sim}{P} + \underset{\sim}{P}\,\underset{\sim}{A} = -\underset{\sim}{I} \qquad (14)$$

where $\underset{\sim}{A}'$ is the transpose of $\underset{\sim}{A}$, and $\underset{\sim}{I}$ is the unit matrix. If a matrix $\underset{\sim}{P}$ that will satisfy this condition can be found, then the system described by Equation (3) is asymptotically stable. <u>This requirement is both necessary and sufficient</u>.

Matrix P is symmetric if $\underset{\sim}{P}' = \underset{\sim}{P}$. It is positive definite if all the principal minors are positive. The elements of $\underset{\sim}{P}$ can be found from the simultaneous algebraic equations that result from expanding the matrix equation, Equation (14). There are (n/2)(n +1) such simultaneous equations. This process is similar to the Routh-Hurwitz test for stability. However, it does not require the explicit determination of the characteristic equation, Equation (6). This determination is a tedious process for systems of high order, and thus the Lyapunov approach may require less manipulation. This is typically the case if the order of the system exceeds four or five.

The figure of merit η for the transient response of the free, linear, stationary system can readily be found. The Lyapunov function for the system is $V(\underset{\sim}{x}) = ||\underset{\sim}{x}||^2_{\underset{\sim}{P}}$, which is Equation (13). Its time derivative can be shown to be given by the relation

$$dV/dt = \dot{V}(\underset{\sim}{x}) = -\ ||\underset{\sim}{x}||^2 \tag{15}$$

Figure of merit η is then

$$\eta = (||\underset{\sim}{x}||^2 / ||\underset{\sim}{x}||^2_{\underset{\sim}{P}})_{min}$$

By additional work, this can be shown to be the minimum eigenvalue for the inverse of matrix $\underset{\sim}{P}$, which is written $\underset{\sim}{P}^{-1}$, where $\underset{\sim}{P}\,\underset{\sim}{P}^{-1} = \underset{\sim}{I}$. Thus, in two alternate forms,

$$\begin{aligned} \eta &= \min \lambda\ (\underset{\sim}{P}^{-1}) \\ 1/\eta &= \max \lambda\ (\underset{\sim}{P}) \end{aligned} \tag{16}$$

The determination of this eigenvalue requires either the expansion of a determinant similar to Equation (5), or the use of an appropriate numerical procedure directly with the matrix.

Example 1

A simple example of the use of this technique is provided by the electric circuit of Figure 1. The elements of this linear circuit

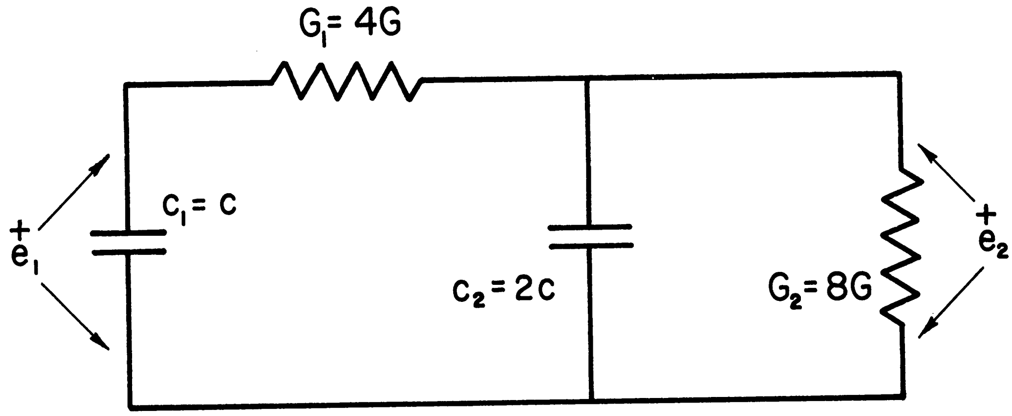

Figure 1. Stable second-order electric current for Examples 1 and 2.

have the values shown. The equations for the circuit may be written

$$
\begin{aligned}
de_1/dt &= \dot{e}_1 = -4ke_1 + 4ke_2 \\
de_2/dt &= e_2 = 2ke_1 - 6ke_2
\end{aligned}
\tag{17}
$$

where e_1 and e_2 are the instantaneous voltages across the two capacitors and the definition has been made, $k = G/C$. Written in the form of Equation (3), these relations are

$$
d\underset{\sim}{e}/dt = \dot{\underset{\sim}{e}} = \underset{\sim}{A}\,\underset{\sim}{e} \tag{18}
$$

where

$$
\underset{\sim}{A} = \begin{bmatrix} -4k & 4k \\ 2k & -6k \end{bmatrix}
$$

It should be noted that the two dependent variables here, e_1 and e_2, are both voltages and thus have the same physical dimensions. This is necessary for the application of Equation (14).

In a conventional solution, the characteristic equation is found first as

$$
\underset{\sim}{A} - \lambda\,\underset{\sim}{I} = \begin{vmatrix} (-4k \;\; -\lambda) & 4k \\ 2k & (-6k \;\; -\lambda) \end{vmatrix} = 0
$$

Its roots yield the two eigenvalues, $\lambda_1 = -2k$, $\lambda_2 = -8k$. Since these are both real and negative, the system is stable. If initial conditions are chosen arbitrarily as $e_1 = -E$, $e_2 = 2E$ at $t = 0$, the solution for the system is easily determined to be

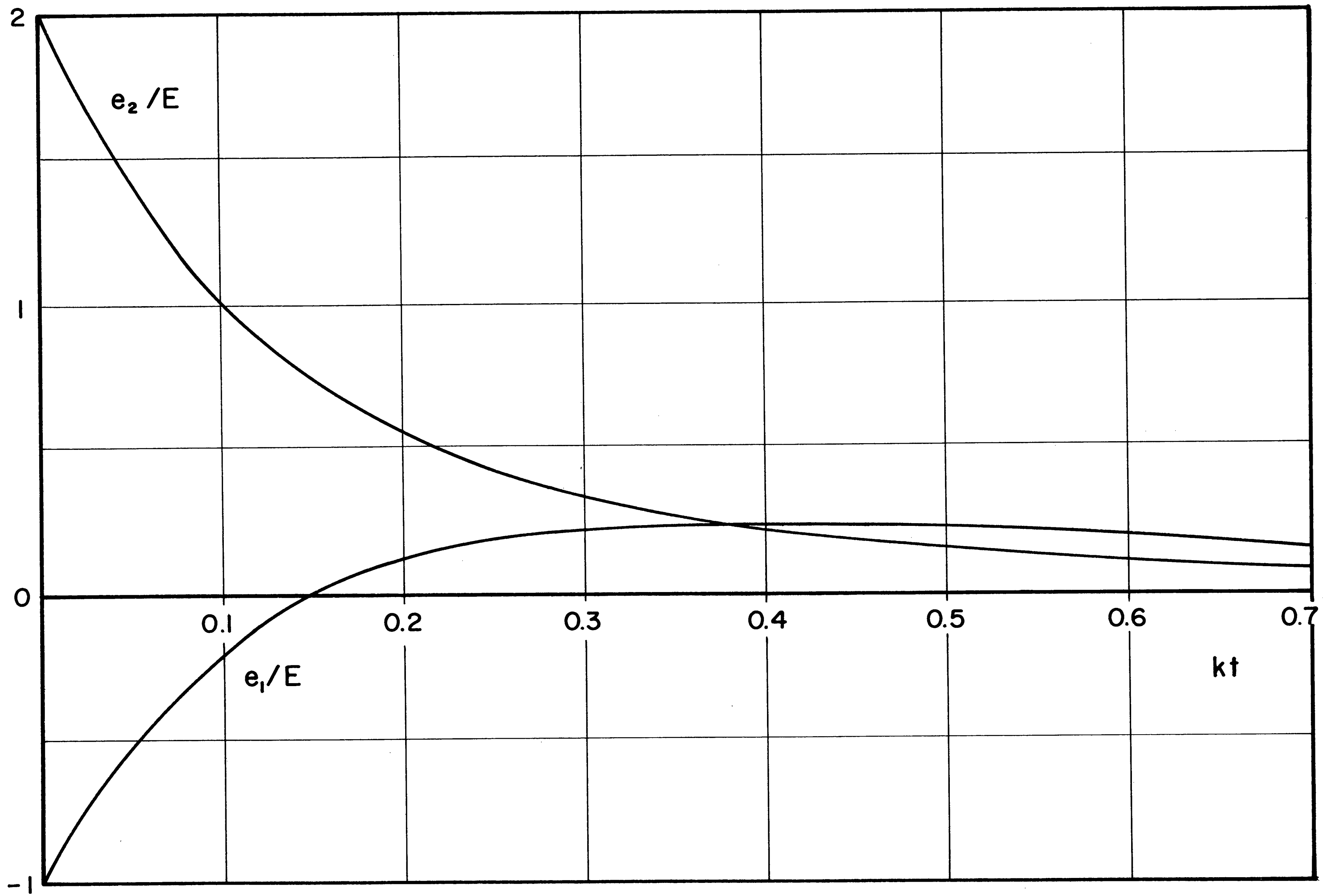

Figure 2. Variation of voltages e_1 and e_2 with time t for Examples 1 and 2. Initial conditions are such that e_1 undergoes a polarity reversal.

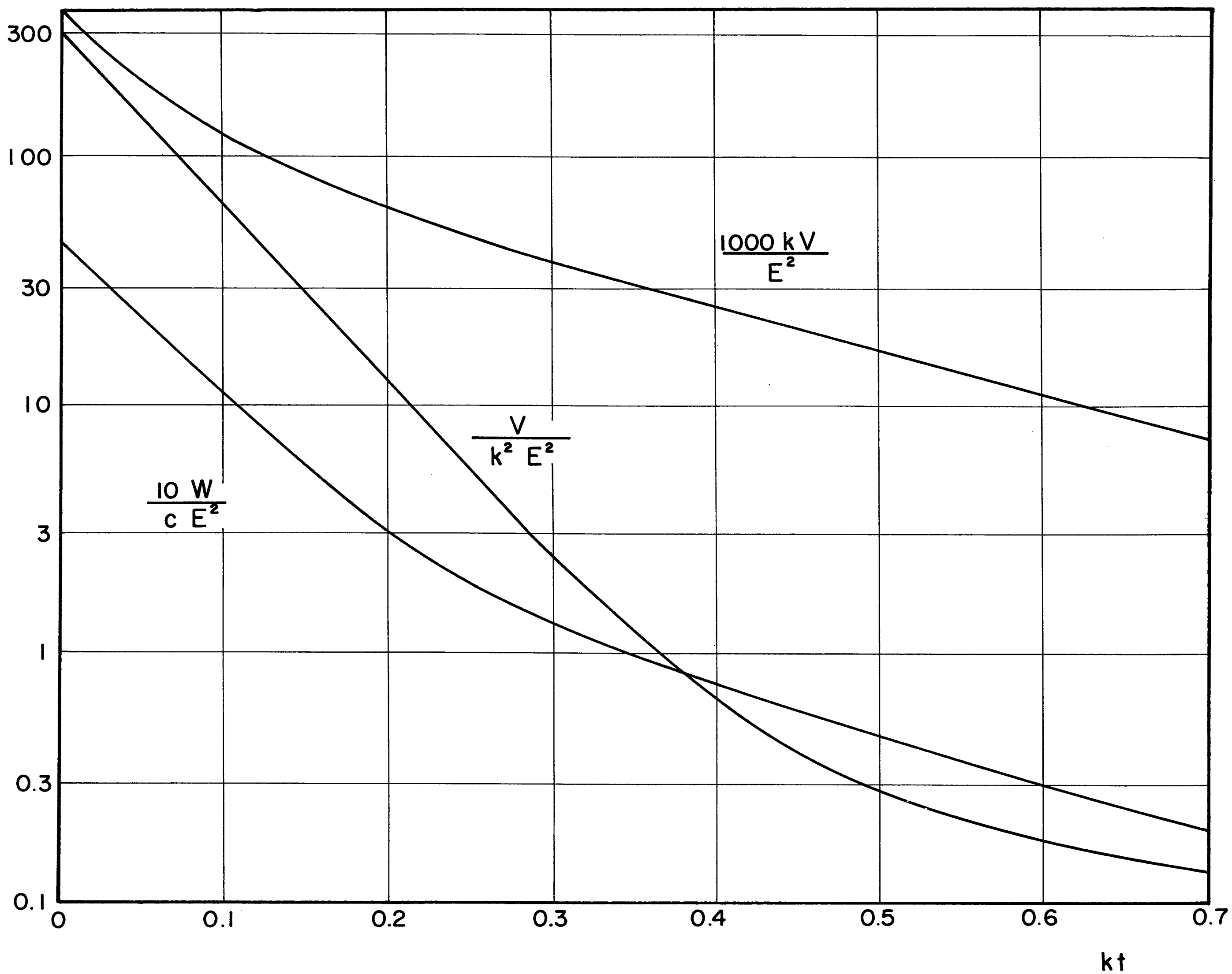

Figure 3. Functions relating to the stability for Examples 1 and 2. The energy, W from Equation 20, is plotted as $10W/CE^2$. One Lyapunov function, V from Equation 22, found using Lyapunov's theorem, is plotted as $1000kV/E^2$. A second Lyapunov function, V from Equation 27, found using Krasovskii's theorem, is plotted as V/k^2E^2. All three functions are positive, with negative time derivatives.

$$e_1/E = 2/3 \exp(-2kt) - 5/3 \exp(-8kt)$$
$$e_2/E = 1/3 \exp(-2kt) + 5/3 \exp(-8kt) \tag{19}$$

The two voltages are plotted against time in Figure 2. The initial conditions have been chosen so that the polarity of voltage e_1 reverses as time progresses. Both voltages decay toward zero with increasing time, but their ratio becomes $e_2/e_1 = 1/2$.

The energy stored in the circuit at any instant is given by

$$W = \tfrac{1}{2} C_1 e_1^{2} + 1/2 \; C_2 e_2^{2} \tag{20}$$

This energy W is plotted logarithmically in Figure 3. It is clear that the energy is a positive quantity, but that it has a negative time derivative, as would be expected for a stable system. This is so, even when one voltage goes through zero and reverses sign, as is the case here.

A Lyapunov function for this same system can be found by applying Equation (14), which then appears as

$$\begin{bmatrix} -4k & 2k \\ 4k & -6k \end{bmatrix} \begin{bmatrix} q & r \\ r & s \end{bmatrix} + \begin{bmatrix} q & r \\ r & s \end{bmatrix} \begin{bmatrix} -4k & 4k \\ 2k & -6k \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

where symmetric matrix $\underset{\sim}{P}$ has been written with elements q, r, s that are to be determined. This matrix equation leads to the three simultaneous equations

$$
\begin{aligned}
-8kq \;+\; 4kr \quad\quad\quad\quad &= -1 \\
4kq \;-\; 10kr \;+\; 2ks &= 0 \\
8kr \;-\; 12ks &= -1
\end{aligned}
$$

Solution for the quantities q, r, s determines P as

$$
\underset{\sim}{P} = (1/40k) \begin{vmatrix} 7 & 4 \\ 4 & 6 \end{vmatrix} \tag{21}
$$

Matrix $\underset{\sim}{P}$ is positive definite, since both $|7| > 0$ and $\begin{vmatrix} 7 & 4 \\ 4 & 6 \end{vmatrix} > 0$, and thus the system is asymptotically stable.

A Lyapunov function is, from Equation (13),

$$
\begin{aligned}
V(\underset{\sim}{e}) &= ||\underset{\sim}{e}||^{2}_{\underset{\sim}{P}} = (1/40k)(7e_1^2 + 8e_1e_2 + 6e_2^2) \\
&\doteq (E^2/72k) \;[10 \exp(-4kt) - 8 \exp(-10kt) + 25 \exp(-16kt)]
\end{aligned} \tag{22}
$$

where Equation (19), the exact solution, has been used. The time derivative is, from Equation (15)

$$
\begin{aligned}
\dot{V}(\underset{\sim}{e}) &= - ||e||^2 = - e_1^2 - e_2^2 \\
&= - (E^2/9 \;[5 \exp(-4kt) - 10 \exp(-10kt) + 50 \exp(-16kt)]
\end{aligned} \tag{23}
$$

This Lyapunov function is plotted in Figure 3. It is, of course, positive, with a negative time derivative.

The ratio $-\dot{V}/V$ is initially 16k, but becomes 4k as time increases indefinitely. Thus, the figure of merit is $\eta = 4k$, as found from $-\dot{V}/V$ making use of exact solutions for e_1 and e_2. This result corresponds to the more slowly-varying component of voltage, which varies as exp(-2kt), so that the component of energy associated with it varies as exp(-4kt). The figure of merit can also be found directly from Equation (16), which does not require the use of exact solutions for $\underset{\sim}{e}$. The eigenvalues for matrix $\underset{\sim}{P}$ are found from the relation

$$\begin{vmatrix} (7/40k - \lambda) & 4/40k \\ 4/40k & (6/40k - \lambda) \end{vmatrix} = 0$$

The two values are $\lambda_1 = 0.263/k$ and $\lambda_2 = 0.062/k$. The reciprocal of the larger of these is the figure of merit, $\eta = 1/(0.263/k) = 3.8k$. This result is similar to that found using exact solutions for $\underset{\sim}{e}$, although it has been obtained without the need of these solutions.

7. KRASOVSKII'S THEOREM FOR NONLINEAR SYSTEM

The Lyapunov function of Equation (13), and the test for stability employing Equation (14), apply only to linear systems. The stability of a nonlinear system near an equilibrium point may be investigated using this technique, providing the system is appropriately linearized. This linearization may be carried out by applying the first method of Lyapunov, as described in Equations (8) and (9). An alternate approach to a nonlinear system is based upon work by Krasovskii.[4]

A free, stationary, nonlinear system is described by Equation (2) as

$$d\underset{\sim}{x}/dt = \underset{\sim}{f}(\underset{\sim}{x}) \qquad (2)$$

where $\underset{\sim}{f}(\underset{\sim}{x})$ is differentiable, but generally nonlinear, and it is assumed that $\underset{\sim}{f}(\underset{\sim}{0}) = \underset{\sim}{0}$. The Jacobian matrix for the system is

$$\underset{\sim}{F}(\underset{\sim}{X}) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots \\ \vdots & & \end{bmatrix}$$

A matrix $\hat{\underset{\sim}{F}}(\underset{\sim}{x})$ is defined as $\hat{\underset{\sim}{F}}(\underset{\sim}{x}) = \underset{\sim}{F}(\underset{\sim}{x}) + \underset{\sim}{F}'(\underset{\sim}{x})$ where $\underset{\sim}{F}'(\underset{\sim}{x})$ is the transpose of $\underset{\sim}{F}(\underset{\sim}{x})$. Matrix $\hat{\underset{\sim}{F}}(\underset{\sim}{x})$ is evidently symmetric. If $-\hat{\underset{\sim}{F}}(\underset{\sim}{x})$ is positive definite for all values of x, the equilibrium point $\underset{\sim}{x}_e = \underset{\sim}{0}$ is asymptotically stable in the large, and a Lyapunov function for the system is

$$V(\underset{\sim}{x}) = \|\underset{\sim}{f}(\underset{\sim}{x})\|^2 \tag{24}$$

In order that $-\hat{\underset{\sim}{F}}(\underset{\sim}{x})$ be positive definite, all its principal minors must be positive.

This criterion of stability is readily applied because the mathematical manipulations required are simple. Such simplicity is an evident necessity where a nonlinear system is involved. If the criterion holds for all values of the state variables $\underset{\sim}{x}$, the system is stable for any state. There is no limitation to small departures from the equilibrium condition, as is the case with linearization as used with the Lyapunov first method. On the other hand, it must be recognized that the criterion is a rather restrictive one. While it is sufficient to assure asymptotic stability, it may not be necessary for such stability.

In other words, a particular state may actually be stable even though the criterion is not satisfied. The criterion does represent, however, one of the few known general criteria applicable to nonlinear systems.

The time derivative of $V(\underset{\sim}{x})$ of Equation (24) can be shown to be given by the relation

$$\dot{V}(\underset{\sim}{x}) = \| \underset{\sim}{f}(\underset{\sim}{x}) \|^{2} \hat{\underset{\sim}{F}}(\underset{\sim}{x}) = \underset{\sim}{f}' \hat{\underset{\sim}{F}} \underset{\sim}{f} \qquad (25)$$

A comparison of Equations (24) and (25) with Equations (13) and (15) indicates that figure of merit η is

$$\eta = \min \lambda \left[-\hat{\underset{\sim}{F}}(\underset{\sim}{x}) \right] \qquad (26)$$

<u>Example 2</u>

This method of Krasovskii may be applied to the circuit of Figure 1, which is, of course, a linear system. Matrix F(e) is the same as $\underset{\sim}{A}$ of Equation (18), and is

$$\underset{\sim}{F}(\underset{\sim}{e}) = \begin{bmatrix} -4k & 4k \\ 2k & -6k \end{bmatrix}$$

Matrix $\hat{\underset{\sim}{F}}(\underset{\sim}{e})$ is then

$$\hat{\underset{\sim}{F}}(\underset{\sim}{e}) = \underset{\sim}{F} + \underset{\sim}{F}' = \begin{bmatrix} -4k & 4k \\ 2k & -6k \end{bmatrix} + \begin{bmatrix} -4k & 2k \\ 4k & -6k \end{bmatrix} = \begin{bmatrix} -8k & 6k \\ 6k & -12k \end{bmatrix}$$

$$-\hat{\underset{\sim}{F}}(\underset{\sim}{e}) = \begin{bmatrix} 8k & -6k \\ -6k & 12k \end{bmatrix}$$

This matrix is positive definite since both $|8k| > 0$ and

$$\begin{bmatrix} 8k & -6k \\ -6k & 12k \end{bmatrix} > 0,$$ and thus the system is asymptotically stable.

A Lyapunov function is, from Equations (17) and (24)

$$V(e) = ||f(e)||^2 = \dot{e}_1{}^2 + \dot{e}_2{}^2 = k^2(20e_1{}^2 - 56e_1e_2 + 52e_2{}^2) \qquad (27)$$

Its time derivative is, from Equation (25),

$$\dot{V}(e) = ||f(e)||^2 \; \hat{F}(e) = k(-8\dot{e}_1{}^2 = 12\, \dot{e}_1\dot{e}_2 - 12\dot{e}_2{}^2) \qquad (28)$$

This Lyapunov function, Equation (27), is plotted in Figure 3 where it may be compared with that found by the first method applicable only to linear systems and given by Equation (22). Both these functions, as well as the energy given by Equation (20), are positive and have negative time derivatives. Any one provides a valid test for stability. All three curves of Figure 3 have similar shapes in that they are asymptotic to two straight lines, with slopes dependent upon $\lambda = 16k$ and $\lambda = 4k$.

An estimate for the figure of merit is obtainable from Equation (26). This leads to the relation

$$\begin{vmatrix} (8k - \lambda) & -6k \\ -6k & (12k - \lambda) \end{vmatrix} = 0$$

which gives $\lambda_1 = 16.3\ k$, $\lambda_2 = 3.7\ k$. The smaller of these is taken as η, giving $\eta = 3.7k$, which is similar to values found previously. The Krasovskii method is readily applied to this example and generally verifies its stability and yields a value for the figure of merit.

Example 3

A somewhat different example is given by the circuit of Figure 4. The box of this figure contains a nonlinear, voltage-controlled, negative resistance. The current in this resistance is assumed to be related to the voltage across it as $i = -ae_1 + be_1^3$, where a and b are positive constants. This relation is shown in Figure 5. Also shown in Figure 5 is a load line constructed to determine operating conditions if just a single positive resistance R were connected across the terminals of the negative resistance. The three intersection points of this line as drawn indicate three equilibrium points.

The equations for the circuit of Figure 4 may be written

$$\begin{aligned} de_1/dt &= \dot{e}_1 = (a/C)e_1 - (b/C)e_1^3 + (1/RC)e_2 \\ de_2/dt &= \dot{e}_2 = -(R/L)e_1 - (R/L)e_2 \end{aligned} \tag{29}$$

where the two variables, e_1 and e_2, are both voltages with the same dimensions, as is required. The symbols have the meanings identified in Figure 4. There are generally three equilibrium conditions, $\underset{\sim}{e} = \begin{Bmatrix} e_{1e} \\ -e_{1e} \end{Bmatrix}$ where $e_{1e} = \pm [(a - 1/R)/b]^{1/2}$ or $e_{1e} = 0$. If $R<1/a$, two of these conditions are imaginary.

Matrix $f(e)$ is

$$f(\underset{\sim}{e}) = \begin{bmatrix} (a/C)e_1 - (b/C)e_1^3 + (1/RC)e_2 \\ -\ (R/L)e_1 - (R/L)e_2 \end{bmatrix}$$

and matrix $\underset{\sim}{F}(\underset{\sim}{e})$ is

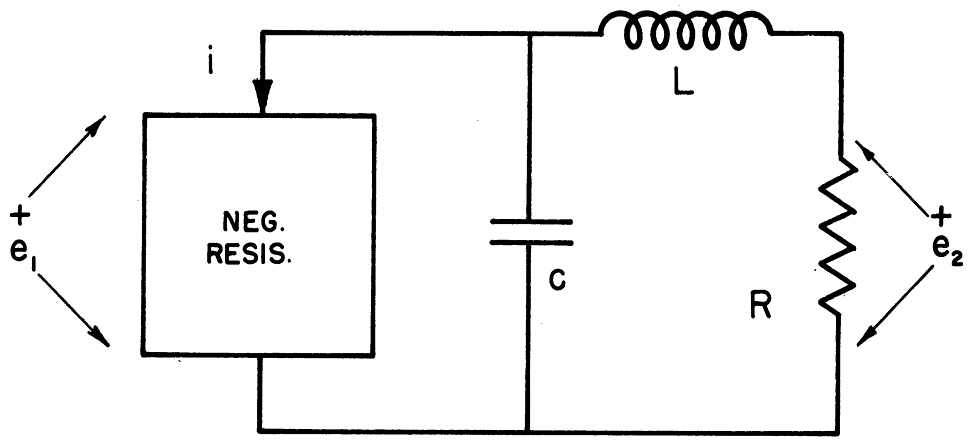

Figure 4. Electric circuit for Example 3. The box contains a nonlinear, voltage-controlled negative resistance. The circuit may, or may not, be stable.

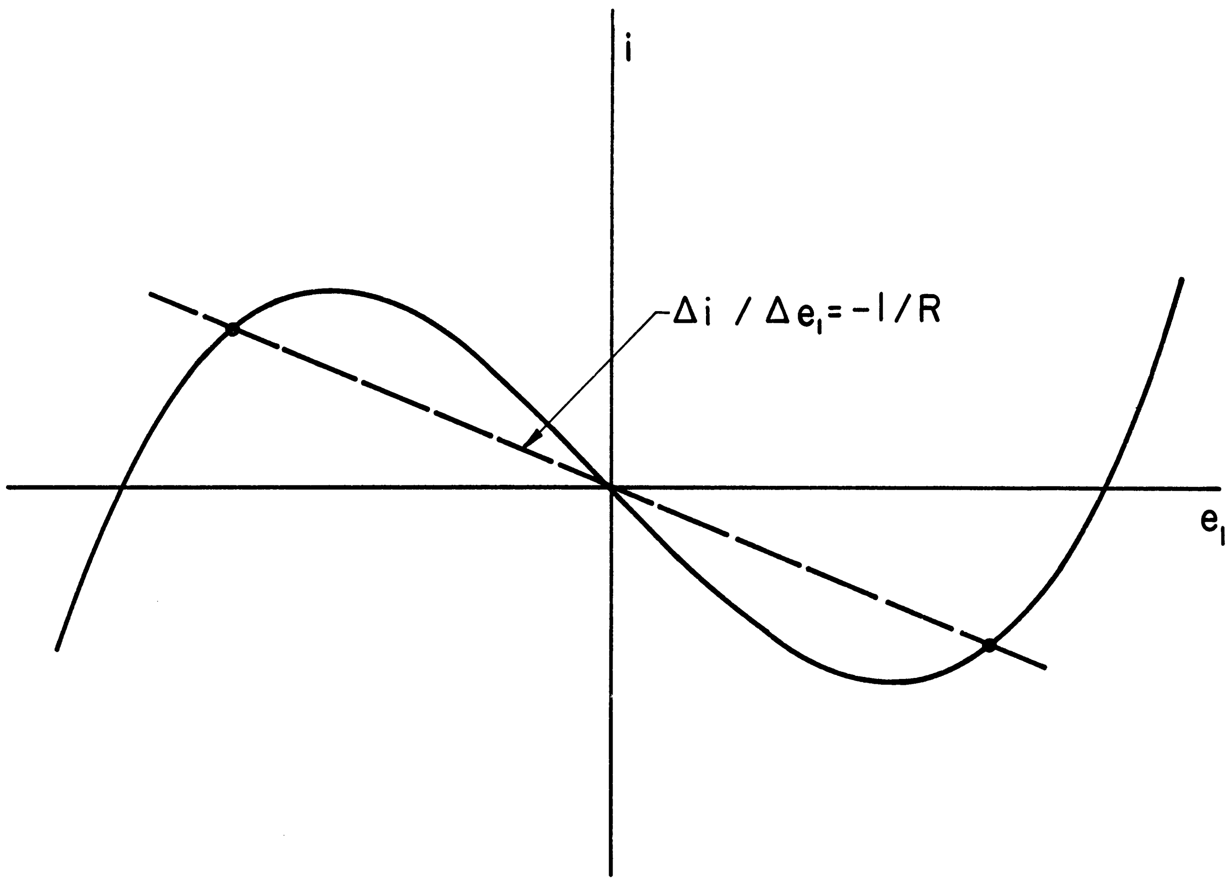

Figure 5. Nonlinear current-voltage characteristic for the negative resistance in the circuit of Figure 4. Also shown is a load line for a positive resistance R connected to the terminals of the negative resistance.

$$\underset{\sim}{F} = \begin{bmatrix} (a/C - 3be_1^{\,2}/C & 1/RC \\ -R/L & -R/L \end{bmatrix}$$

so that matrix $-\hat{\underset{\sim}{F}}(\underset{\sim}{e})$ becomes

$$-\hat{\underset{\sim}{F}} = \begin{bmatrix} -2(a/C - 3\,be_1^{\,2}\;/C & (R/L - 1/RC) \\ (R/L - 1/RC) & 2R/L \end{bmatrix} \qquad (30)$$

This matrix is never positive definite near $\underset{\sim}{e} = \underset{\sim}{0}$, so that there is no assurance that the circuit can be stable near its rest condition.

On the other hand, elimination of e_2 in Equation (29) yields the equivalent single equation in e_1

$$\ddot{e}_1 = (R/L - a/C + 3be_1^{\,2}/C)\dot{e}_1 + (1/LC)(1 - aR + bRe_1^{\,2})e_1 = 0 \qquad (31)$$

This equation does have a stable solution near $e_1 = 0$, provided both a $<$ RC/L and a $<$ 1/R. These two conditions are equivalent to the requirements that both the d-c load line, governed by the resistance R, and the a-c load line, governed by the dynamic resistance L/RC, intersect the negative-resistance characteristic of Figure 5 only at the origin. Thus, the circuit may be stable under appropriate conditions, although this may not be predicted from the matrix $-\hat{\underset{\sim}{F}}$ of Equation (30).

If the negative resistance element is removed entirely from the circuit of Figure 4, the resulting passive circuit clearly must be stable. Removal of the negative resistance causes both coefficients a and b of Equation (30) to vanish, leaving

$$-\hat{\underset{\sim}{F}} = \begin{bmatrix} 0 & (R/L - 1/RC) \\ (R/L - 1/RC) & 2R/L \end{bmatrix} \qquad (32)$$

This matrix can never be positive definite and again there is no assurance of stability.

The equations for the circuit of Figure 4 can be set up in a different way. If the definition is made, $\omega_o^2 = 1/LC$, a dimensionless time may be introduced as $\tau = \omega_o t$. Derivatives may be written in terms of this dimensionless time as $de_1/dt = \omega_o\, de_1/d\tau$ and $d^2e_1/dt^2 = \omega_o^2\, d^2e_1/d\tau^2$. Also, the definitions can be made, $x_1 = e_1$ and $x_2 = dx_1/d\tau$. With these definitions, Equation (31) can be written

$$\begin{aligned} dx_1/d\tau &= x_2 \\ dx_2/d\tau &= -(1 - aR + bRx_1^2)x_1 - (R/L\omega_o - a/C\omega_o + 3bx_1^2/C\omega_o)x_2 \end{aligned} \tag{33}$$

Equations (33) are different from Equation (29) even though they describe the same physical system. Here, one variable, x_2, is simply the time derivative of the other, x_1. That is not the case with the first set of equations. Again, however, because of the use of dimensionless time, the dimensions of both variables are the same.

Matrix $\hat{\underset{\sim}{F}}$ arising from Equation (33) is

$$\hat{\underset{\sim}{F}} = \begin{bmatrix} 0 & (-aR + 3bRx_1^2 + 6bx_1x_2/C\omega_o) \\ (-aR + 3bRx_1^2 + 6bx_1x_2/C\omega_o) & 2(R/L\omega_o - a/C\omega_o + 3bx_1^2/C\omega_o) \end{bmatrix}$$

This matrix, just as that of Equation (30) found previously, is never positive definite near $x = 0$, so once more there is no assurance that stability can exist.

If the negative resistance is removed, Equation (34) becomes

$$-\underset{\sim}{\hat{F}} = \begin{bmatrix} 0 & 0 \\ 0 & 2R/L\omega_o \end{bmatrix} \qquad (35)$$

This matrix is positive semidefinite, in that the principal minors are zero, and at least are not negative. The test for stability is almost satisfied.

It should be evident from the results of this example that because the Krasovskii method gives only a sufficient condition for stability, it may lead to erroneous conclusions about the behavior of any given system. Furthermore the conclusions that are received by applying the method may depend upon just how the analysis is carried out.

8. BARBASHIN'S THEOREM FOR THIRD-ORDER NONLINEAR SYSTEM

Specific stability criteria have been obtained for one particular third-order system by Barbashin[5,6]. The system is described by the equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -\ f(x_1) - g(x_2) - a_2 x_3 \end{aligned} \qquad (36)$$

where $f(0) = 0$ and $g(0) = 0$, and both $f(x_1)$ and $g(x_2)$ are differentiable. If written as a single third-order equation, this system is equivalent to

$$\dddot{x}_1 + a_2\ddot{x}_1 + g(\dot{x}_1) + f(x_1) = 0 \qquad (37)$$

The equilibrium point, $\underset{\sim}{x}_e = \underset{\sim}{0}$, is asymptotically stable in the large if

$$\begin{aligned} &(i) \quad a_2 > 0 \\ &(ii) \quad f(x_1)\, x_1 > 0, \; x_1 \neq 0 \\ &(iii) \quad a_2 g(x_2)/x_2 - f'(x_1) > 0, \; x_2 \neq 0 \end{aligned} \tag{38}$$

where $f'(x_1) = d/dx_1 \, [f(x_1)]$. While Equations (36) and (37) are written with dots indicating differentiation, the independent variable must be a dimensionless time in order that the x's be of the same dimensions and the criteria of Equation (38) be directly applicable. A Lyapunov function for the system is

$$V(\underset{\sim}{x}) = a_2 F(x_1) + f(x_1) x_2 + G(x_2) + 1/2(a_2 x_2 + x_3)^2 \tag{39}$$

where $F(x_1) = \int_0^{x_1} f(x_1)\, dx_1$ and $G(x_2) = \int_0^{x_2} g(x_2)\, dx_2$.

The time derivative is

$$\dot{V}(\underset{\sim}{x}) = - [a_2 g(x_2)/x_2 - f'(x_1)]\, x_2^{\,2} \tag{40}$$

Variable x_3 does not appear in $\dot{V}(\underset{\sim}{x})$. If the system is stable, $V(\underset{\sim}{x}) > 0$, $\dot{V}(\underset{\sim}{x}) < 0$, except at $\underset{\sim}{x} = \underset{\sim}{0}$, and the conditions of Equation (38) apply.

While Equation (37) is a nonlinear, third-order equation, with rather general nonlinearities allowed in both the dependent variable x_1 and its first derivative $\dot{x}_1$, it is necessary that there be no products of these two sorts of terms, and that the second and third derivatives appear only in linear terms. These requirements tend to limit somewhat the applicability of Equation (37) to third-order systems as they arise in physical situations.

There have been efforts made toward extending the idea used in this case to systems of fourth[6], and higher, orders. The mathematical complications tend to increase very rapidly when this is done, however.

It is worth noting that the stability criteria of Equation (38) contain two different types of linearization, each of which is sometimes used with nonlinear functions. One of these is the derivative $f'(x) = d/dx\,[f(x)]$, which corresponds geometrically to the slope of the tangent to the curve representing $f(x)$, at the point in question. The other is the ratio $f(x)/x$, which corresponds geometrically to the slope of the chord from the origin to the point in question on the curve of $f(x)$. These two slopes are shown in Figure 6 for a case in which they are evidently quite different. For a nonlinear electric resistance for example, variable x might represent voltage and function $f(x)$ would represent current. The derivative $f'(x)$ would then be the variational, or a-c, conductance, while the ratio $f(x)/x$ would be the steady, or d-c, conductance. Often an attempt is made to analyze a nonlinear system by introducing some kind of linearization, and often just which of these types of linearization should be used is not self evident. In this case of Barbashin's theorem, both types appear.

<u>Example 4</u>

An example of the application of Barbashin's theorem is given by the electric circuit of Figure 7, which represents a kind of phase-shift oscillator. The phase-shifting network consists of a three-section resistance-capacitance network composed of elements as shown in the figure. Its differential equation may be shown to be

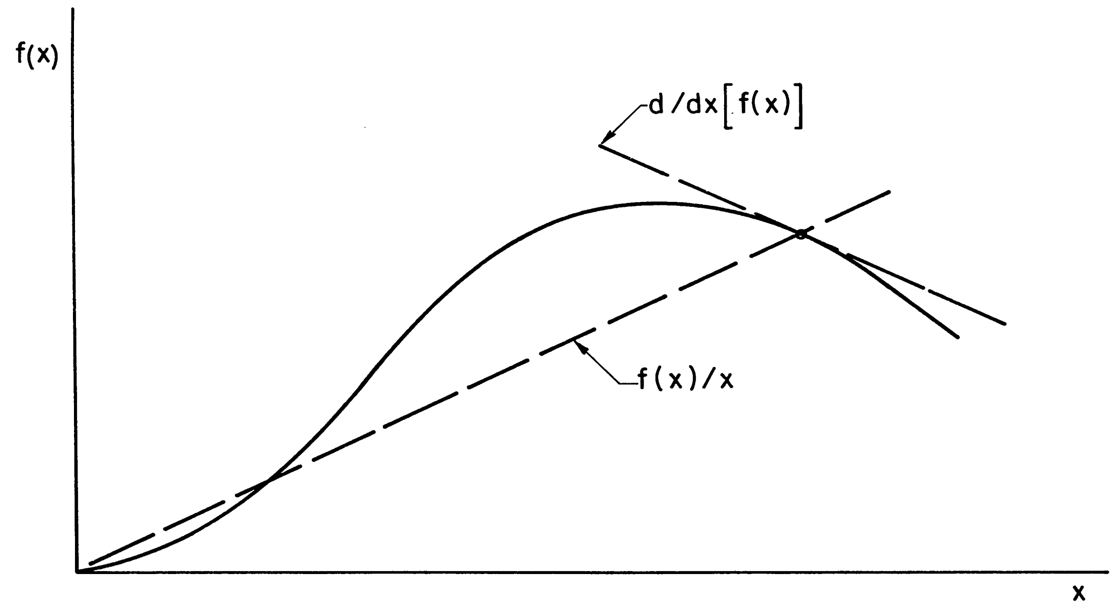

Figure 6. Two ways of linearizing nonlinear function f(x) at a given point. One makes use of the derivative d/dx[f(x)], the other makes use of the ratio f(x)/x. The two methods here give widely different results.

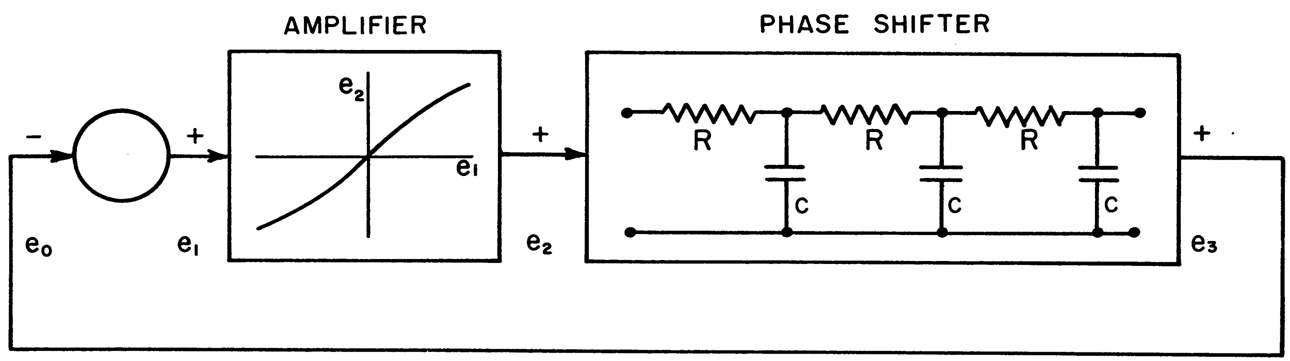

Figure 7. Phase-shift oscillator circuit for Example 4. The amplifier has the nonlinear characteristic shown.

$$e_2 = R^3C^3 \dddot{e}_3 + 5R^2C^2 \ddot{e}_3 + 6RC\, \dot{e}_3 + e_3 \tag{41}$$

where dots indicate derivatives with respect to time, and voltages e_2 and e_3 have the meaning shown in the figure. An amplifier is used in conjunction with the phase-shift network. This amplifier is assumed to saturate as its input voltage increases, so that the following equation might apply

$$e_2 = A\,(1 - be_1{}^2)\, e_1 \tag{42}$$

where A and b are positive constants, and A is the small-signal voltage amplification. This relation holds only moderately well for an actual amplifier, since it predicts that the output voltage first increases, and then decreases with an ultimate change in sign, as the input voltage indefinitely increases. As the circuit is commonly used, there is a polarity reversal, as represented by the relation

$$e_1 = - e_o \tag{43}$$

Finally, of course,

$$e_o = e_3 \tag{44}$$

If Equations (41) - (44) are combined, the result is

$$R^3C^3 \dddot{e}_o + 5R^2C^2 \ddot{e}_o + 6RC\, \dot{e}_o + e_o = - A\,(1 - be_o{}^2)e_o \tag{45}$$

Again, the stability criteria of Equation (38) require the use of a dimensionless time variable. It is convenient here to define $\tau = t/RC$, in which case Equation (45) becomes

$$e_o''' + 5\,e_o'' + 6\,e_o' + (1+A)\,e_o - bA\,e_o^3 = 0$$

where primes indicate differentiation with respect to the dimensionless time τ. Requirements that $e = 0$ be stable are given by Equation (38) as

$$\begin{aligned} &\text{(i)}\quad a_2 = 5 > 0 \\ &\text{(ii)}\quad f(e_o)/e_o = [(1 + A - bAe_o^2] > 0 \\ &\text{(iii)}\quad a_2\, g(\dot{e}_o)/\dot{e}_o - f'(e_o) = [5(6) - (1 + A) + 3bA\, e_o^2] > 0 \end{aligned} \tag{46}$$

Of these relations, (i) is obviously satisfied, while (ii) and (iii) can be written respectively as $A(1 - be_o^2) > -1$ and $A(1 - 3be_o^2) < 29$. If the circuit is initially at rest, e_o is near zero, and stability is predicted if $29 > A > -1$. These are well known conditions for this circuit.

If the circuit is to be used as an oscillator, A is chosen to exceed 29, and for example it might be assumed that $A = 35$. The important condition now is (iii) which becomes $be_o^2 > (1 - 29/35)/3$. Thus, if e_o is initially small, the circuit is unstable and any disturbance builds up, ultimately leading, of course, to an oscillation. However, if e_o is initially large, the circuit is initially stable, and the large initial voltage first decays. This decay brings the circuit into an unstable condition, and oscillation again takes place.

9. CONCLUSION

In conclusion, it should be pointed out that the second method of Lyapunov is valuable, in part, because it provides a useful concept for considering the stability of a system. The idea of the Lyapunov function is somewhat similar to that of energy, but it is more general and more widely applicable. In some cases, a specific expression is known for the appropriate Lyapunov function. When this is so, stability can be explored and information about the speed of transient response can be obtained. Ingenuity is required to apply the concept to systems of other than a few simple types, and there is wide opportunity for further work in this area.

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LYAPUNOV APPROACH TO STABILITY AND PERFORMANCE OF NONLINEAR CONTROL SYSTEMS

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LYAPUNOV APPROACH TO STABILITY AND PERFORMANCE
OF NONLINEAR CONTROL SYSTEMS

In the theory of stability of dynamical systems, the second method of Lyapunov should be considered as a philosophy of approach rather than a systematic method. A unified approach to the whole theory of control systems is made possible by using the basic concept of a Lyapunov function. This relatively new point of view offers much promise to the further development of control theory, particularly as regards nonlinear systems. In the nonlinear case, intuition must be used to obtain suitable Lyapunov functions because no straightforward methods are available at present for doing this.

This paper presents an application of Lyapunov's second method to the study of stability and performance of control systems which may be described by the following nonlinear differential equations:

$$D^n e + p_1 D^{n-1} e + \dots\dots + p_n e = N\ (\ D^{n-1}e,\ D^{n-2}e,\ \dots\ e,\ t\ ) \quad (1)$$

where e represents the error signal, N represents a nonlinear function, $D^n = d^n/dt^n$, and p_j's are constant coefficients. A simple example is a second-order nonlinear control system of the type shown in Fig. 1. If $G_2(S)$ is a second-order transfer function of the form $JS^2 + KS + L$, as it will be for a simple motor and load, and $G_1(S)$ is first order, the differential equation will be of the form

$$D^2 e + k\ De + le + f\ (\ De,\ e\ ) = D^2 r + k\ Dr + lr \quad (2)$$

where $k=K/J$, $l=L/J$.

Equation (2) can be rewritten

$$D^2 e + k\ De + le = N\ (\ De,\ e,\ t\ ) \quad (3)$$

for any given input r(t).

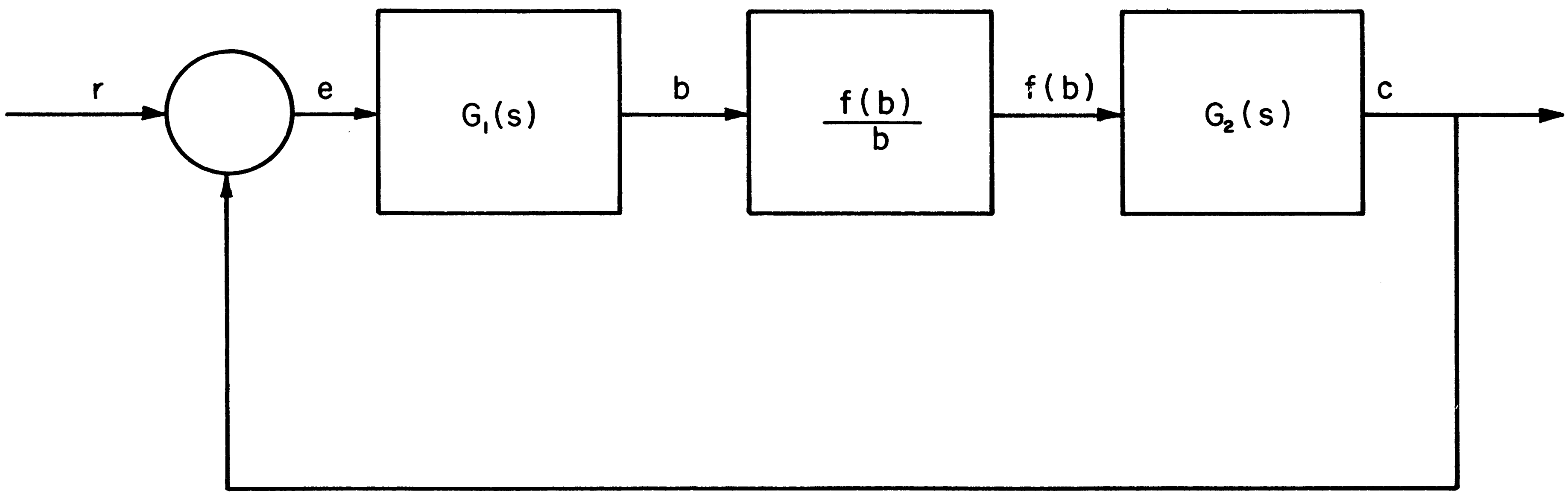

Figure 1. A second-order nonlinear control system.

The Second Method of Lyapunov.

It is usually not difficult to define what is meant by stability in a linear system. Because of the new types of phenomena which arise in a nonlinear system, it is not possible to use a single definition for stability which is meaningful in every case. Kalman and Bertram, 1, stated that concepts of stability are closely related to concepts of convergence. When there are as many of the latter, there are correspondingly many types of stability.

Lyapunov, 2, divided the methods which could be used to indicate the solution of the problem of stability into two categories. He included in the first category those methods which reduce to a direct consideration of the equation of motion, that is, to the explicit determination of the general or a particular solution of the equation. It is usually necessary to search for these solutions by successive approximations. Lyapunov calls the totality of all methods of this category the " first method ".

It is possible, however, to indicate other methods of solution of the stability problem which do not require the calculation of a solution of equation, but which reduce to the search for certain functions known as Lyapunov functions that possess special properties. Lyapunov calls the totality of the methods belonging to this second category the " second method ". Some authors, among whom are Lefschetz, 3, and Hahn, 4, call it the " direct " method of Lyapunov.

For an autonomous system,

$$\dot{x}= f(x) \quad \text{i.e.} \quad \dot{x}_i= f_i(x_1,x_2....x_n) \quad i=1,2,....n \tag{4}$$

where $\dot{x}=dx/dt$, $x=(x_1,x_2,......x_n)$ and $f_i = (f_1, f_2, ... f_n)$.

Lyapunov's second method consists in finding a real, continuous scalar function $V=V(x) = V (x_1,x_2,.....x_n)$ in the neighborhood U of a point of equilibrium (which may be assumed to be $x=0$ without loss of generality), or in the whole phase space, satisfying the following two conditions:

(i) $V(0) =0$;

(ii) $V(x)$ is positive definite.

The first condition means that the function we are interested in vanishes only for $x_i=0$, $i=1,2,....n$. The second condition requires that $V(x)$ has continous partial derivatives and that $\dot{V}(x) > 0$ for all $x=0$; and $\dot{V}(x) = \sum_i \dot{x}_i \; (\partial V / \partial x_i) \; = \sum_i f_i(x) \; (\partial V/\partial x_i) < 0$ for all $x \neq 0$. $V(x)$ and its time derivative $\dot{V}(x)$ are opposite in sign. This function $V(x)$ is called a Lyapunov function, and the Lyapunov theorem has shown that the existence of such a function implies the stability of the system. It may be noted that $\dot{V}$ is actually the total derivative of V with respect to t. For non-autonomous systems, $\dot{V} = \frac{\partial V}{\partial t} + \sum_i f_i(\,t,\,x\,)\frac{\partial V}{\partial x_i}$. Hence, $\dot{V} < 0$ means that V is a decreasing function of t.

<u>Transformation to Canonical Form.</u>

Consider the system described by equation (1), let the phase space variables be given by the matrix equation

$$\{e_1,\, e_2, \ldots\ldots e_n\} = [\, e,\, De,\, \ldots\ldots\, D^{n-1}e\,] \qquad (5)$$

The equivalent system of first order is

$$D\begin{bmatrix} e_1 \\ e_2 \\ \cdot\cdot \\ \cdot\cdot \\ e_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot\cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot\cdot & 0 & 1 \\ -p_n & -p_{n-1} & -p_{n-2} & \cdot\cdot & -p_2 & -p_1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \cdot\cdot \\ \cdot\cdot \\ e_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ N \end{bmatrix} \qquad (6)$$

which may written as

$$D\{e\} = \{A\}\{e\} + \{N\} \qquad (7)$$

where $[A]$ is the square matrix in equation (6).

It is obvious that the latent roots of the square matrix $[A]$ are the characteristic roots of linear portion of the system. Let these roots be distinct and given by $S_1, S_2, \ldots. S_n, \bar{S}_1 \bar{S}_2, \ldots. \bar{S}_m$, and $\lambda_{2m-1} \ldots\ldots \lambda_n$ where S_j and $\bar{S}_j$ are complex conjugate pair and

λ_j's are real roots. Equation (6) can be transformed to canonical form by the following Vandermonde matrix

$$[T] = \begin{bmatrix} 1 & 1 & & 1 & 1 & 1 & \cdots & 1 \\ S_1 & \bar{S}_1 & & S_m & \bar{S}_m & \lambda_{2m+1} & & \lambda_n \\ S_1^2 & \bar{S}_1^2 & & S_m^2 & \bar{S}_m^2 & \lambda_{2m+1}^2 & & \lambda_n^2 \\ \cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots \\ S_1^{n-1} & \bar{S}_1^{n-1} & & S_m^{n-1} & \bar{S}_m^{n-1} & \lambda_{2m+1}^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \tag{8}$$

Thus, transforming the phase variables e_j's to the phase variables x_j's by the equation

$$[e] = [T][X] \tag{9}$$

we obtain

$$D[x] = [B][x] + [T]^{-1}[N] \tag{10}$$

where

$$[B] = [T]^{-1}\ [A]\ [T] \tag{11}$$

is a diagonal matrix having latent roots as its diagonal elements.

A Lyapunov Function.

The problem of determining a Lyapunov function for the systems for which the solution $x = 0$ is known to be stable, is called the inverse problem. There are necessary and sufficient conditions for the existence of a Lyapunov function, but there is no general method of solution for the difficult inverse problem. Research had been done in this direction by a number of investigators, 5,6,7,8,9. The case of autonomous systems when the number of the unknown functions is equal to 2 has been extensively treated by Malkin and his followers.

Actually, in his attack on the problem of stability, Lyapunov's second method was inspired by Dirichlet's proof of Lagrange's theorem on the stability of stationary states. This has been pointed out by Lefschetz. The geometrical interpretation of a Lyapunov function could be a measure of the " distance " of the state x from the origin in the state space. Suppose the distance between the origin and the instantaneous state is continually decreasing as $t \to \infty$, then $x(t) \to 0$. Since the Langrangian function also has the property of being a measure of the swing of the energy content of the system away from the equilibrium point, it is natural to investigate whether it would satisfy the conditions of a Lyapunov function. In fact, in many cases Lyapunov functions are already available, though unrecognized, in standard results in control theory. Kalman and Bertram pointed out that a system whose energy E decreases on the average, but not necessarily at each instant, is stable but E is not a Lyapunov function. A Lyapunov function has to be positive definite. Kang and Fett, 10, suggested the use of the envelope to the Lagrangian function instead of the function itself as a distance function in case the Lagrangian function contains oscillatory terms. Thus, by introducing the following matrix.

$$[W] = \begin{bmatrix} 0 & s_1\bar{s}_1 & & & & & & & & \\ s_1\bar{s}_1 & 0 & & & & & & & & \\ & & 0 & s_2\bar{s}_2 & & & & & & \\ & & s_2\bar{s}_2 & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & s_m\bar{s}_m & & & \\ & & & & & s_m\bar{s}_m & 0 & & & \\ & & & & & & & \lambda^2_{2m+1} & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda^2_n \end{bmatrix} \qquad (12)$$

one establishes the function

$$V(x) = [x]_t \; [W] \; [x] \tag{13}$$

or
$$V(x) = 2\sum_{j=1}^{m} S_j \bar{S}_j x_{2j-1} x_{2j} + \sum_{j=1}^{n-2m} \lambda^2_{2m+j} \, x^2_{2m+j} \tag{14}$$

Equation (14) satisfies the conditions $V(x) > 0$ when $x \neq 0$ and $V(0) = 0$.

Using equation (10) and $[W] = [W]_t$, the time derivative of $V(x)$ can be written as

$$\dot{V}(x) = 2\,[x]_t\,[W][B][x] + 2\,[x]_t\,[W][B]^{-1}[N] \tag{15}$$

The first term on the left hand side of the above equation represents the force free case to the linear system, while the second term represents the effect of the auxiliary forcing function and hence the effect of the nonlinearity.

Denote $[B]^{-1}$ by $\{c_{ij}\}$, and $N(e, De, \ldots\ldots\ldots D^{n-1}e, t)$ by $G(t)$, equation (15) becomes

$$\dot{V}(x) = \dot{V}_{\ell}(x) + \dot{V}_{n}(x) \tag{16}$$

where

$$\dot{V}_{\ell}(x) = 2\sum_{j=1}^{m} (S_j^2 \bar{S}_j + S_j \bar{S}_j^2) x_{2j-1} x_{2j} + 2\sum_{j=1}^{n-2m} \lambda^3_{2m+j} \, x^2_{2m+j} \tag{17}$$

$$\dot{V}_{n}(x) = 2G(t)\left\{\sum_{j=1}^{m} S_j \bar{S}_j (C_{2j,n} x_{2j-1} + C_{2j-1,n} x_{2j}) + \sum_{j=1}^{n-2m} \lambda^2_{2m+j} C_{2m+j,n} x_{2m+j}\right\} \tag{18}$$

$$= 2\,G(t)\,H$$

Consideration on Stability and Performance.

The necessary and sufficient condition that the basic linear system is stable is when the value of its Lyapunov function, along its force free trajectory tends to zero as time approches infinity. In terms of error coordinates e_j's, the Lyapunov function given in equation (13) becomes

$$V\,(x) = [e]_t\,[Y]\,[e] \tag{19}$$

where

$$[Y] = [T^{-1}]_t \quad [W] \quad [T^{-1}] \tag{20}$$

If the characteristic roots of the basic linear system are all real, then the characteristic roots of the matrix [Y] are all positive. It follows that V = constant is an ellipsoidal surface, and the e space is topologically Euclidean. It is clear from equation (17) that if all the characteristic roots of the basic linear system have negative real parts, $\dot{V}_{\ell}(x)$ is always negative and the linear system is stable. Since the behavior of the non-linear system is expressed in terms of some auxilliary forcing function on a linear system, whether the effect of nonlinear terms is to make $\dot{V}\,(x)$ more negative or not will depend on the sign of $\dot{V}_n(x) = G(t)\,H$. H is a linear function of the coordinates; hence $H = 0$ is a plane through the origin partitioning the phase space into two subspaces. The effect of nonlinearity can then be studied in the light of this criterion.

Among all the states in the error coordinates, the necessary condition that one state is better than the other is that the distance of this state from the origin is smaller than those of the other states. Therefore, V(x) can be used as an ordering relation to define a preference among all states. One can also monitor the state of the system or simply the sign of H in the expression of $\dot{V}_n(x)$ by direct measurements, and then adjust the system properly to really improve the performance. This is analogous to control systems with system-parameter adaptation, whereby the parameter are adjusted in accordance with input-signal characteristics or measurements of the system variables. No actual design of specific systems is attempted in this paper. However it is clear that Lyapunov method is not merely an abstract tool for studying the stability of dynamical systems; it is also a concrete one to facilitate the study and design of a general control system on the basis of performance.

Nonlinear Control Systems with Random Inputs.

There is still a lack in the literature on the application of Lyapunov's method to nonlinear control systems with random inputs. Research in this direction has much promise for the development of a satisfactory theory concerning both the analysis and synthesis of such systems. Consider the simple second-order nonlinear systems described by equation (2) and assume random behavior for input r(t). Equation (2) can be written

$$\ddot{e} + F(e, \dot{e}) = \ddot{r} + k\dot{r} + l r \tag{21}$$

where the single and double dots refer to first and second time derivatives respectively. By one of various methods of construction, 11, the phase-phane trajectories can be drawn for the case of zero input

$$\ddot{e} + F(e, \dot{e}) = 0 \tag{22}$$

by writing in the form

$$\frac{d\dot{e}}{de} = \frac{-F(e, \dot{e})}{\dot{e}} \tag{23}$$

The finite input r(t) will increase the slope of the trajectory by

$$-\frac{\ddot{r} + k\dot{r} + lr}{\dot{e}}$$

or

$$\frac{d\dot{e}}{de} = \frac{-F(e,\dot{e})}{\dot{e}} + \frac{\ddot{r} + k\dot{r} + lr}{\dot{e}} \tag{24}$$

If the system starts at the point P_o ($\dot{e}_o, e_o$) corresponding to the input conditions $\ddot{r}_o, \dot{r}_o, r_o$, this point will move in the direction P_oP_1 instead of along the trajectories of equation (23). After a short time interval δt, P_1 will travel along another path that makes an angle with the previous path dependent upon the new values of $\ddot{r}, \dot{r}, r$. This continuous variation of the slope from that of the trajectories drawn for r(t) = 0 results in a drunkard's walk about one of the force-free trajectories, see Fig. 2.

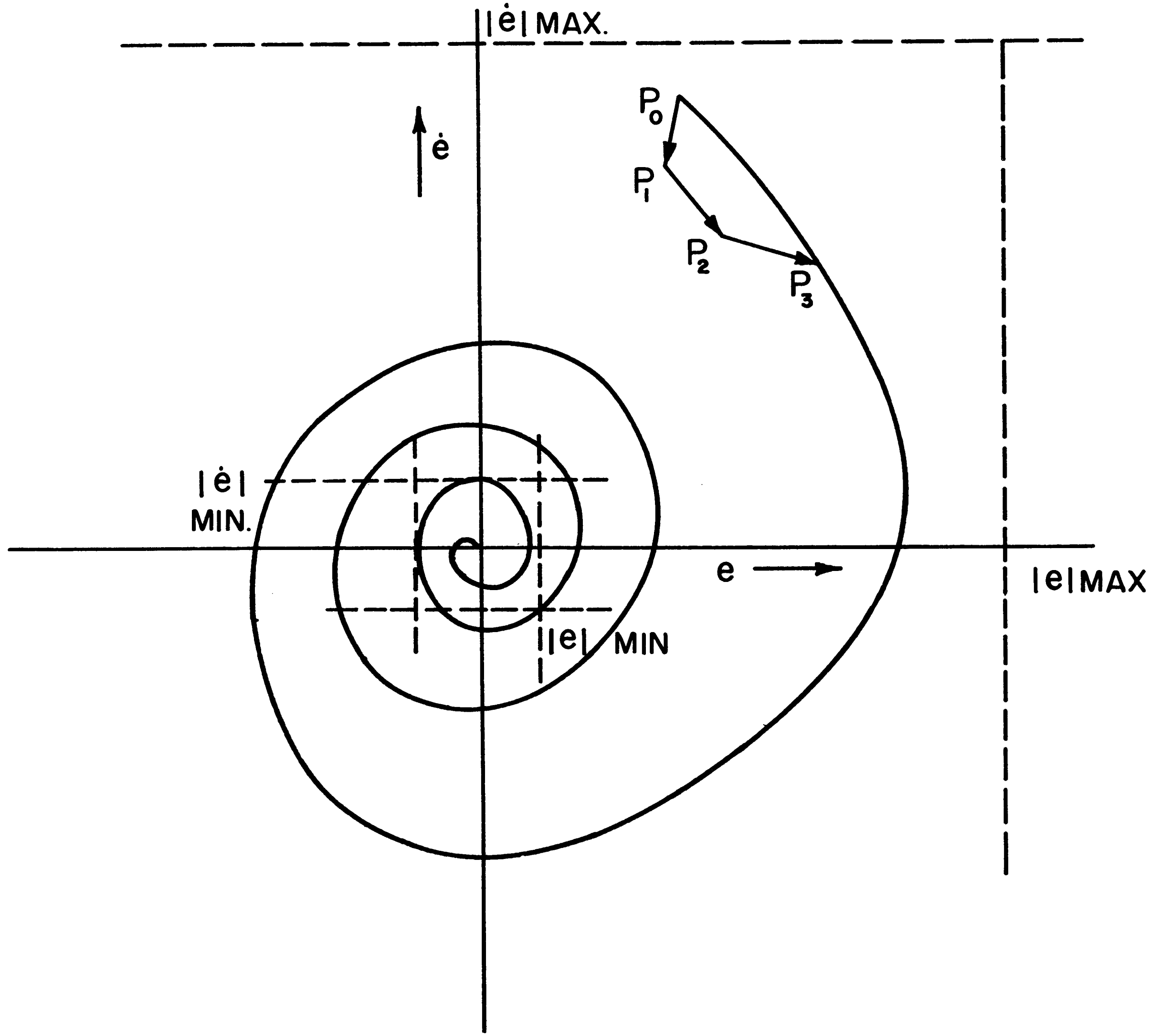

Figure 2. Error trajectory of system with random inputs.

In mathematical statistics, the variates are mostly assumed to be unbounded, and given a sufficiently long time infinite values will be obtained. In actual physical control systems this is not the case, as both the input and output and their derivatives, and therefore the error function and all its derivatives, are bounded. If the control system is a useful one and stable, the actual trajectory in spite of its flucuations will tend to the origin after a sufficient long time. The upper and lower limits of r(t) then set a definite limit to the magnitude of excursions of the drumkard's walk from one of the force-free trajectories. As a result, it may be possible to put down limits on $|e|_{max}$ and $|\dot{e}|_{max}$ as shown in Fig. 2. In general, some small value e_{min} may exist below which no matter how e varies, $\dot{e}$ can be just so large to force the system into alignment so that e does not exceed $|e|_{min}$. It is evident that the above discussions can be extended to systems of high order. One may infer, therefore, that it is almost certain the space trajectory of error for zero input can be used to investigate the stability of nonlinear control systems with random inputs, provided the bounds of both input and output can be determined. In this sense, Lyapunov's approach to system stability and performance as presented in this paper would be useful to tackle problems of this nature.

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PRINCIPAL DEFINITIONS OF STABILITY

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INTRODUCTION

To those familiar with linear control system design the concept of stability seems so obvious as to need no more than a casual definition. This statement is not quite the same in the field of rigid body dynamics. It is well known that a rigid body has two stable axes and one unstable axis. A few experiments at throwing a book in the air will verify this. Obviously one axis is unstable, for the book will not rotate about it. But what does the instability mean? Is it the same as for an unstable control system? In a sense it is, but a detailed discussion of the meaning of stability is required to see the connection.

The original ideas on the stability of dynamic systems were advanced by Poincare [1] and Lyapunov [2]. These were extensions of the concepts of stability for static equilibrium under small disturbances. For many years they were deemed adequate for classical dynamics. In the development of linear feedback theory this early work was not used. Stability was based on the exponential decay of solutions of linear differential equations. Finally, with the revival of the Lyapunov method of analysis and its application to control theory, the general formulation of stability and its correlation among various disciplines was attempted.

Definitions of stability that are suitable for automatic control may be stated in many ways, not all of which are equivalent. The concept is somewhat arbitrary depending on the particular requirements for the system. The definitions given here are those found useful by the author in applying the Lyapunov method to nonlinear control problems. Certain aspects that are more of a mathematical than an engineering nature are neglected. References 3, 4 and 5 contain discussions of various other definitions used in conjunction with the Lyapunov method.

DESCRIPTION OF PHYSICAL SYSTEMS

From an engineering standpoint the problem of stability begins with a physical system which is capable of changing its state, in some sense of the word, from time to time. This eliminates many of the questions of mathematical concern, such as existence and uniqueness of solutions, escape times etc. Specifically this discussion is concerned with a system of the type shown in Figure 1. It is a plant -- a dynamical system or otherwise -- whose outputs are to be regulated by means of a set of inputs which are compared to certain output variables. The differences are operated upon by a controller which in turn supplies signals to the plant. In addition, both the plant and controller may be subjected to the influence of various uncontrolled or free inputs. These represent such things as temperature variations, component aging or other parametric excitation. This is essentially a conventional control system.

There is an equation, either differential or difference, which describes the nature of the changes that the variables of such a system undergo. One description might be the normal set of n first order differential equations;

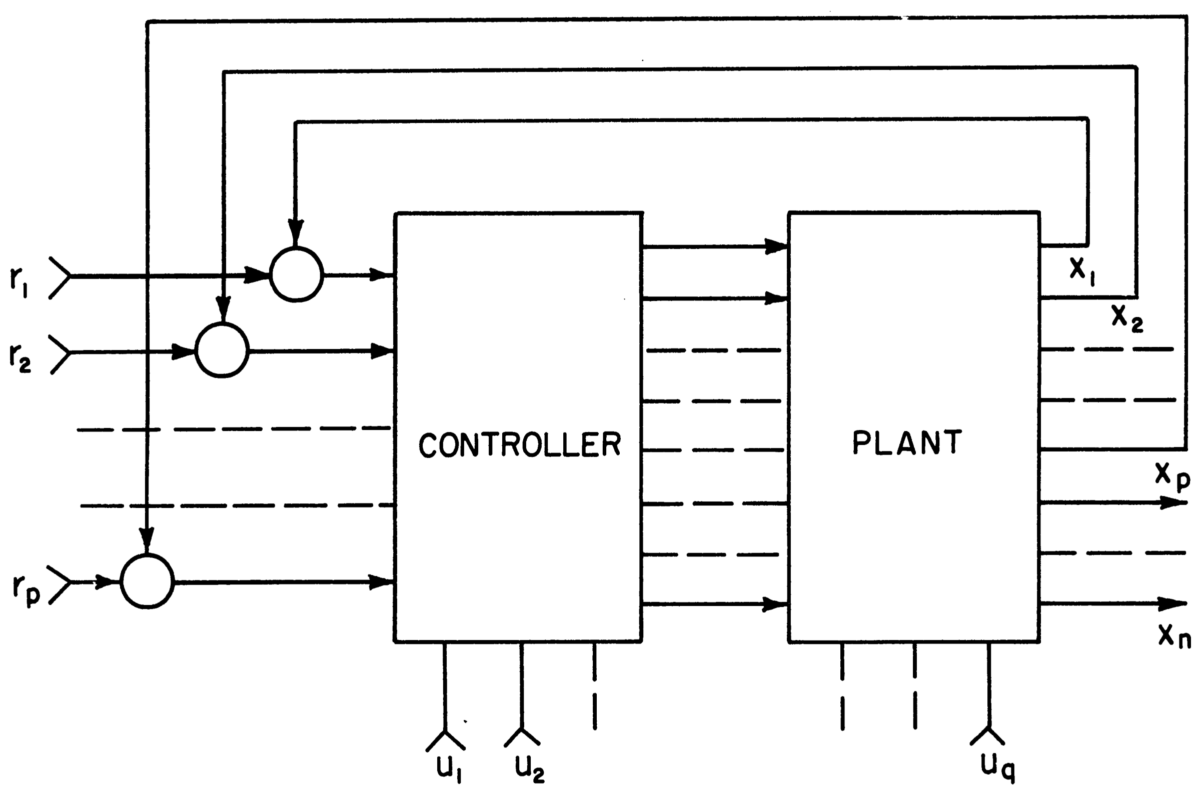

Figure 1. General control system

$$\frac{dx_i}{dt} = f_i(x_j, r_k(t), u_s(t)) \qquad \begin{aligned} i &= 1,2,\ldots\ldots,n \\ j &= 1,2,\ldots\ldots,n \\ k &= 1,2,\ldots\ldots,p \\ s &= 1,2,\ldots\ldots,q \end{aligned} \tag{1}$$

These will be used as a specific reference. The explicit dependence of the equations on time is broken into two parts, the reference inputs $r_k(t)$ and the free inputs $u_s(t)$. The x_j represent the magnitudes of n different variables which completely specify the state of the plant and controller. They are called the state variables or, from a geometrical viewpoint, they form the components of the state vector.

Any condition where all of the state variables are constant is called an equilibrium position. With regard to Equations (1) this is characterized by all of the functions f_i being equal to zero. The dependence on time of a physical system may be divided into those factors which influence the equilibrium position and those which do not. In the set of reference equations it is assumed that the $r_k(t)$ affect the equilibrium position while the $u_s(t)$ do not. An equilibrium position can exist only when the r_k are constant.

The equation describing the motion of a system has a solution defined as any set of functions $\phi_i(t)$ which, if they are substituted for the state variables x_i, satisfy the equation. Solutions are dependent on time in the manner prescribed by the equation of motion and also upon an initial state $x_i(t_0)$ and the time t_0 when this state occurs. Thus the notation $\phi_i(t)$ is understood to imply a set of functions which have the value $x_i(t_0)$ at a specified time t_0; $\phi_i(t) = \phi_i(t_0, x_j(t_0); t)$.

To interpret the implications associated with general concepts of stability the idea of an n dimensional state space is used. Each of the state variables is imagined to represent a length along an axis in this hyperspace. If the variables are functionally independent no three axes lie in a plane. Under special circumstances this space reduces to the conventional phase space, in which case each succeeding axis represents the rate of change of the quanity measured along the one preceding it.

Any point in state space is represented by a vector having the various state variables as components. The totality of all points in the space represents all possible states which the system may assume. With the passage of time the state vector traces a curve in the space, known as a system trajectory. Figure 2 is a possible trajectory in three dimensional space. It may be thought of as the projection of a solution of the equations, plotted in a space containing the n state axes and a time axis, onto the state space. Such a space and the projection is shown in Figure 3.

STABILITY CONCEPTS AND DEFINITIONS

Solutions completely specify the motion of a system. When these are known any more general properties of the motion are determined. However, if the solutions are divided into general classes certain properties of each class may be found without knowledge of the solutions. In some cases this is a simpler task than the

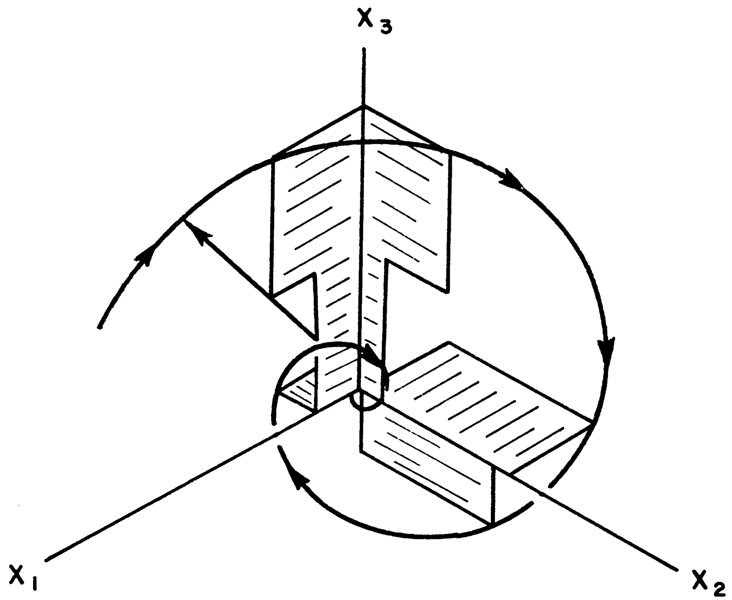

Figure 2. System trajectory

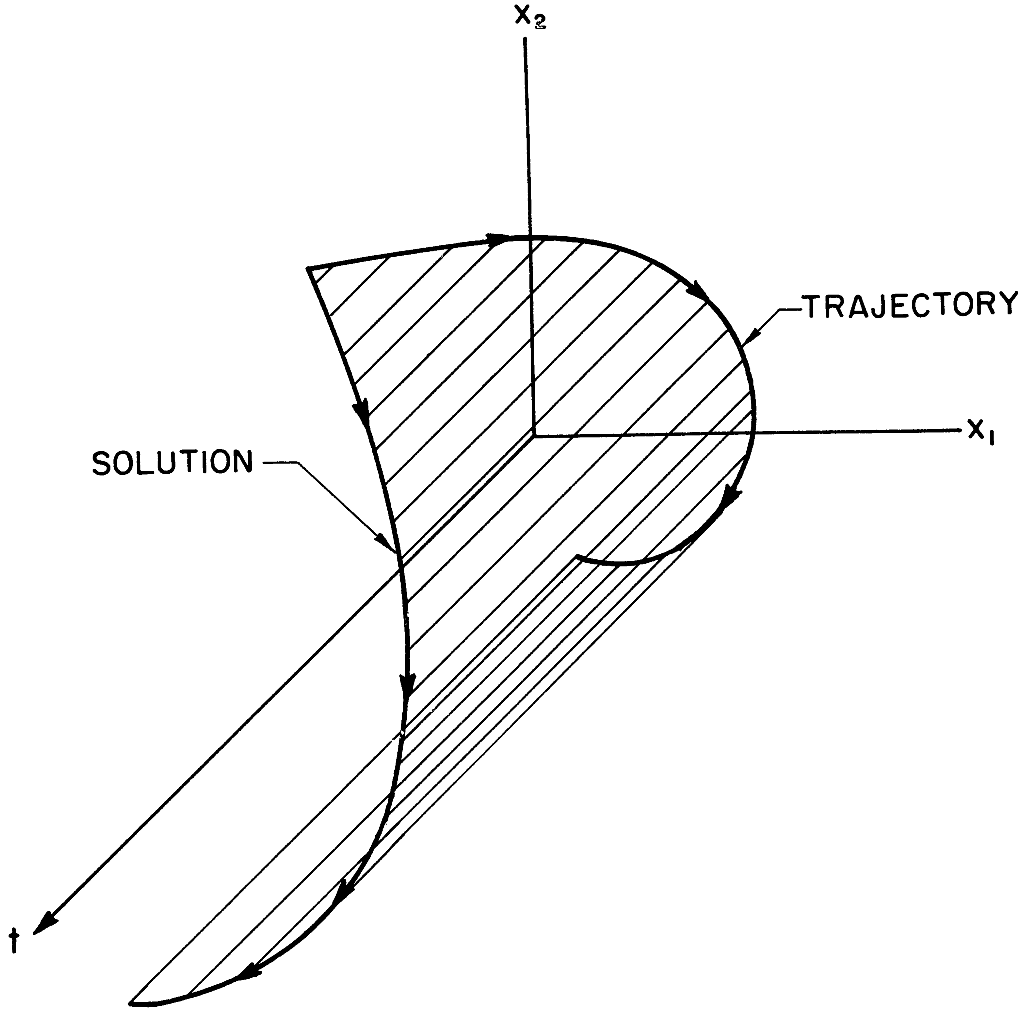

Figure 3. Projection of a solution

determination of solutions. One useful division is into the classes "stable" and "unstable."

The primary purpose of the Lyapunov method is to furnish a criterion by which an investigator can decide into which of these classes the solutions of a particular system fall. To make the decision a defintion of stability is required. The definition must supply both necessary and sufficient conditions for a system to have stable solutions. On the other hand a criterion for stability, based on the definition, may provide only a sufficient condition, as is the case in most applications of the Lyapunov method.

The concept of stability of a dynamic system is basically the question of whether or not it will return to a particular state after it has been disturbed in some way. Actually some state, either stationary or dynamic, is always stable and the question is as to the stability of a specified state. Various definitions of stability are available depending on the nature of the state and the manner in which the system approaches or deviates from it.

Stability in the Sense of Lyapunov

The mathematical definition of the stability of Equation (1) as stated by Lyapunov is as follows. Let $\phi_i(t)$ be a solution of the equation and define a new set of variables; $q_i = x_i - \phi_i$. If these are substituted into (1) a new set of equations in q_i results which has an equilibrium position at $q_i = 0$. This is true whether the original set of equations possess an equilibrium position or not.

The equilibrium position is called stable in the sense of Lyapunov if for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that, $|q(t)| < \epsilon$ whenever $|\underline{q}(t_0)| < \delta$ for all $t > t_0$. In other words, the equilibrium position, $q_i = 0$, is stable if the magnitude of the new state vector $\underline{q}$ can be made to remain permanently below an arbitrary upper bound by choosing its initial magnitude sufficiently small. If $\underline{q}$ approaches zero as t approaches infinity the equilibrium position is called asymptotically stable.

Extending this to (1) the solution $\phi_i(t)$ is called stable in this sense if, $|x_i(t)-\phi_i(t)| < \epsilon$ whenever $|x_i(t_0)-\phi_i(t_0)| < \delta$ for $t > t_0$. Thus if one solution remains arbitrarily close to another when their initial values are sufficiently close together that solution is called stable.

A brief consideration of the aspects of control system design makes it evident that these definitions are not completely satisfactory. First, the question of what happens when the initial disturbance cannot be made sufficiently small arises. For instance, the state of a stable system, by this definition, can go into a limit cycle or increase without limit if the initial disturbance is above a specified bound. Second, the fact that one solution remains close to another is of little value if both deviate greatly from the desired solution. Fortunately in the case of linear systems the definitions above are coincident with broader aspects of stability.

More General Definitions of Stability

A more general set of stability classifications is defined in the following discussion. The basic definition is concerned with the stability of an equilibrium position. Let the system considered be one that has an equilibrium position; for example, in the set of Equations (1) all of the forcing functions $r_k(t)$ are constant. Let $\underline{x}_e$ be the vector representing the equilibrium position and let R be some region in state space, bounded by a hypersurface and containing the point $\underline{x}_e$. It is desired to classify solutions for which the initial state vector represents a point in R at a time t_0 as stable or unstable.

A state vector at some point in R at t_0 will be at some other point in state space at a later time t_1. Another state vector initially in R will be at another point at t_1. In terms of two such vectors an analytic definition of stability is;

Definition: 1) An equilibrium position $\underline{x}_e$ is <u>stable</u> in a region R at a time t_0 if for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $|\underline{\phi}_1(t_0)-\underline{\phi}_2(t_0)| < \delta$ implies $|\underline{\phi}_1(t)-\underline{\phi}_2(t)| < \epsilon$ for all $\underline{\phi}_1(t_0)$ and $\underline{\phi}_2(t_0)$ in R and all $t > t_0$.

This definition is illustrated in Figure 4. All trajectories originating in a sphere of radius δ arrive at some other set of points at t_1. These lie within a sphere of radius ϵ. Regardless of the size of ϵ it must be possible to choose δ small enough so that the trajectories remain within it for all time and this must be true for all possible states originating in R. These requirements are very weak as far as the stability of control systems is concerned. They effectively eliminate the three unstable phenomens, unbounded variations, other equilibrium states, and limit cycles.

By this definition no state originally in R can become infinite at infinite time for if it were possible two adjacent initial

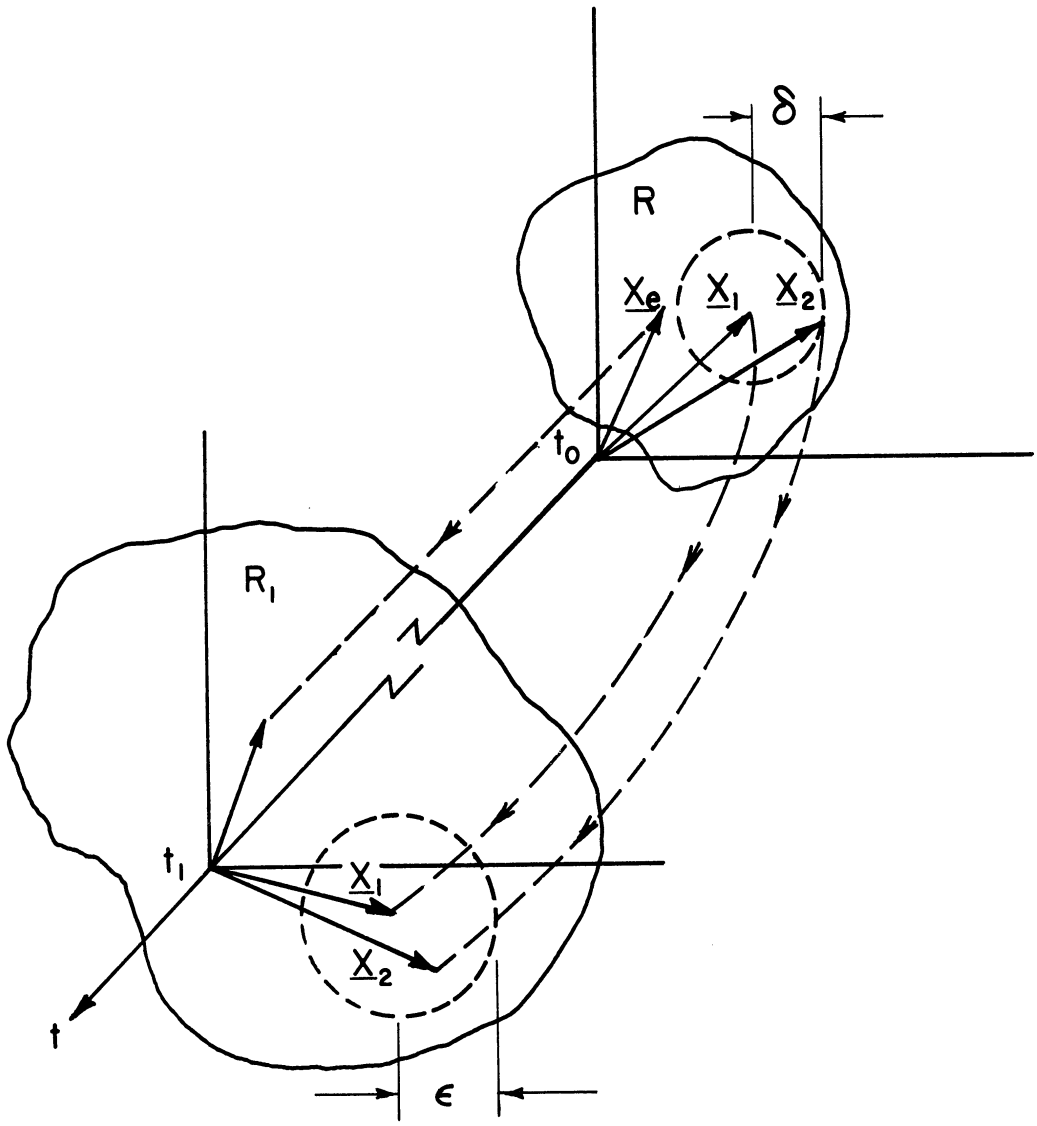

Figure 4. Stability of solutions

states could exist, one of which remains finite while the other becomes infinite. At least one state, $\underline{x}_e$, remains finite. If a finite value of ϵ is chosen it is impossible to choose a δ small enough to keep the difference between these two states less than ϵ after a sufficiently long time.

No trajectories initiating in R can go to other equilibrium positions, if they exist, for if this were possible two adjacent initial states could exist, one of which would approach another equilibrium state while the other would not. Again at least one trajectory exists which remains at a specified position. Then for a sufficiently small value of ϵ it is impossible to choose a δ small enough that the variation between the states will not exceed this value. A similar argument proves that if any trajectories initiate in R and go to a limit cycle the requirements of the definition are violated.

It is noted that under this definition a stable system does not necessarily return to the equilibrium position. A typical example is a second order conservative system such as might be represented on an analog computer by two integrators each of whose outputs feeds the input of the other. The poles of the transfer function are purely imaginary and any disturbance results in a continued oscillation of fixed amplitude. Unlike a limit cycle, however, the amplitude is dependent on the initial disturbance.

A stronger kind of stability is given by the following statement.

Definition: 2) An equilibrium position $\underline{x}_e$ is asymptotically stable in a region R at a time t_0 if it is stable and if $\underline{\phi}(t) \rightarrow \underline{x}_e$ as $t \rightarrow \infty$ for all $\underline{\phi}(t_0)$ in R.

In applying the Lyapunov method it is customary to distinguish stability for cases where (2) is not satisfied by the statement;

Definition: 3) An equilibrium position $\underline{x}_e$ is neutrally stable in a region R at a time t_0 if it is stable and not asymptotically stable.

Geometrically, asymptotic stability means that the region R, which becomes a region R_1 at time t_1 as shown in Figure 4, eventually becomes arbitrarily small. In other words, no matter what initial disturbance is given to the system within the limits specified at t_0 the system will return to its equilibrium position. As far as control systems are concerned this behavior is usually necessary to achieve their objectives.

Neutral stability is primarily of interest in these applications for interpretation of analytical results. It represents the boundary between stability and instability. However, in other applications, such as the motion of rigid bodies or planets in orbit, it is the only kind of stability possible.

So far the definitions have been concerned with the behavior of solutions after a long time has elapsed since the initial disturbance. Trajectories on the boundary of R at t_0 are on the boundary of R_1 at t_1. Thus a system which is asymptotically stable in R at t_0 is also asymptotically stable in R_1 for all t_1 after t_0. However, R_1 can become considerably larger than R before the trajectories begin to converge.

In effect an asymptotically stable system can behave as an unstable one for an indefinite length of time. In practical problems not only the asymptotic character of the solutions but also the manner in which they exhibit this behavior is of importance.

A convenient way to state a definition of stability having more desirable properties for automatic control is to use a construction similar to the formulation of the second method itself. Thus the Lyapunov method is especially suited for providing the desired information. It will be called monotonic stability because of the similarity to a monotonic sequence.

Imagine the space inside of the region R_1, derived from R as above, to be divided into a continuous nest of non-intersecting hypersurfaces inclosing the point $\underline{x}_e$. The innermost surface reduces to a point at $\underline{x}_e$. This is shown for two dimensions in Figure 5. Designate one of the surfaces by S_1 and another inside of it by S_2.

> Definition: 4) The equilibrium position is <u>monotonically stable</u> in R at t_0 if it is possible to choose a set of surfaces in R_1 such that, if $\underline{\phi}(t_1)$ is a state vector representing a point on any surface S_1 at t_1, $\underline{\phi}(t_2)$ represents a point on some surface S_2 inside of S_1 for all $t_2 > t_1$ and all $t_1 > t_0$.

This definition is concerned with the behavior of the system at all times rather than only after a long time since t_0. It necessarily implies asymptotic stability, since if the solutions go inside of every surface they must eventually converge to $\underline{x}_e$. It is a stronger requirement than the "exponential stability" defined in reference 4; for even

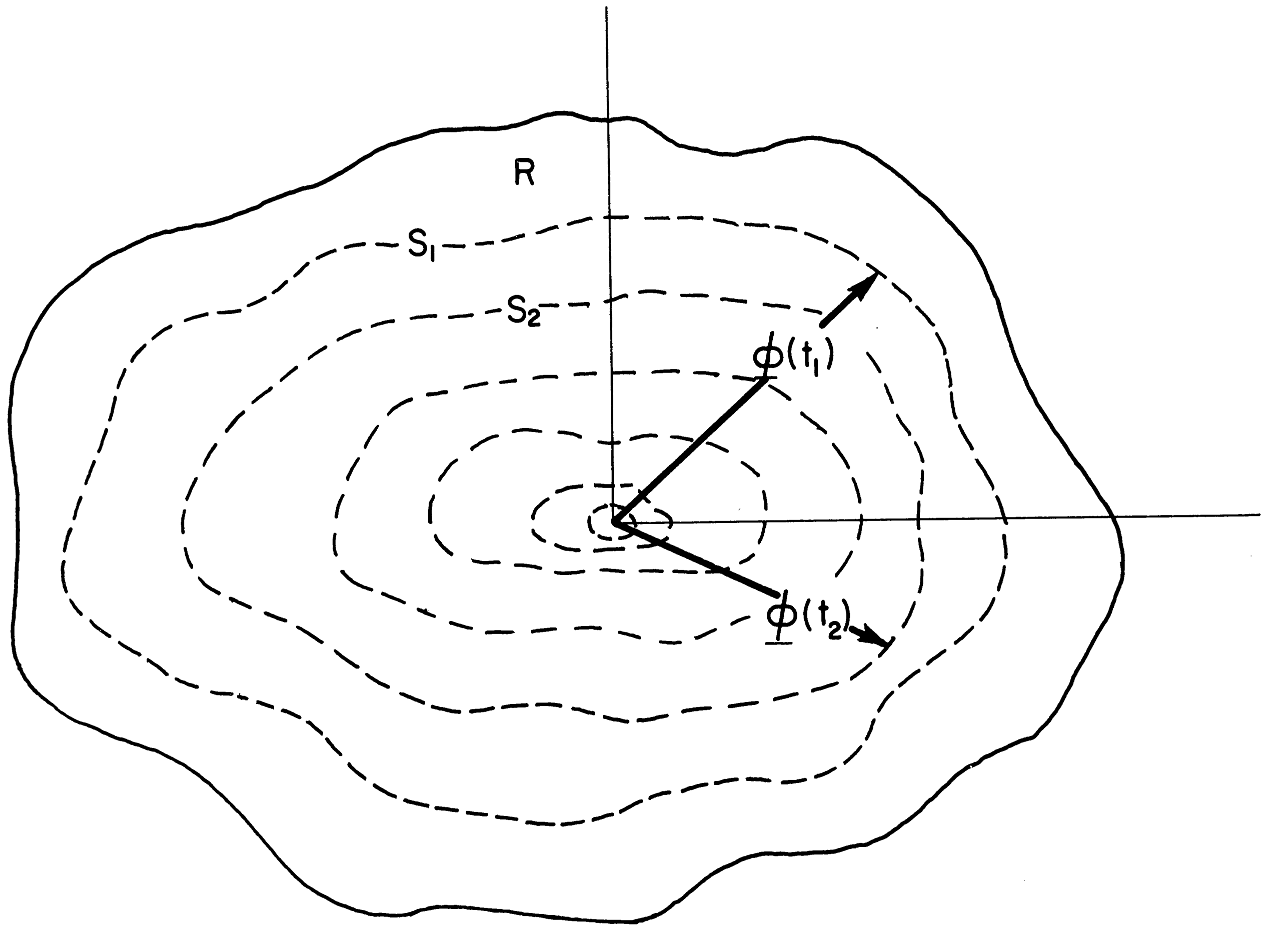

Figure 5. Monotonic stability

though the solutions remain bounded by an exponential function they may increase at times, whereas by this definition they must always tend toward equilibrium. It is a property of autonomous systems that this is the only kind of asymptotic stability possible, but it is not necessarily restricted to autonomous systems.

Stability As a Function of Initial Conditions

Definitions 1 to 4 are concerned with stability for a given set of initial conditions, namely for an initial state in a region R at a time t_0. These are usually dependent upon the choice of R and t_0. It may be that the system is such that the definitions hold irrespective of the time t_0.

> Definition: 5) The equilibrium position is called respectively uniformly stable, uniformly asymptotically stable, uniformly neutrally stable, or uniformly monotonically stable if the choice of the region R in definitions 1, 2, 3, or 4 is independent of the initial time t_0.

Specifically the requirements of (5) are always satisfied for autonomous systems. For non-autonomous systems they may be fulfilled for certain regions and not for others. Uniform stability is a desirable feature for control systems. For example, if it is known that a control is unstable when the initial disturbance exceeds a specified bound a limit stop may be added below the bound to prevent this occurrence. However, if the stability is not uniform it may be difficult to specify the bound or design the limit stop.

So far nothing has been said about the size of the region R. Sometimes it may be chosen as small as is necessary to satisfy the particular requirements. When this is all that is required the investigation of stability is a comparatively simple task. Usually a linearization about the equilibrium position is sufficient.

Definition: 6) The equilibrium position is called respectively A-stable, asymptotically A-stable, neutrally A-stable or monotonically A-stable at t_0 if definitions 1, 2, 3, or 4 are fulfilled for initial states within an arbitrarily small region A about the equilibrium position.

This condition is referred to as "stability in the small" in Russian literature. The "stability in the sense of Lyapunov" defined previously is of this kind. It has considerable mathematical interest but is of little use for control system design since the possibility of rather large disturbances always exists.

A more practical condition is given by;

Definition: 7) The equilibrium position is called respectively B-stable, asymptotically B-stable, neutrally B-stable, or monotonically B-stable at t_0 if definitions 1, 2, 3, or 4 are fulfilled for initial states within a specified and finite region B containing the equilibrium position.

This is usually called "stability in the large" by Russian authors. The major problem in nonlinear systems where a finite region of stability exists is in determining the largest boundary within which the solutions tend toward equilibrium. This is usually very difficult but if a boundary is determined which is so large that no disturbance is likely to exceed it the problem is solved for practical purposes. The Lyapunov method gives such information more or less precisely.

A special case of the above occurs when the stability requirements are satisfied for all possible initial states.

> Definition: 8) The equilibrium position is called respectively <u>B_∞-stable</u>, <u>asymptotically B_∞-stable</u>, <u>neutrally B_∞-stable</u>, or <u>monotonically B_∞-stable</u> at t_0 if definitions 1, 2, 3, or 4 are fulfilled for all possible initial states.

This clearly implies B-stability as B-stability implies A-stability. It is alternatively called "stability in the whole" and "stability in the large" in various translations of Russian papers. This is often considered to be the only kind of stability of interest for control systems. With anything less there is always the risk of some random disturbance putting the system into an unwanted mode of operation. Of course in linear systems the nature of the stability is always independent of the magnitude of the initial disturbance. For these there is no distinction between A and B or B_∞-stability.

The process of classification of solutions could be carried on to considerably greater length but the definitions given so far serve to outline the principal distinctions among the ways solutions can behave. The strongest classification is uniform-monotonic-B_∞ stability while the weakest is A-stability. The rest fall at intervals between these two. Finally, instability is defined by the statement;

Definition: 9) If the equilibrium position is not stable it is unstable.

Figure 6 is a flow diagram illustrating the various classifications. The restrictions on the system become progressively stronger in going from left to right on the diagram or from the bottom to the top. The implications of each class can be noted by following the flow back to the solution. Thus B-stability implies A-stability but neutral B-stability does not imply asymptotic A-stability nor does monotonic A-stability necessarily follow from asymptotic B-stability.

Stability of Control Systems with Forcing Functions

When the system is subjected to dynamic inputs which affect the equilibrium position, as when the $r_k(t)$ in Equations (1) are not constant, the stability of an equilibrium position has no meaning. Then it is customary, in mathematical discussions, to follow the method of Lyapunov described previously. Some solution is called the "undisturbed motion" and all other solutions, called "disturbed motion", are classed according to whether or not they converge toward this one.

The procedure followed here is similar in meaning but it is stated differently. A control system is designed to follow a particular set of inputs. The information that is desired is the stability for these inputs. For this purpose the description "reference inputs" is understood to include all functions of time which influence an equilibrium position whether they are specifically intended as references or not.

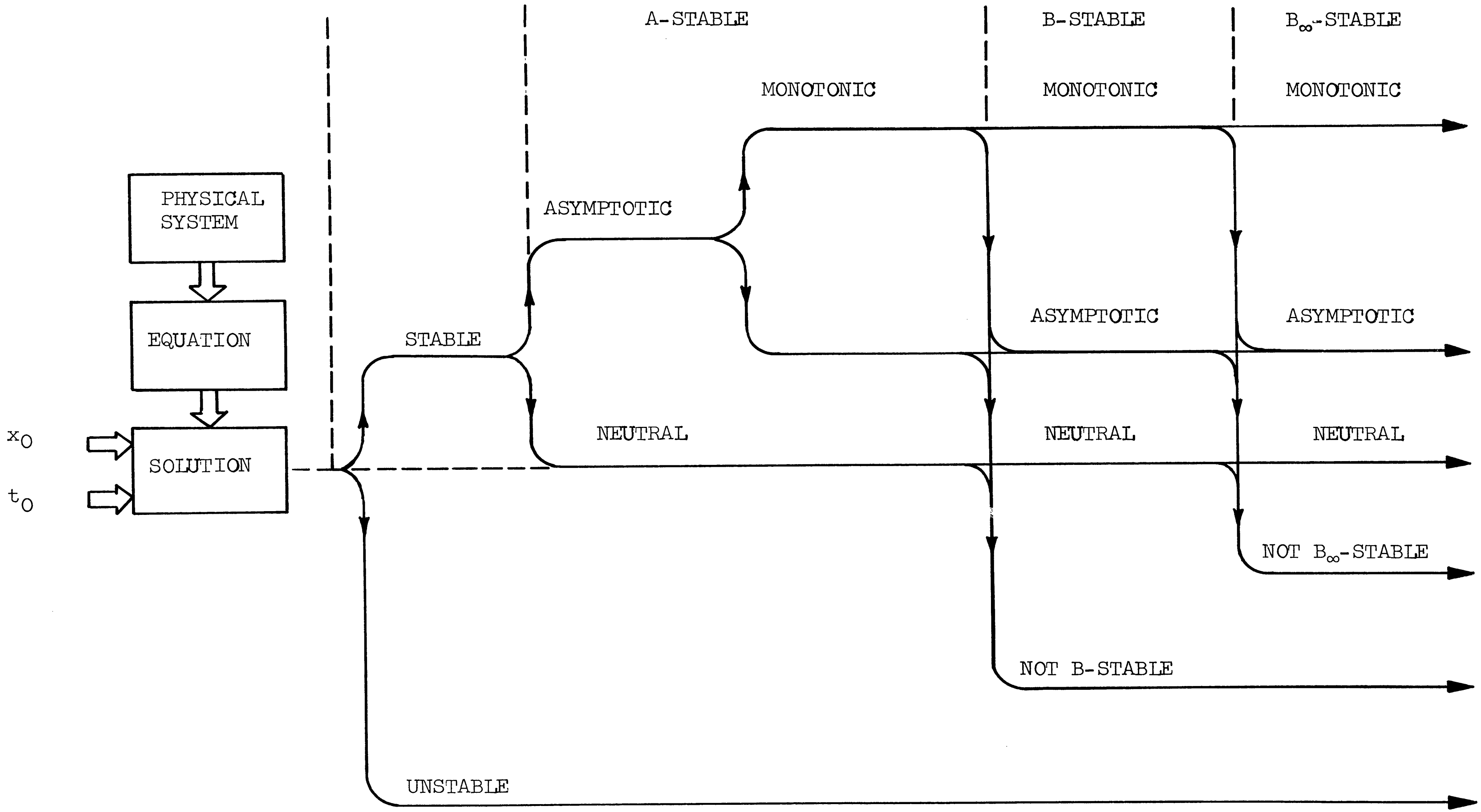

Figure 6. Stability Classifications

When the reference inputs are identically zero the system has some equilibrium position which may be classed as stable or unstable. Similarly when they have a constant set of values there is another equilibrium position which may be considered. For physical systems this position depends continuously on the set of values assumed by the inputs. Thus at any time there can be said to be an equilibrium position corresponding to the instantaneous values of the references.

A time varying set of references may be considered as an effective movement of an equilibrium state from point to point in state space. If they vary slowly enough the state of the system will follow a stable position. Of course the references may affect other aspects of the motion as well as the equilibrium state.

> Definition: 10) Let $\underline{x}_e(t)$ be the equilibrium position corresponding to a set of reference inputs $r_k(t)$ and let $\underline{\phi}(t_0) = \underline{x}_e(t_0)$. A system is <u>stable at t_0 with respect to this set of inputs</u> if there exists a $T > t_0$ such that $\underline{\phi}(t_1)$ falls inside of a region for which the equilibrium position corresponding to $r_k(t_1)$ is stable by definition 1 for all $t_1 > T$.

In effect this definition states that the motion is stable if there is some time after which the references may be held constant and the ensuing motion is stable. It differes from the usual one in that no initial disturbance is specified. This is taken into account in specifying the reference themselves and also the possibility of continuously acting disturbances is admitted. The solution may temporarily go into an unstable region provided that the inputs are such that it returns to a stable region and remains there. This is somewhat risky when classes of inputs rather than specific inputs are considered.

To avoid such behavior a stronger restriction is imposed.

Definition: 11) A system is called <u>continuously stable with respect to a set of inputs</u> if definition 10 is satisfied for all $t_1 > t_0$.

Here the inputs may be fixed at any time after t_0 and the resulting motion must be stable. Both definitions (10) and (11) may be extended to include asymptotic stability, uniform stability, etc., by inserting the appropriate one of definitions (2) to (8) instead of defintion (1). Thus a set of classifications for systems with forcing function inputs is defined. Many more refined distinctions exist which could be used as a basis for further subdivision but these describe the major courses that a state of motion may follow.

CONCLUSIONS

Various classifications of stability for equilibrium and dynamic states have been discussed. It is emphasized that these are not unique. The variation from the strongest kind of stability to instability is a continuous process which may be graduated by a variety of demarcations. For most applications in automatic control only the most restrictive categories are of value; however, there are circumstances where weaker classifications may be used.

In general it can be said that the strongest kinds of stability are also the easiest to investigate. For example, a Lyapunov function for a neutrally stable system is unique, whereas a variety are possible for a monotonically stable one. The investigation of anything but $B_{\overline{\infty}}$-stability for dynamic states of a nonlinear system is particularly difficult since a set of solutions must be known or at least approximated by suitable upper bounds.

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THE DIRECT METHOD OF LYAPUNOV IN THE ANALYSIS AND DESIGN OF DISCRETE-TIME CONTROL SYSTEMS

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THE DIRECT METHOD OF LYAPUNOV IN THE ANALYSIS AND DESIGN OF DISCRETE-TIME CONTROL SYSTEMS

1. Introduction:

Lyapunov's Direct Method often called the Second Method is one of the most general methods known for the study of the stability of equilibrium solutions of dynamic systems described by ordinary differential or difference equations. The present paper is restricted to the study of dynamic systems in which the time variable changes discretely, i.e. those governed by ordinary difference equations. With the advent of digital components, pulsed radar, and analytical instruments employing sampling techniques, discrete-time systems have become quite common to the control engineer. The objective of the paper is to present methods rather than the rigorous development of the mathematical structure. Good sources for the statements and proofs of the mathematical theorems underlying the method can be found in [1], [2], [3], and [4]. These references also contain extensive bibliographies.

2. Description of Discrete-Time Dynamic Systems:

A large class of discrete-time dynamic systems may be described by the vector difference equation

$$\underline{x}(t_{k+1}) = \underline{f}(\underline{x}(t_k), \underline{u}(t_k)) , \tag{2.1}$$

where the t_k (k an integer) indicate discrete values of time

$$-\infty < \dots, t_{k-1} < t_k < t_{k+1} \dots < \infty \quad ; \quad t_k \to \infty \text{ as } k \to \infty$$

at which the behavior of the system can be or is observed; t_k is regarded as an independent variable analogous to t in continuous-time systems. Equation (2.1) is equivalent to the set of n scaler difference equation

$$x_i(t_{k+1}) = f_i(x_1(t_k), \dots, x_n(t_k), u_1(t_k), \dots, u_m(t_k)), \tag{2.2}$$

$$i = 1, \dots, n$$

The vector $\underline{x}$ is the state of the system (2.1); its components x_i are the state variables. The vector $\underline{u}$ is the control input of the system; its components u_i are the control variables. The system is specified by the vector valued function $\underline{f}$. The integer n is the order of the system. Usually the number of control inputs, m, is less than the order of the system, n, $(m < n)$.

Example 2.1.

To illustrate how equations in the form of (2.1) are obtained from control problems consider the block diagram of the sampled-data system in Figure 1. It consists of the following:

(a) The plant and the feedback instruments, which are governed by ordinary, linear, time-invariant, differential equations. As is customary in the control system literature, the plant and the instruments are described by rational tranfer functions.

(b) The sample-and-hold element, which replaces the continuous error signal e(t) with a piecewise constant sampled signal $e^*(t)$ described by

$$e^*(t) = e(t_k) \ , \ t_k \leq t < t_{k+1} \qquad k = 0, 1, 2, \ldots \quad (2.3)$$

The sampling is periodic with period T.

(c) The amplifier which as shown has a gain of k for $| e | \leq e_1$ and saturates for $| e | > e_1$. The transfer characteristic of the amplifier is given by the function f_s so that

$$m(t) = f_s(e(t)) \qquad (2.4)$$

The first step in obtaining the equations (2.1) is to redraw the block diagram in such a way as to make the state variables accessible. This can be done by simulating the plant and the feedback instruments on an analog computer. One such simulation of the system is shown in the block diagram of figure 2. The state variables are the outputs of the two integrators and are labeled x_1 and x_2. At the sampling instant $t = t_{k+1}$ in terms of the state variables and the input at the sampling instant $t = t_k$ are given by

$$\begin{aligned} x_1(t_{k+1}) &= x_1(t_k) + T\, x_2(t_k) + T^2/2 \; m(t_k) \\ x_2(t_{k+2}) &= x_2(t_k) + T\, m(t_k) \end{aligned} \qquad (2.5)$$

where

$$m(t_k) = f_s(e(t_k)) \qquad (2.6)$$

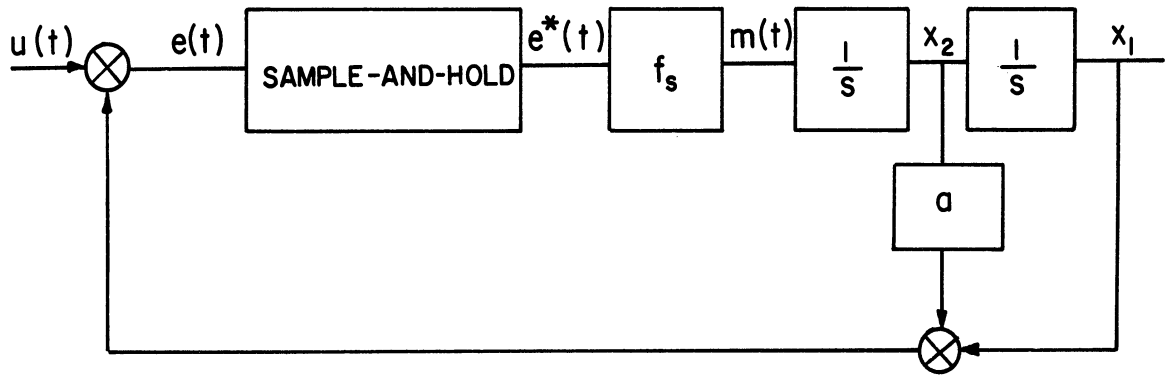

Figure 1. Sampled-data-system

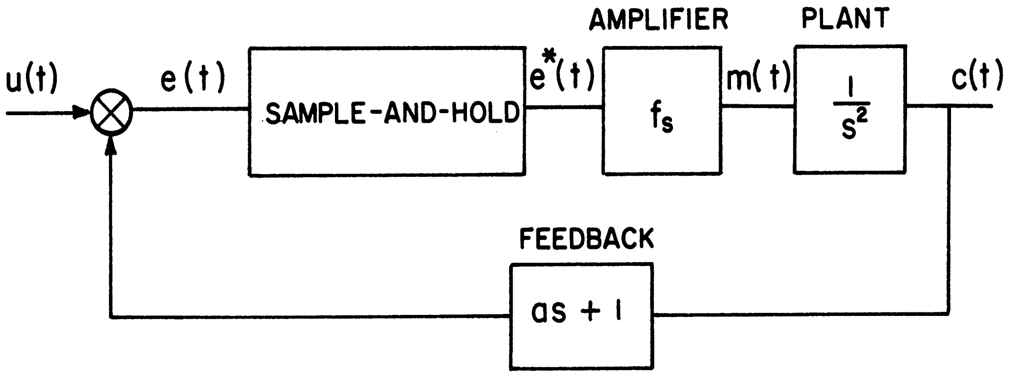

Figure 2. Simulation of sampled-data system

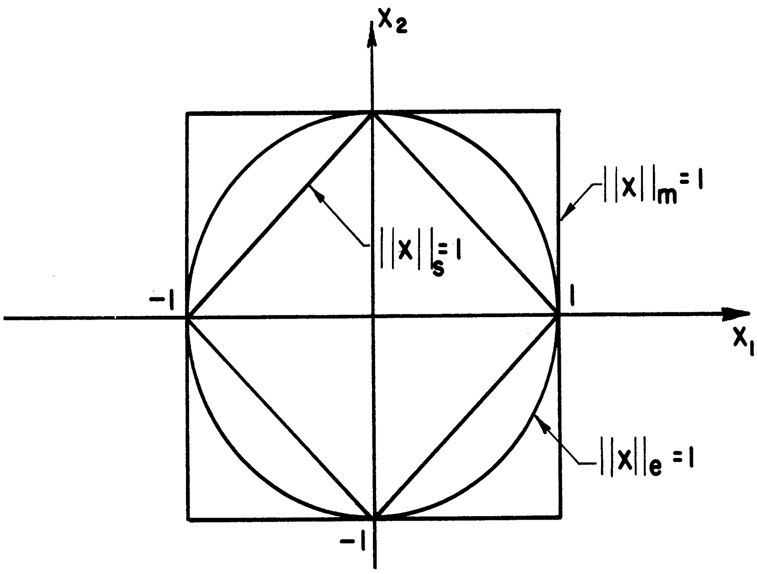

Figure 3. Examples of Norm

and

$$e(t_k) = u(t_k) - x_1(t_k) + a x_2(t_k) \tag{2.7}$$

Therefore substituting (2.6) and (2.7) in (2.5) the system is described by the equations

$$x_1(t_{k+1}) = x_1(t_k) + T x_2(t_k) + T^2/_2\ f_s(u(t_k) - x_1(t_k) + a x_2(t_k))$$

$$x_2(t_{k+1}) = \qquad x_2(t_k) + T f_s(u(t_k) - x_1(t_k) + a x_2(t_k)) \tag{2.8}$$

which are in the form of (2.1) and (2.2) where

$$f_1(x_1, x_2, u) = x_1 + T x_2 + T^2/_2\ f_s(u - x_1 + a x_2)$$

and

$$f_2(x_1, x_2, u) = x_2 + T\ f_s(u - x_1 + a x_2)$$

3. Concepts of Stability:

If the control input $\underline{u}(t_k) \equiv \underline{0}$ for all t_k, we say that the system (2.1) is unforced.

$$\underline{x}(t_{k+1}) = \underline{f}(\underline{x}(t_k)) \tag{3.1}$$

A state $\underline{x}_e$ of the unforced dynamic system (3.1) is an equilibrium state if

$$\underline{x}_e = \underline{f}(\underline{x}_e) \tag{3.2}$$

Thus if the unforced system (3.1) is started in the equilibrium state, $\underline{x}_e$, it remains in this state for all t_k. This is, of course, a mathematical statement, and the actual physical behavior raises the problem of stability. It is never physically possible to start the system exactly in its equilibrium state and, in addition, the system is always subject to outside forces which are not considered in the mathematical description. Thus the system is constantly being disturbed and displaced from its equilibrium state. Roughly speaking, if it remains near the equilibrium state we say that the system is stable. If the system remains near to the equilibrium state and, in addition, tends to return to equilibrium we say that the system is asymptotically stable.

Now to make these notions more precise let us assume that the equilibrium state being investigated is located at the origin: $\underline{x}_e = \underline{0}$. (This can always be accomplished by a translation of coordinates). Let $|| \underline{x} ||_e$ be the Euclidean length of the vector $\underline{x}$ ($|| \underline{x} ||_e = (x_1^2 + \ldots x_n^2)^{1/2}$). If S(R) denotes the spherical region of radius R about the origin, then S(R) consists of all points $\underline{x}$ such that $|| \underline{x} ||_e \leq R$.

The equilibrium state at the origin is said to be stable if corresponding to each S(R) there is a spherical region S(r) of smaller radius such that for an initial state $\underline{x}(t_o)$ starting in S(r) the solution $\underline{x}(t_k)$ does not leave S(R) for all $t_k \geq t_o$.

If, in addition, there exists a third spherical neighborhood of the origin $S(R_o)$ such that every solution starting in $S(R_o)$ approaches the origin as $t_k \to \infty$, the system is said to be asymptotically stable.

Stability and asymptotic stability are local concepts. In practice some knowledge as to size of the rgion in which stable behavior is to be expected is required. In many control system applications it is important to assure that no matter how large the perturbation, the system tends to return to its equilibrium state. This is asymptotic stability in the large.

4. Principal Result:

If within the neighborhood of the equilibrium state the total energy of the unforced system is always decreasing, we should expect that the equilibrium state is asymptotically stable. Lyapunov's direct method generalizes this idea. Suppose that within some neighborhood S(R) of the origin it is possible to construct a scaler function $V(\underline{x})$, continuous in $\underline{x}$, and such that $V(\underline{0}) = 0$ and $V(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$. Define, with reference to the unforced system (3.1)

$$\Delta V(\underline{x}(t_k)) = V(\underline{x}(t_{k+1})) - V(\underline{x}(t_k)) \tag{4.1}$$

Now, if $\underline{x}(t_k)$ is the solution of (3.1), the change of the Lyapunov function $V(\underline{x})$ along the solution sequence $\{ \underline{x}(t_k) \}$ is

$$\Delta V(\underline{x}(t_k)) = V(\underline{f}(\underline{x}(t_k))) - V(\underline{x}(t_k)) \tag{4.2}$$

which is obtained without any knowledge of the solutions but directly from the structure of the difference equations. If such a Lyapunov function $V(\underline{x})$ can be constructed in a neighborhood S(R) of the equilibrium state and if in that neighborhood $V(\underline{x}) > 0$ for $\underline{x} \neq 0$, then the equilibrium state is asymptotically stable if $\Delta V(\underline{x}) < 0$. This is one form of Lyapunov's Asymptotic Stability Theorem for difference equations. The proof of this theorem is quite simple and can be found in [3].

A few additional conditions are necessary to assure asymptotic stability in the large. Because of its many applications in the control field I shall state it as a theorem.

Theorem 1.

Consider the unforced, discrete-time, dynamic system

$$\underline{x}(t_{k+1}) = \underline{f}(\underline{x}(t_k))$$

where $\underline{f}(\underline{0}) = \underline{0}$. Suppose there exists a scaler function $V(\underline{x})$ such that $V(\underline{0}) = 0$ and

(i) $V(\underline{x}) > 0$ when $\underline{x} \neq \underline{0}$;

(ii) $\Delta V(\underline{x}) < 0$ when $\underline{x} \neq 0$;

(iii) $V(\underline{x})$ is continuous in $\underline{x}$;

(iv) $V(\underline{x}) \rightarrow \infty$ when $||\ \underline{x}\ || \rightarrow \infty$

Then the equilibrium state $\underline{x} = \underline{0}$ is asymptotically stable in the large and $V(\underline{x})$ is is a Lyapunov function.

Since the major difficulty in applying the Second Method is the construction of a suitable Lyapunov function, it is desirable to weaken condition (ii) of Theorem 1 and thereby enrich the class of functions. Actually $\Delta V(\underline{x})$ need only be non-positive $(0 \geq \Delta V(\underline{x}))$ as long as it doesn't vanish identically on any solution sequence of the difference equation. Thus (ii) can be replaced by the conditions

(ii_1) $\Delta V(\underline{x}) \leq 0$ for all $\underline{x}$

(ii_2) $\Delta V(\underline{x})$ does not vanish identically for any sequence $\{\underline{x}(t_k)\}$ satisfying the difference equation being studied.

5. Applications:

Traditionally, the study of automatic control systems proceeds along the following path. After an adequate mathematical model of the system is obtained, conditions for stability are sought. Once the system is stabilized attention is usually focused on the transient behavior. Then the effect of arbitrary inputs and noise disturbances are considered. Finally when the analysis problems related to a class of control problems are understood, effort is directed toward deriving a synthesis or design procedure which ensures a stable system whose transient behavior and response to inputs is optimal in some specified sense. In this section, with the aid of numerous examples, it will be shown how the "direct method" may be used for the study of discrete-time automatic control systems of the regulator type.

(a) The Mathematical Model.

We are concerned with the analysis and design of automatic control systems in which the plant to be controlled is described by the ordinary difference equation

$$\underline{x}(t_{k+1}) = \underline{g}_1(\underline{x}(t_k), \underline{u}(t_k), \underline{v}(t_k)) \tag{5.1}$$

where $\underline{x}$ is the state vector of the plant, $\underline{u}$ is the control vector, and $\underline{v}$ is an uncontrolled input or disturbance vector. The plant is described by the vector valued vector function $\underline{g}_1$. It is assumed that $\underline{g}_1(\underline{0}, \underline{0}, \underline{0},) = \underline{0}$.

The regulator type control problem is concerned with generating the control vector,$\underline{u}$, by means of a feedback structure so that the state of the plant is near in some sense to the equilibrium state $\underline{x}_e = \underline{0}$, In function notation

$$\underline{u}(t_k) = \underline{g}_2(\underline{x}(t_k)) \tag{5.2}$$

Thus the closed-loop, regulator system that we are considering is of the form

$$\begin{aligned} \underline{x}(t_{k+1}) &= \underline{g}_1(\underline{x}(t_k), \underline{g}_2(\underline{x}(t_k), \underline{v}(t_k)) \\ &= \underline{f}(\underline{x}(t_k), \underline{v}(t_k)) \end{aligned} \tag{5.3}$$

Example 5.1.

In many practical situations the dynamic behavior of the plant is adequately approximated by an ordinary, linear, time-invariant, difference equation

$$\underline{x}(t_{k+1}) = \Phi\, \underline{x}(t_k) + \Delta\, \underline{u}(t_k) + \underline{v}(t_k) \tag{5.4}$$

(i.e. $\underline{g}_1$ is a linear function of $\underline{x}$, $\underline{u}$, and $\underline{v}$.)

With such plants the control input, $\underline{u}(t_k)$, is often made a linear function of the state, $\underline{x}(t_k)$

$$\underline{u}(t_k) = B\, \underline{x}(t_k) \tag{5.5}$$

Thus the overall system is described by the relation

$$\underline{x}(t_{k+1}) = \Psi\, \underline{x}(t_k) + \underline{v}(t_k) \tag{5.6}$$

where $\Psi = \Phi + \Delta B$.

Example 5.2.

Even when the plant is described by a nonlinear difference equation it is often found that the dynamic behavior is close to that of a linear system. Thus it is convenient to describe such systems in the following way

$$\begin{aligned}\underline{x}(t_{k+1}) &= \underline{f}\,(\underline{x}(t_k),\ \underline{u}(t_k),\ \underline{v}(t_k)\,) \\ &= \Phi\, \underline{x}(t_k) + \Delta\, \underline{u}(t_k) + \underline{g}\,(\underline{x}(t_k),\ \underline{u}(t_k),\ \underline{v}(t_k)\,)\end{aligned} \tag{5.7}$$

where the elements of Φ and Δ, ϕ_{ij} and Δ_{ij}, are $\frac{\partial f_i}{\partial x_j}$ and $\frac{\partial f_i}{\partial u_j}$ evaluated at the equilibrium point, $\underline{x} = 0$, $\underline{u} = 0$. The vector valued vector function $\underline{g}$ is defined by (5.7).

Because of its simplicity, such systems are often stabilized as in the linear case by linear feedback of the state $\underline{x}(t_k)$. That is

$$\underline{u}(t_k) = B\, \underline{x}(t_k) \tag{5.8}$$

In this case the overall system is governed by the difference equation

$$\underline{x}(t_{k+1}) = \Psi \, \underline{x}(t_k) + \underline{g}(\underline{x}(t_k), \, \underline{v}(t_k)) \tag{5.9}$$

where as before $\Psi = \Phi + \Delta B$.

In example 5.1 we are interested in the conditions on the matrix Ψ so that the overall system performance is satisfactory in the presence of the disturbance $\underline{v}$. Example 5.2 is similar, except that the disturbance has become a nonlinear term.

(b) Conditions for Stability.

As we previously pointed out, the major problem envolved in applying the direct method is to find a Lyapunov function. For discrete-time systems, a class of functions which has proven extremely useful in a number of cases is just the norm of the state vector $\underline{x}$.

The norm of $\underline{x}$ denoted by $|| \underline{x} ||$ may be thought of as a measure of the length of the vector. To see that a norm might make a useful Lyapunov function, let us define the concept more precisely. A norm is a function which assigns to every vector $\underline{x}$ in a given Euclidean space a real number denoted by $|| \underline{x} ||$ such that

(i) $|| \underline{x} || = \underline{0}$ if, and only if $\underline{x} = \underline{0}$

(ii) $|| \underline{x} || > \underline{0}$ for all $\underline{x} \neq \underline{0}$

(iii) $|| \alpha \underline{x} || = | \alpha | \cdot || \underline{x} ||$ for all $\underline{x}$ and α a real constant

(iv) $|| \underline{x} + \underline{y} || \leq || \underline{x} || + || \underline{y} ||$ for all $\underline{x}$ and $\underline{y}$

The best known norm is probably the Euclidean measure of length

$$|| \underline{x} ||_e = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} = (\underline{x}' \underline{x})^{1/2}$$

Some other commonly used norms are

$$|| \underline{x} ||_m = \max_i \{ | x_i | \}$$

$$|| \underline{x} ||_s = \sum_{i=1}^{n} | x_i |$$

It is easily shown that all three satisfy the four conditions of a norm.

The use of the norm as a Lyapunov function is facilitated by the geometrical picture of the locus of $\underline{x}$ for a constant value of the norm. In Figure 3 the loci in the two dimensional case for the three norms set equal to unity are shown. Note that for $|| \underline{x} ||_e = 1$ the locus is a circle of radius, one centered at the origin; for $|| \underline{x} ||_m = 1$ the locus is a square with sides of length, two parallel to the coordinate axes and centered at the origin; and for $|| \underline{x} ||_s = 1$ the locus is again a square but in this case the sides are of length $\sqrt{2}$ and the verticies lie in the coordinate axes one unit from the origin.

It is often desirable to emphasize certain elements of the vector $\underline{x}$ so that in the cases discussed the circle becomes an ellipse, the first square a rectangle, and the second square a diamond. This is accomplished by weighting each of the elements x_i by positive, scaler constants. In this way we obtain the norms

$$|| \underline{x} ||_{e,\, c} = \left(\sum_{i=1}^{n} c_i x_i^2 \right)^{1/2}$$

$$|| \underline{x} ||_{m,\, c} = \max_i \{ c_i | x_i | \}$$

$$|| \underline{x} ||_{s,\, c} = \sum_{i=1}^{n} c_i | x_i |$$

where the c_i are positive scaler constants.

In the examples which follow it will be convenient to rotate the major and minor axes of the ellipse. For any rotation of the type described it is always possible to find a nonsingular, linear transformation T acting on the vector $\underline{x}$ which produced it. From this point of view we can define a <u>generalized Euclidean norm</u>

$$|| \underline{x} ||_p = [(T\underline{x})' (T\underline{x})]^{1/2} = [\underline{x}' T' T\underline{x}]^{1/2}$$

$$= \underline{x}' P \underline{x} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_i p_{ij} x_j \right)^{1/2}$$

The matrix P generated in this way is always symmetric and has the property that $\underline{x}' P \underline{x} > 0$ for all $\underline{x} \neq 0$. A matrix with this property is said to be positive definite. Thus the generalized Euclidean norm is defined for any positive definite matrix P. For a complete discussion of the concept of norm and normed linear spaces see [5].

Now returning to the stability problem we consider the unforced system

$$\underline{x}(t_{k+1}) = \underline{f}(\underline{x}(t_k)) \tag{5.10}$$

where $\underline{f}(\underline{0}) = \underline{0}$. Setting the Lyapunov function equal to the norm of $\underline{x}$

$$V(\underline{x}(t_k)) = \| \underline{x}(t_k) \| \tag{5.11}$$

we satisfy all conditions of Theorem 1 except for condition (ii). In this case the difference

$$\begin{aligned} \Delta V(\underline{x}(t_k)) &= \| \underline{x}(t_{k+1}) \| - \| \underline{x}(t_k) \| \\ &= \| \underline{f}(\underline{x}(t_k)) \| - \| \underline{x}(t_k) \| \end{aligned} \tag{5.12}$$

From (5.12) we see that condition (ii) is satisfied and the equilibrium solution $\underline{x} = \underline{0}$ of the system (5.10) is asymptotically stable in the large if

$$\| \underline{f}(\underline{x}) \| < \| \underline{x} \| \quad \text{for all } \underline{x} \neq 0$$

A function $\underline{f}$ which has this property for some norm is said to be a contraction. Unfortunately considerable ingenuity is often required to find such a norm.

The following examples show the type of results that may be obtained from the norms previously defined.

Example 5.3.

For the linear, time-invariant, discrete-time, dynamic system

$$\underline{x}(t_{k+1}) = \Phi \, \underline{x}(t_k) \tag{5.13}$$

a convenient Lyapunov function is the square of the generalized Euclidean norm. Thus we set

$$V(\underline{x}) = \underline{x}' \, P \, \underline{x} = \| \underline{x} \|_p^2 \tag{5.14}$$

where as previously noted P is a symmetric, positive definite matrix. From (5.13) and (5.14) the difference is seen to be

$$\begin{aligned} \Delta V(\underline{x}(t_k)) &= \underline{x}(t_k)' \Phi' P \Phi \underline{x}(t_k) - \underline{x}'(t_k) P \underline{x}(t_k) \\ &= - \underline{x}'(t_k) Q \underline{x}(t_k) \end{aligned} \tag{5.15}$$

where $Q = P - \Phi' P \Phi$ is a symmetric matrix. From (5.15) we see that $\Delta V < 0$ for all $\underline{x} \neq 0$ if Q is positive definite. The positive definiteness of Q can be checked from Sylvester's inequalities for it is necessary and sufficient that the n determinants

$$D_k = \begin{vmatrix} q_{11} & \cdots & q_{1k} \\ \vdots & \ddots & \vdots \\ q_{k1} & \cdots & q_{kk} \end{vmatrix} > 0, \quad k = 1, 2, \ldots, n \tag{5.16}$$

all be positive in order that Q be positive definite.

Since in the present example it is well known that the null solution is asymptotically stable in the large if, and only if, all roots of the characteristic equation

$$\det [\Phi - I\lambda] = 0 \tag{5.17}$$

are less in magnitude than unit ($|\lambda_i| < 1$, $i = 1, 2, \ldots, n$), it is possible to obtain similar results which are both necessary and sufficient. Since this result is used so often it will be stated as Theorem 2.

Theorem 2.

The discrete-time, unforced, linear, time-invariant dynamic system of (5.13) is asymptotically stable in the large if, and only if, given any symmetric, positive-definite matrix Q there exists a symmetric, positive-definite matrix P which is the unique solution of the linear equation

$$-Q = \Phi' P \Phi - P \tag{5.18}$$

and

$$V(\underline{x}) = \underline{x}' \, P \, \underline{x}$$

is a Lyapunov function with $\Delta V(\underline{x}) = -\underline{x}' \, Q \, \underline{x}$.

The proof of Theorem 2 is analogous to the proof in the continuous-time case of reference [3]. Note carefully that the theorem does not say that if the system is stable, then given any symmetric, positive definite P the resulting Q is positive definite.

Example 5.4.

Consider the nonlinear system

$$\underline{x}(t_{k+1}) = \Phi(\underline{x}(t_k))\, \underline{x}(t_k) \tag{5.19}$$

where $\Phi(\underline{x}(t_k))$ is an (n x n) matrix whose elements ϕ_{ij} are functions of $\underline{x}(t_k)$.

(a) If we let

$$V(\underline{x}) = \|\underline{x}\| = \max_i \{ c_i \, | x_i | \} \tag{5.20}$$

where $c_1, c_2, \ldots, c_n$ are positive constants, then from the principle of a contraction, the system (5.13) is asymptotically stable in the large when

$$\| \Phi(\underline{x}(t_k))\, \underline{x}(t_k) \| < \| \underline{x}(t_k) \|$$

For the norm chosen

$$\| \Phi(x)\, \underline{x} \| = \max_i \Big\{ c_i \, \Big| \sum_{j=1}^{n} \phi_{ij}(\underline{x})\, x_j \Big| \Big\}$$

$$\leq \max_i \Big\{ \sum_{j=1}^{n} \frac{c_i}{c_j} \, | \phi_{ij}(\underline{x}) | \cdot c_j \, | x_j | \Big\}$$

$$\leq \max_i \Big\{ \sum_{j=1}^{n} \frac{c_i}{c_j} \, | \phi_{ij}(\underline{x}) | \Big\} \cdot \max_j \{ c_j \, | x_j | \}$$

Consequently if

$$\max_i \left\{ \sum_{j=1}^{n} \frac{c_i}{c_j} \mid \phi_{ij}(\underline{x}) \mid \right\} < 1 \quad , \text{ for all } \underline{x} \tag{5.21}$$

the system of (5.13) is asymptotically stable in the large.

(b) If we let the norm be

$$V(\underline{x}) = \| \underline{x} \| = \sum_{i=1}^{n} c_i \mid x_i \mid \tag{5.22}$$

then

$$\| \Phi(\underline{x}) \underline{x} \| = \sum_{i=1}^{n} c_i \mid \sum_{j=1}^{n} \phi_{ij}(\underline{x}) x_j \mid$$

$$\leq \sum_{i=1}^{n} \frac{c_i}{c_j} \sum_{j=1}^{n} \mid \phi_{ij}(\underline{x}) \mid \cdot c_j \mid x_j \mid$$

$$\leq \max_j \left\{ \sum_{i=1}^{n} \frac{c_i}{c_j} \mid \phi_{ij}(\underline{x}) \mid \right\} \cdot \sum_{j=1}^{n} c_j \mid x_j \mid$$

Consequently for this norm if

$$\max_j \left\{ \sum_{i=1}^{n} \frac{c_i}{c_j} \mid \phi_{ij}(\underline{x}) \mid \right\} < \underline{1} \quad , \text{ for all } \underline{x} \tag{5.23}$$

the system of (5.13) is asymptotically stable in the large.

Therefore we conclude that the system of (5.13) is asymptotically stable in the large if it is possible to find n positive constants $c_1, c_2, \ldots, c_n$ such that maximum absolute sum of either the rows or columns of the weighted matrix

$$\begin{bmatrix} \mid \phi_{11}(\underline{x}) \mid & \frac{c_1}{c_2} \mid \phi_{12}(\underline{x}) \mid & \cdots & \frac{c_1}{c_n} \mid \phi_{1n}(\underline{x}) \mid \\ \frac{c_2}{c_1} \mid \phi_{21}(\underline{x}) \mid & \mid \phi_{22}(\underline{x}) \mid & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{c_n}{c_1} \mid \phi_{n1}(\underline{x}) \mid & \cdot & \cdot & \mid \phi_{nn}(\underline{x}) \mid \end{bmatrix} \tag{5.24}$$

is less than unity.

Example 5.4. b.

To illustrate the application of this result, consider the sampled control system of Figure 4. In the sampled-data controller the storage element denoted by z^{-1} saturates as would be expected in any physical system. The saturating storage is represented by the function of Figure 5. From this figure it is seen that the instantaneous gain $f(x) / x$ is such that

$$0 < \frac{| f(x) |}{| x |} \leq 1 \tag{5.25}$$

for $0 < x < \infty$. The problem is to determine if the system is asymptotically stable in the large with this saturating element. Since our methods only give sufficient conditions a negative result would be inconclusive.

From an inspection of Figure 4 the state equation of the unforced system can be written as

$$\begin{aligned} x_1(t_{k+1}) &= x_1(t_k) - 0.2\, x_2(t_k) + 0.6 F_s\left(-x_1(t_k) + 0.3\, x_2(t_k)\right) \\ x_2(t_{k+1}) &= F_s\left(-x_1(t_k) + 0.3\, x_2(t_k)\right) \end{aligned} \tag{5.26}$$

It is desirable to make a transformation of variables in this case so that the argument of the function F_s is a single variable. One such transformation is to let

$$\begin{aligned} y_1(t_k) &= x_1(t_k) \\ y_2(t_k) &= -x_1(t_k) + 0.3\, x_2(t_k) \end{aligned} \tag{5.27}$$

With this transformation the system equations are easily put in the form of (5.13)

$$\begin{bmatrix} y_1(t_{k+1}) \\ \\ y_2(t_{k+1}) \end{bmatrix} = \begin{bmatrix} 1/3 & (-2/3 + 3/5 \dfrac{F_s(y_2(t_k))}{y_2(t_k)} \\ \\ -1/3 & (2/3 - 3/10 \dfrac{F_s(y_2(t_k))}{y_2(t_k)} \end{bmatrix} \begin{bmatrix} y_1(t_k) \\ \\ y_2(t_k) \end{bmatrix} \tag{5.28}$$

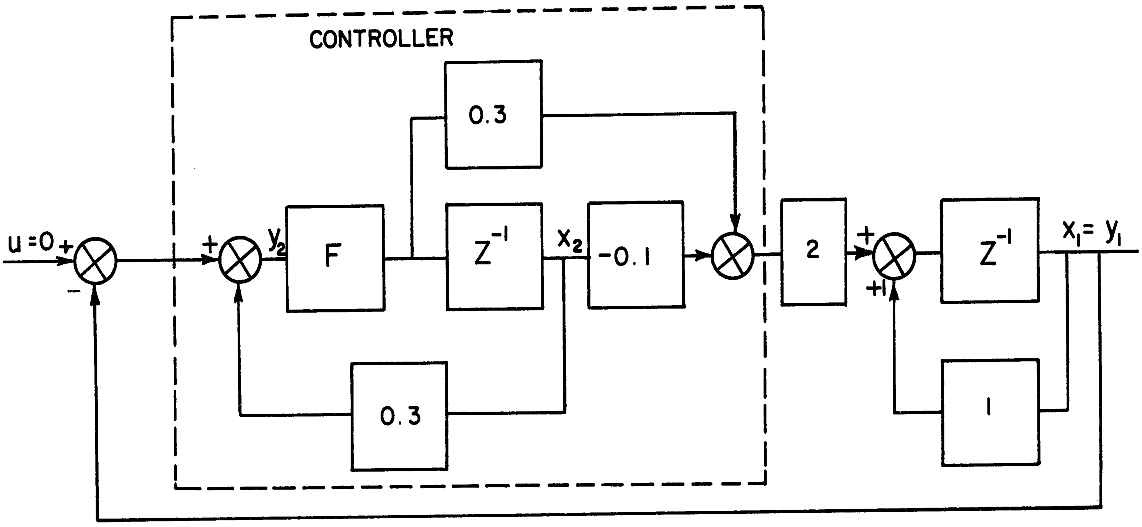

Figure 4. Sampled system with saturating storage

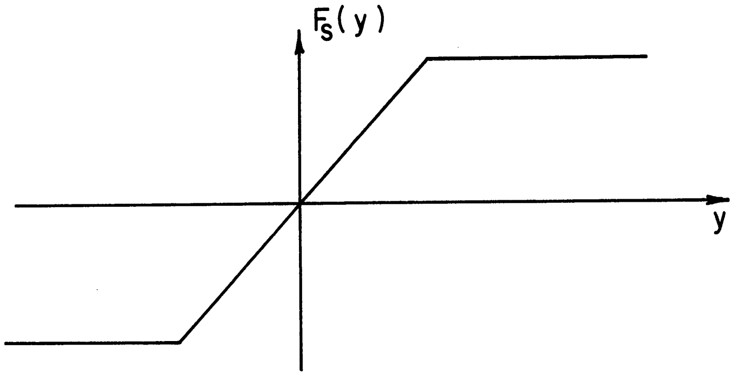

Figure 5. Saturation function

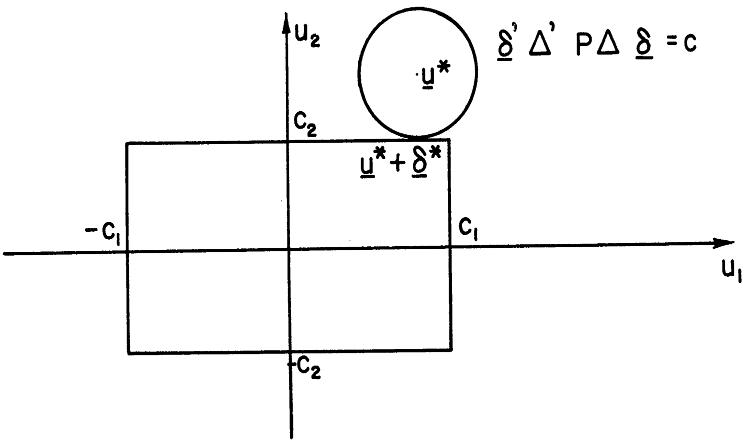

Figure 6. Input space--Example 5.10

Considering just the columns, we conclude that the system is asymptotically stable in the large if positive constants c_1 and c_2 can be found such that

$$| 1/3 | + \frac{c_1}{c_2} | 1/3 | < 1$$

$$\frac{c_2}{c_1} \left| - 2/3 + 3/5 \frac{F_s(y_2)}{y_2} \right| + \left| 2/3 - 3/10 \frac{F_s(y_2)}{y_2} \right| < 1 \tag{5.29}$$

A study of these inequalities show that for $\frac{| F_s(y_2) |}{| y_2 |}$ restricted to the range $0 < \frac{| F_s(y_2) |}{| y_2 |} < 3.34$, the inequalities are satisfied for $\frac{c_1}{c_2} = 2 - \epsilon$, where $\epsilon > 0$. Thus the system is asymptotically stable in the large.

(c) Estimation of Transient Behavior.

If the asymptotic stability in the large of a discrete-time dynamic system has been established by means of a Lyapunov function, it is possible to estimate the transient behavior. This depends upon regarding the value of the Lyapunov function for every point $\underline{x}$ as a measure of the distance from the equilibrium state $\underline{x}_e = \underline{0}$. Consider the obvious equality for $\underline{x} \neq 0$

$$\Delta V(\underline{x}(t_k)) = V(\underline{x}(t_{k+1}) - V(\underline{x}(t_k)) = [\Delta V(\underline{x}(t_k)) / V(\underline{x}(t_k))] \; V(\underline{x}(t_k)) \tag{5.30}$$

from which we obtain the difference equation for V

$$V(\underline{x}(t_{k+1}) = [1 + \Delta V(\underline{x}(t_k)) / V(\underline{x}(t_k))] \; V(\underline{x}(t_k)) \tag{5.31}$$

Since the system is asymptotically stable in the large the ratio $-1 < \Delta V(\underline{x}(t_k)) / V(\underline{x}(t_k)) \leq 0$; and if condition (ii) of Theorem 1 rather than the alternate condition (ii_1) and (ii_2) are satisfied, then $-1 < \Delta V(\underline{x}(t_k)) / V(\underline{x}(t_k)) < 0$. Letting

$$\eta_1 = \max_{\underline{x}} \left| \frac{\Delta V(\underline{x})}{V(\underline{x})} \right|$$

$$\eta_2 = \min_{\underline{x}} \left| \frac{\Delta V(\underline{x})}{V(\underline{x})} \right| \tag{5.32}$$

we see that

$$(1 - \eta_1)\, V(\underline{x}(t_k)) \le V(\underline{x}(t_{k+1})) \le (1 - \eta_2)\, V(\underline{x}(t_k)) \tag{5.33}$$

Thus if the system starts in initial state $\underline{x}(t_0)$ the "distance" from the origin, as measured by the Lyapunov function decreases in the following way

$$(1 - \eta_1)^k\, V(\underline{x}(t_0)) \le V(\underline{x}(t_k)) \le (1 - \eta_2)^k\, V(\underline{x}(t_0)) \tag{5.34}$$

We may consider $(1 - \eta_2)$ the largest time constant of the discrete-time system and $(1 - \eta_1)$ the smallest. Note however that η_1 and η_2 depend on the Lyapunov function choosen. In fact if the function were choosen so as to satisfy only condition (ii_1) and (ii_2) of Theorem 1, then $\eta_2 = 0$. In this case the transient behavior would have to be studied in regions of the state space and not the entire space.

Example. 5.5.

From Theorem 2 we know that if the linear system

$$\underline{x}(t_{k+1}) = \Phi\, \underline{x}(t_k) \tag{5.35}$$

is asymptotically stable in the large, then by choosing any symmetric, positive definite Q and solving the linear equation

$$Q = P - \Phi' P \Phi \tag{5.36}$$

for P, we obtain a satisfactory Lyapunov function

$$V(\underline{x}) = \underline{x}' P \underline{x} \tag{5.37}$$

The determination of η_1 and η_2 is well known [3] in this case.

$$\eta_1 = \max_{\underline{x}} \frac{\underline{x}' Q \underline{x}}{\underline{x}' P \underline{x}} = \lambda_{max} (Q P^{-1})$$

$$\eta_2 = \min_{\underline{x}} \frac{\underline{x}' Q \underline{x}}{\underline{x}' P \underline{x}} = \lambda_{min} (Q P^{-1}) \tag{5.38}$$

where $\lambda_{max} (Q P^{-1})$ and $\lambda_{min} (Q P^{-1})$ denote the largest and smallest characteristic roots of the equation $\det [Q P^{-1} - I \lambda] = 0$.

Example 5.6.

In example 5.4 if the saturation effect were not present $(F_s(y_2)/y_2 = 1)$ the conditions for asymptotic stability in the large could have been achieved with $c_1 = 1$ and $c_2 = 2$. With these values we see that the Lyapunov function, in this case the norm of $\underline{x}$, as a function of time is bounded by

$$|| \underline{x}(t_k) || \leq (\tfrac{1}{2})^k \; || \underline{x}(t_0) || \tag{5.39}$$

This means that in each sampling period the norm is at least halved.

Now let us consider the effect of the nonlinear saturating term. Restricting $| y_2 | \leq 10$ confines $0.1 \leq | F_s(y_2) | / | y_2 | \leq 1$. In this case the least upper bound on the transient behavior is obtained with $c_1 = 1.86$ and $c_2 = 1$ which insures asymptotic stability in the large. The bound on the transient behavior is

$$|| \underline{x}(t_k) || \leq (29/30)^k \; || \underline{x}(t_0) || \tag{5.40}$$

Thus we see that the effect of the saturation is to slow down the transient behavior. Further the slowing down action increases with increased $|y_2|$. The same conclusion is easily obtained from physical reasoning in this example.

(d) Effect of Inputs.

Also, when the system stability has been determined by means of a Lyapunov function, it is possible to estimate the effect of bounded input disturbances. The following example illustrates the method.

Example 5.7.

Consider the nonlinear discrete-time regulator system

$$\underline{x}(t_{k+1}) = \underline{f}(\underline{x}(t_k)) + \underline{v}(t_k) \tag{5.41}$$

where $\underline{f}(\underline{0}) = 0$. The function $\underline{f}$ is a contraction with respect to a certain norm so that

$$|| \underline{f}(\underline{x}) || \leq c_1 || \underline{x} || \quad , 0 < c_1 < 1 \tag{5.42}$$

Therefore the system is asymptotically stable in the large. The disturbance, $\underline{v}$, is bounded so that $|| \underline{v} || \leq c_0$. The effect of this disturbance is to perturb the system away from the equilibrium point, $\underline{x} = \underline{0}$. We are interested in determining the extent of these perturbations. To do this, note that the difference ΔV is

$$\begin{aligned} \Delta V(\underline{x}(t_k)) &= || \underline{x}(t_{k+1}) || - || \underline{x}(t_k) || \\ &= || \underline{f}(\underline{x}(t_k)) + \underline{v}(t_k) || - || \underline{x}(t_k) || \\ &\leq c_1 || \underline{x}(t_k) || + c_0 - || \underline{x}(t_k) || \end{aligned} \tag{5.43}$$

The difference $\Delta V < 0$ as long as

$$|| \underline{x} || \geq c_0/(1 - c_1) \tag{5.44}$$

This implies that for any $\underline{x}$ such that $|| \underline{x} || \geq c_0/(1 - c_1)$ the effect is to force $\underline{x}$ into the region bounded by $|| \underline{x} || = c_0/(1 - c_1)$, but once $\underline{x}$ is in this region there is no restoring effect. Thus the perturbations resulting from the disturbance input are confined to this region. Note that this region becomes smaller as $c_1 \rightarrow 0$ (as the transient behavior becomes faster).

Example 5.8.

As a second and more complex example consider the system

$$\underline{x}(t_{k+1}) = \Phi \underline{x}(t_k) + \underline{f}(\underline{x}(t_k)) + \underline{v}(t_k) \tag{5.45}$$

where the nonlinear term $\underline{f}$ represents parameter variations ($\underline{f}(\underline{0}) = \underline{0}$). With $\underline{f}$ and $\underline{v}$ equal to zero the system is asymptotically stable in the large. It is assumed that the parameter variations and the disturbances are such that

$$\|\underline{f}(\underline{x})\| \leq c_1 \|\underline{x}\|$$

$$\|\underline{v}\| \leq c_0$$

As a Lyapunov function in this example we choose the form $V = \underline{x}' P \underline{x}$ where P is the solution of the linear matrix equation $Q = P - \Phi' P \Phi$. Since the linear part of (5.45) is stable, for any positive definite Q the solution P is positive definite.

The difference ΔV is

$$\Delta V(\underline{x}) = -\underline{x}' Q \underline{x} + 2\underline{f}' P \Phi \underline{x} + 2\underline{v}' P \Phi \underline{x} + 2\underline{f}' P \underline{v} + \underline{f}' P \underline{f} + \underline{v}' P \underline{v} \tag{5.46}$$

Obviously

$$\Delta V(\underline{x}) \leq -\min_{\underline{x}} \{\underline{x}' Q \underline{x}\} + 2|\underline{f}' P \Phi \underline{x}| + 2|\underline{v}' P \Phi \underline{x}| + 2|\underline{f}' P \underline{v}| + |\underline{f}' P \underline{f}| + |\underline{v}' P \underline{v}| \tag{5.47}$$

Furthermore from the Schwarz inequality and relation (5.38)

$$|\underline{f}' P \Phi \underline{x}| \leq (\underline{f}' \underline{f})^{1/2} (\underline{x}' \Phi' P^2 \Phi \underline{x})^{1/2} \leq c_1 \|\underline{x}\|^2 \lambda^{1/2}_{max} (\Phi' P^2 \Phi)$$

$$|\underline{v}' P \Phi \underline{x}| \leq (\underline{v}' \underline{v})^{1/2} (\underline{x}' \Phi' P^2 \Phi \underline{x})^{1/2} \leq c_0 \|\underline{x}\| \lambda^{1/2}_{max} (\Phi' P^2 \Phi)$$

$$|\underline{f}' P \underline{v}| \leq (\underline{f}' \underline{f})^{1/2} (\underline{v}' P^2 \underline{v})^{1/2} \leq c_1 \|\underline{x}\| c_0 \lambda_{max}(P)$$

$$|\underline{f}' P \underline{f}| \leq (\underline{f}' \underline{f}) \lambda_{max}(P) \leq c_2 \|\underline{x}\|^2 \lambda_{max}(P)$$

$$|\underline{v}' P \underline{v}| \leq (\underline{v}' \underline{v}) \lambda_{max}(P) \leq c_0^2 \lambda_{max}(P)$$

while

$$\min_{\underline{x}} \{\underline{x}' Q \underline{x}\} = \lambda_{min}(Q) \; || \underline{x} ||^2$$

Consequently $\Delta V < 0$ if

$$- a || \underline{x} ||^2 + b || \underline{x} || + c < 0 \tag{5.48}$$

where

$$a = [\lambda_{min} (Q) - c_1 \lambda^{1/2}_{max} (\Phi' P^2 \Phi) - c_1^2 \lambda_{max} (P)]$$

$$b = [c_0 \lambda^{1/2}_{max} (\Phi' P^2 \Phi) + c_0 c_1 \lambda_{max}(P)]$$

$$c = c_0^2 \lambda_{max} (P)$$

If the system is to be stable then a must be a positive number. Note that this condition is satisfied if the nonlinear effect is sufficiently small (i.e. c_1 sufficiently small). If this condition is satisfied, since b and c are always psoitive, it follows that ΔV is negative for

$$|| \underline{x} || \geq \frac{b + \sqrt{b^2 + 4 ac}}{2a} \tag{5.49}$$

Thus in this rather complicated example the effect of the disturbance is confined to a sphere of radius $(b + \sqrt{b^2 + 4 ac}\;) / 2a$.

(e) Design and Optimization.

In complex systems it is almost a necessity that the design procedure be so devised that it ensure system stability. Procedure based on the direct method have this property. First consider the methods of the next example.

Example 5.9.

If a system is asymptotically stable in the large then the sum

$$- \sum_{k=0}^{\infty} \Delta V(\underline{x}(t_k)) = V(\underline{x}(t_0)) \tag{5.50}$$

Thus if the $-\Delta V(\underline{x})$ is considered as the penalty for system deviation and the sum of such penalities along a solution sequence as the performance index, then (5.50) gives a simple method for evaluation of this index. In the linear case

$$\underline{x}(t_{k+1}) = \Phi \underline{x}(t_k)$$

if the penalty function $-\Delta V(\underline{x}(t_k)) = \underline{x}'(t_k) Q \underline{x}(t_k)$ then

$$I(\underline{x}(t_0)) = - \sum_{k=0}^{\infty} \underline{x}'(t_k) Q \underline{x}(t_k) = \underline{x}'(t_0) P \underline{x}(t_0) \tag{5.51}$$

where P is the solution of $-Q = \Phi' P \Phi - P$.

Thus if some parameters of Φ denoted by α are adjustable we can select them so as to minimize the performance $I(\underline{x}(t_0))$

$$\min_{\alpha} \{ I(\underline{x}(t_0)) \} = \min_{\alpha} \{ \underline{x}'(t_0) P \underline{x}(t_0) \} \tag{5.52}$$

Note that this minimum is a function of the initial state $\underline{x}(t_0)$ so that the minimizing parameters α are different for each initial state.

The design objective might be to select the parameters α so as to minimize the maximum performance index for the initial state $\underline{x}(t_0)$ in a certain region X of the state space

$$\min_{\alpha} \{ \max_{\underline{x}(t_0) \in X} I(\underline{x}(t_0)) \} = \min_{\alpha} \max_{\underline{x}(t_0) \in X} \{ \underline{x}'(t_0) P \underline{x}(t_0) \} \tag{5.53}$$

In this way α is made independent of the initial state $\underline{x}(t_0)$.

Example 5.10.

As a final design example consider the problem of designing a regulating input for the plant described by

$$\underline{x}(t_{k+1}) = \Phi \underline{x}(t_k) + \Delta \underline{u}(t_k) \tag{5.54}$$

and where the control inputs are constrained such that

$$| u_i(t) | \leq c_i \ , \ c_i > 0 \ , \ (i = 1, 2, \ldots, n) \tag{5.55}$$

Assume that the unforced system is <u>asymptotically stable in the large</u> so that given any symmetric positive definite Q the linear equation $-Q = \Phi' P \Phi - P$ has an unique solution the positive definite matrix P.

Then $V(\underline{x}) = \underline{x}' P \underline{x}$ is clearly a Lyapunov function for the unforced part of the system. For the forced system the difference

$$\Delta V = -\underline{x}' Q \underline{x} + 2\underline{u}' \Delta' P \Phi \underline{x} + \underline{u}' \Delta' P \Delta \underline{u} \tag{5.56}$$

The transient behavior relative to the Lyapunov function (norm) choosen is made as fast as possible by choosing $\underline{u}$, the control input, so ΔV is as negative as possible. If there are no constraints on the input magnitude,by ordinary calculus we find that the optimum $\underline{u}$ in this sense

$$\underline{u}^* (t_k) = -(\Delta' P \Delta)^{-1} \Delta' P \Phi \underline{x}(t_k) \tag{5.57}$$

This solution exists if $\Delta' P \Delta$ is nonsingular. Actually $\Delta' P \Delta$ is positive definite (and thus nonsingular) provided that the columns of Δ are linearly independent. Physically this condition requires that the effects of the control inputs be linearly independent. If this is not so we can always consider a subset of the inputs which are independent. Therefore in what follows $\Delta' P \Delta$ is assumed to be positive definite (nonsingular).

When the input magnitudes are constrained in magnitude, it is convenient to consider the space of the inputs. For example, with two inputs ($m = 2$) the constraints require that a permissible input lie in the shaded rectangle of Figure 6. If the optimum control input u^*, calculated by (5.57), lies inside this rectangle then it is the solution. If the optimum control input u^* lies outside the rectangle, then the constrained solution lies somewhere on the boundary of the rectangle.

We have previously noted that ΔV assumes its largest negative value when $\underline{u} = \underline{u}^*$. The first term in ΔV is independent of the choice of $\underline{u}$. The last two terms of ΔV become more positive as $\underline{u}$ moves away from $\underline{u}^*$. To study this effect let $\underline{u} = \underline{u}^* + \delta$. Then

$$V = \underline{x}' Q \underline{x} - \underline{u}^{*'} \Delta' P \Delta u^* + \underline{\delta}' \Delta' P \Delta \underline{\delta} \tag{5.58}$$

where $\underline{\delta}' \Delta' P \Delta \underline{\delta} > 0$ for all $\underline{\delta} \neq 0$ and $-u^{*'} \Delta' P \Delta \underline{u}^* < 0$ for $\underline{u}^* \neq 0$, since $\Delta' P \Delta$ is positive definite. The loci of $\underline{\delta}' \Delta' P \Delta \underline{\delta} = c$ (a constant) is a quadratic surface (in two dimensions an ellipse). The optimum control input in the sense of making Δ V as negative as possible and with constraints on the input magnitudes is to set

$$\underline{u} = \underline{u}^* + \underline{\delta}^* \tag{5.59}$$

where that $\underline{\delta}^*$ is that value of $\underline{\delta}$ which produces the smallest term $\underline{\delta}'\Delta' P \Delta \ \underline{\delta}$ and is just tangent to the rectangle containing permissible inputs. This optimum solution requires extensive computation for each $\underline{x}(t_0)$.

Note that it is always possible to insure stable operation with the constrained inputs (saturation) by setting

$$\underline{u}(t_k) = \alpha \, \underline{u}^* (t_k) \tag{5.60}$$

where α is choosen such that

$$\max_i \ \{\alpha \mid u_i \mid\} \leq c_i \tag{5.61}$$

This solution is not optimal but it has the advantage of being easy to calculate and insures stable operation.

These two examples illustrate only a few of the ways in which the direct method may be used in the design of discrete-time systems. As more workers in the control field become familiar with the techniques I expect that such design methods will become very numerous.

6. Acknowledgement.

This lecture very closely follows reference [4] by Dr. R. E. Kalman and the author. Many of the methods were first suggested by Dr. Kalman.

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STABILITY ANALYSIS OF NUCLEAR REACTORS BY LIAPOUNOFF'S SECOND METHOD

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I. INTRODUCTION

In the first several of a series of related interesting papers,[1-8] the nature of the inherent stability, and the power and temperature time-responses to a step-function input of excess reactivity, of several types of nuclear chain reactors (initially in an equilibrium state and operating under certain prescribed conditions as noted below), characterized by non-linear differential equations of performance, are determined by classical procedures: primarily as based on use of the Hamiltonian function, on use of Green's function, etc. In a later paper[9] the natures of the inherent stability of the reactors so constructed and operated that the Hamiltonian analysis is applicable are redetermined somewhat more elegantly and concisely by use of Liapounoff's Direct Method (hereafter designated, for conciseness of expression, by LDM). And in most recently published papers,[10-17] consideration has been extended to obtain knowledge not merely of ordinary local stability (i.e., whether the equilibrium state is asymptotically or locally stable under disturbances resulting from the introduction of an excess reactivity of "sufficiently small" magnitude), but also of bounds on the range of reactivity under which the equilibrium point is yet stable.

A rather considerable interest and scope of application attends such use of LDM: both to the general student, teacher or worker in control systems engineering and to the specialist in nuclear reactor control systems theory and design. To the latter it is of obvious value to know the theory and use of this powerful phase of analysis as it applied in his particular field of work. To the former it provides as example of the use of LDM whereof the pertinent physical phenomena requires analysis, perspective and interpretation rather unique as compared with the (electrical, mechanical, hydraulic, etc.) systems more familiar to him.

It is in such thought that the present paper, exhibiting the nature and use of LDM in the kinetic analysis of nuclear reactors, is advanced. As the paper is tutorial in purpose, it comprises in the large a connected exposition and interweaving of analysis and results to be found disseminated through the cited papers: thus, it is instructive, rather than original, in content. And in that prime interest re instruction centers on the application of LDM, rather than on account of reactor kinetics, the latter is enfolded only as needed to make understandable the nature of the problem and the procedure attending investigation by LDM. For detail of reactor kinetics and of the investigation of more complex reactors than considered in this paper, the reader so particularly interested will have little difficulty reading the above cited references after assimilating the content of the present paper (in a systematic reading of the cited references, it is advantageous to read them in the order published, because of the interlinking of their contexts).

II. THE GENERAL PROBLEM

If -- as in early prototypes -- a reactor is large in structure, low in power, and operates at equilibrium at a temperature well below that at which appreciable temperature-effect damage would result, the rate of temperature rise resulting from suddenly introduced excess reactivity (say by a step-function motion of control rods) is -- usually -- rather slow, a considerable safety margin of temperature exists, and manual or simple automatic control may prove satisfactory. But as with view of increasing rating and efficiency, demands for minimizing size and operating at temperatures close to the permissible maximum are conjoined with increase of power rating -- as is sought in modern power reactor design -- the associated control system becomes correspondingly more complex in construction and operation: to the end that knowledge of the inherent stability and the character of its responses to suddenly-introduced changes in reactivity become increasingly important relative to effecting optimal design of the control system.[18-20]

Such knowledge is gained by solution of the nonlinear differential equations of performance characterizing reactor performance and the associated initial and boundary conditions.[1-30] In general, the equations of simple reactors are not formidable. Thus, a well-investigated class of homogeneous reactors, whereof in the first consideration delayed neutron effects are neglected (but can be enfolded by an elaboration of analysis), is characterized by the pair of nonlinear differential equations

$$\frac{d \log P}{dt} = -(\alpha/\tau)T \tag{1}$$

$$\epsilon \frac{dT}{dt} = (P-P_e) \tag{2}$$

where

P is the total power generated in the reactor (necessarily positive)

P_e is the total power extracted from the reactor

T is the reactor temperature, on a scale whereat $T=0$ at equilibrium

$-\alpha$ is the temperature coefficient of reactivity

τ is the mean lifetime of the neutrons

ϵ is the thermal capacity

and the notation and terminology are as used in most of the cited references.

Specified physical conditions of operation correspond to certain functional expressions of P_e. Thus, to consider two of interest in this paper;

A. Constant power extraction, $P_e = P_0$

B. Newton's law of cooling, $P_e = \lambda(T-T_0)$

wherein P_0 is necessarily positive, λ is a positive constant, and T_0 is the ambient temperature of the surrounding medium, necessarily negative since $T = 0$ is reactor temperature at equilibrium, and this temperature must be greater than that of the ambient medium if an outflow of heat is to result.

With, initially, the reactor in a state of equilibrium, let at $t = t_0$ a positive excess reactivity E be introduced suddenly, say by a step-function withdrawal of control rods, as characterized in strength by $E = -\alpha T$ at $t = t_0$. Hence,

$$T = -E/\alpha, \quad \text{at } t = t_0 \tag{3}$$

Then it is desired to know whether or not the resulting kinetic action of the reactor is "stable"; and, although this is not the major interest in this paper, the time responses of the temperature and power may be desired.

An alternative statement of the problem, both somewhat more familiar in form to those not experienced in nuclear reactor control theory and more directly linked to stability analysis by LDM as this is set out in textbooks, is to be gained as follows. Introducing new variables, defined by $y = T$ and $x = \log P$, whence $P = \exp x$ and $P_e = \exp x_e$, in (1) and (2) yields

$$\dot{x} = -(\alpha/\tau)y \tag{4}$$

$$\dot{y} = (\exp x - \exp x_e)/\epsilon \tag{5}$$

For the stated conditions of operation, x_e is respectively $x_e = x_0$ and $x_e = \lambda(y-y_0)$. Then (4) and (5) comprise a pair of first-order non-linear differential equations, with the corresponding equations of first approximation

$$\dot{x} = -(\alpha/\tau)y \tag{6}$$

$$\dot{y} \cong (x-x_e)/\epsilon \tag{7}$$

The associated singular point in the finite plane, $[y = 0,\ x = x_e$ for $y = 0]$, and thus at $[T = 0,\ P = P_e$ for $T = 0]$ in the (P,T)-plane, corresponds to reactor equilibrium. Then it is desired to know the nature of the singular point, and thus the nature of the stability of the reactor relative to the stated equilibrium conditions enfolded in P_e. Such can be ascertained in Liapounoff's <u>local</u> sense of stability (i.e., examination of the nature of the response following an initial "sufficiently small" displacement at $t = t_0$ from the singular point (x_s, y_s) to $(x = x_0, y = y_0)$, corresponding to, say, $(y_0 = -E/\alpha,\ x = x_{e0}$ for $y_0)$ resulting from introduction of the excess reactivity E into the reactor. The pertinent criterion relative to LDM is stated and use is illustrated on, say, pages 491-497 of the text(31) by Gille, Pelegrin and Decaulne, to cite an easily-obtained control engineering text.

General investigation of stability in the large requires a more comprehensive exposition of LDM than as cited at present in control engineering texts (at least in other than the Russian language), which point is discussed in greater detail in Section IV.

III. SOLUTION BY LDM

In general, as emphasized on page 496 of Reference 31, no specific procedure exists for finding the required Liapounoff function for an arbitrarily-specified system. For a broad class of problems the Lurje transform and its generalization by Letov, as discussed in the recently-published text[32] by the above-named authors, can be used to obtain the desired function. Leaving aside these more sophisticated and as yet relatively unfamiliar (in the U.S.) techniques, the commonly-stated textbook approach is construction, as possible, of an appropriate quadratic form $V(x,y)$ -- as exhibited on pages 493-497 of Reference 31. Now in two variables such defines a conic, through $V(x,y) = \text{const}$; and a necessary condition in use of Liapounoff's criterion relative to the above-stated problem is that the Liapounoff function $V(x,y)$ must be positive for all values of x and y except that it may be zero at $x = 0$, $y = 0$. This conjunction suggests that quadratic forms defining ellipses, through $V(x,y) = 0$, may prove suitable choices -- which is the approach used on pages 495-496 of Reference 31. Now ellipses are closed curves in the (x,y)-plane; which suggests, in turn, that yet other closed curves may provide Liapounoff functions similarly; and again in turn, this suggests that the easily-written Hamiltonian function (expressing the sum of the kinetic and potential energies to within an arbitrary constant, say) pertinent to a particular problem may provide the desired function; and that the corresponding function in terms of T and P provides a desired function for the nuclear reactor stability analysis; and such results for the two cases A and B cited in Section II.

The desired Hamiltonian is easiest constructed by noting that (1) and (2) can be interpreted as characterizing the motion of a sphere on a surface, whereof -log P is the horizontal coordinate, $(\alpha/\tau)T$ the velocity, $(\alpha/\tau)\dot{T}$ the acceleration, $\tau\epsilon/\alpha$ the mass, and $(P - P_e)$ the "general" forcing function. In turn, the "general" potential energy is, to within an arbitrary constant, $\int_{\log P_{eo}}^{\log P} (P-P_e)\, d(\log P)$ and the kinetic energy is $(\epsilon\alpha/2\tau)T^2$.

Such interpretation provides obtaining the equations of the trajectories in the (T,P)-phase plane in the usual fashion; i.e.,

$$\int_{\log P_{e0}}^{\log P} (P-P_e)\, d(\log P) + (\alpha\epsilon/2\tau)T^2 = \text{const., say } C_0 \quad (8)$$

whereof C_0 is determined by the initial conditions.

For case A, $P_e = P_0$; and substituting in (8), noting that $d(\log P) = (1/P)dP$, and integrating yields

$$[P-P_0 - P_0 \log P/P_0 + (\alpha\epsilon/2\tau)T^2] = C_0 \quad (9)$$

Now (8) can also be obtained by reversing the members of (2), dividing by ϵ, multiplying the corresponding members of the resulting equation and of (1), shifting all terms in the resulting equality to the left-hand side, multiplying through by dt, and integrating each side of the resulting expression. Accordingly, (9), as a particular case of (8), <u>is</u> pertinent to the system characterized by (1) and (2) from either of the two approaches yielding (8).

Now (9) is positive definite for $P > 0$ and all T; and

$$\dot{H}(t) = \dot{P} - (P_0/P)\dot{P} + (\alpha\epsilon/\tau)T\,\dot{T} \tag{10}$$

which on substituting for T and $\dot{T}$ from (1) and (2) equals zero. Hence the reactor is stable relative to the point of equilibrium ($T = 0$, $P = P_0$); but it is not asymptotically stable (in the local sense) since such requires that $H(t)$ be always negative for all P and T.

For case B, Newton's law of cooling, for which $P_e = \lambda(T-T_0)$ and $T_0 < 0$, substituting in (8), using $-\lambda T\, d(\log P) = -\lambda T(d\log P/dt)dt = -\lambda T(-\alpha/\tau)\dot{T}\,dt$, and integrating gives

$$P + \lambda T_0 + \lambda T_0 \log(P/-\lambda T_0) + (\alpha\epsilon/2\tau)T^2 = -(\alpha/\lambda)\int_{t=0}^{t} T^2 dt + C_0 \tag{11}$$

Now the left-hand member of (11) is positive definite for $P > 0$ and all T; and characterizing it by $H(t)$,

$$\dot{H}(t) = -(\alpha/\lambda)T^2 \tag{12}$$

which is always negative.* Hence the reactor is asymptotically stable relative to the point of equilibrium ($T = 0$, $P = -\lambda T_0$).

Corroboratively, for case A it may be noted that (9) is the Hamiltonian of a conservative system; the phase-plane plots are closed curves in the (T,P)-plane; the value of C_0 is determined by the initial conditions; if these are, as remarked above, furnished by $P = P_0$, $T = -E/\alpha$ at $t = 0$, then the constant $C_0 = (\epsilon/2\alpha\tau)E^2$; the maximum swings on T, corresponding to $dT/dP = 0$, occur at ($P = P_0$, $T = \pm E/\alpha$); the maximum swings on P, corresponding to $dP/dT = 0$, occur at [$T = 0$, $P = P_1$, P_2, the two roots of (9) with $T = 0$]; the point ($T = 0$, $P = P_0$) is the limit point of the closed curves as E approaches zero and thus the singularity is a center; and as is well-known, this type of singularity is stable, but

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* Provided $T \neq 0$. Special investigation for points on the line $T = 0$ is easily effected.

not asymptotically so, in Liapounoff's sense (the interested reader will find corresponding phase-plane plots and time-resonses of P and T, for specified numerical values in References 1 and 2. It would seem that the temperature used in these plots but not in the equations is the temperature increase, thus $T - E/\alpha$).

Again, for case B, it may be noted that the left-hand members of (9) and (11) are of the same form, $-\lambda T_0$ in (11) playing the role of P_0 in (9), both being positive. But the additional integral term in the right-hand member of (11), equivalent to $(\tau/\alpha) \int_{\log P_{e0}}^{\log P} (\alpha T/\tau) d \log P$, is interpretable as energy loss due to viscous friction (frictional force proportional to velocity). Hence the system is damped; and as t approaches infinity, the disturbed system ($T = -E/\alpha$ at $t = 0$) returns to equilibrium at ($T = 0$, $P = -\lambda T_0$), the coordinates of the singular point: which in this case is a node or focus, hence asymptotically stable, depending on assigned numerical values of the parameters of the reactor. (The corresponding plots are given in Reference 33.)

In similar fashion, the interested reader can determine the stability of reactors with more complex programs of operation and/or complex structures by LDM, and compare these with results obtained thusly or otherwise in the literature: say, for adiabatic operation,[2] $P_0 = 0$, which involves the interesting concept of a singularity occurring at ($x=$) infinity; a heterogeneous reactor,[5,6] with two media, wherefore the equilibrium for P_e may be stable or unstable depending on the structure (but proves stable for, say, uranium and heavy water as in the ZOE reactor at Chatillon); the generalization[9] of the latter to the case of two, three and n media with heat generated in each medium; and other special cases.[12,16]

IV. OTHER STABILITY ASPECTS

The analysis outlined in Section III is suitable, in general, for investigation of ordinary "local" stability: i.e., as determined if the initial departures of P and T from equilibrium are "sufficiently small". Often, however, knowledge is desired of the extent (at least in some measure) of a domain about the equilibrium point in which the initial conditions may be arbitrarily established and the system yet be stable. This problem is examined re use of LDM in recent reports by LaSalle[34] and by Smets,[16,17] the latter being especially concerned with nuclear reactor stability investigation. His analysis is illustrated by consideration of several of the above-remarked types of reactors, for a boiling water reactor, and consideration of xenon build-up. Stability in the large is considered (i.e., any initial location) and is of especial interest. Also termed global stability, the problem has been investigated for a particular set of differential equations in a recent paper[13] in a very abstract manner, and application to nuclear reactors then advanced.

Again, analysis of a continuous medium reactor in which parameters are a function of a space variable, thus the temperature is also, would not seem easily (if at all) analyzable by LDM. For such reactors an approach along lines used early by Welton[21] has been used with success.[11,12] This approach can also be applied to reactors treatable by LDM, hence comprises a somewhat more general mode of stability analysis. This method, involving Green's function analysis, has been used to investigate the stability of circulating-fuel reactors.[2-4]

In conclusion, attention may be directed to a Liapounoff function somewhat different than those used in earlier-cited references, which affords somewhat more general results, as suggested by Popov.[10] A discussion is given by Smets,[16] in a report which has somewhat restricted circulation. But no doubt this discussion, and a very thorough and well-integrated account of the use of LDM (and other approaches) for the stability analysis of nuclear reactors will be found in his forthcoming monograph[17] thereon-- to which the attention of the particularly interested reader is directed; as also to exhaustive bibliographies on the kinetics and control of nuclear reactors[35] in the large (from which the references of this paper have been drawn) and on the literature of Liapounoff's theory in particular and of nonlinear system theory in general.[36,37]

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A RESUME OF THE BASIC LITERATURE ON NONLINEAR SYSTEM THEORY
(WITH PARTICULAR REFERENCE TO LYAPUNOV'S METHODS)

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A RESUME OF THE BASIC LITERATURE ON NONLINEAR SYSTEM THEORY (WITH PARTICULAR REFERENCE TO LYAPUNOV'S METHODS)

In detailing the time stream of the development of nonlinear control systems theory in a 1957 paper [46], the author noted that "the fifth and present stage of work originated about 1950. On one side, through the start of a considerable effort directed toward improving the performance of control systems by the deliberate inclusion of nonlinear elements and effects (as in MacDonald's work on multiple-mode switching); on the other side, through attempt to study inherent nonlinear effects in control systems in the "large", rather than just in the small wherefor linearization yields useful [but limited] solutions." Scan, now, several years later, of the proceedings of subsequently held-conferences and symposia[132-135], of recent review papers[33,40,109,118,121] and selective bibliographies and summary papers on adaptive[9], sampled-data[49,50,114,146,148], time-lag[25,125] and relay-types[129] of specialized control systems and on continuous systems in general[29,49,62,70,90,95,123,137-139,141,149] both supports the validity and strengthens the tone of this two-fold statement: For in each of the several mentioned areas of control activity, a concentrated, fervid and accelerating interest now centers on enlarging the scope of nonlinear aspects and on broadening the domain over which analysis and synthesis can be effected with increased "exactness".

To achieve this latter in the fullest possible sense requires consideration of the actual nonlinear differential equations of performance, with consequent attendant need of the fullest possible knowledge of available analytic methods of solution (and of supporting graphic, numeric and computer techniques). An obvious approach to gain of such is assimilation of theory as set-out in the excellent recently-published texts wholly[76,106] or in part[8,26,65,84,92] devoted to nonlinear differential equation theory and/or associated linear differential equation theory and to more specialized texts relative to asymptotic behavior, particularly as linked with considerations of stability of solution.[12,15,20,98]

In this connection a logical course of procedure, for one not already well-versed in these texts, is to read each of the first group of remarked books as language facility permits: and then to take in turn the clearly written, easily-grasped book by Bellman[12], which advances an admirable treatment of the bases of the vector approach to differential equations; follow this with Cesari's[20] glowingly-reviewed, exhaustively-detailed four-chapter text enfolding: 1) various concepts of stability and linear systems with constant coefficients, 2) variable and periodic coefficient systems, with emphasis on second-order systems, 3) account of the first and second methods of Lyapunov and various analytic methods well-illustrated by solution of some of the better-known (by name) differential equations, and conjoined analytical-topological methods, 4) the general theory of asymptotic development, the whole comprising a unique synthesis of content enfolded in the 69-page, 700-odd item bibliography, which comprises an exhaustive listing of pertinent periodical literature, including numerous papers on control systems theory; and then end with a reading of the rather abstract and well-complementing work by Bogolyubov and Mitropol'skv.[15]

In a connected area of theory, attention is to be directed to several recent texts particularly pertinent to solution of nonlinear control systems containing distributed parameter elements, which give rise to characterization by difference-differential equations.[86, 88, 99] Finally, in the thought that a powerful method of solving system problems characterized by differential or difference-differential equations is to so recouch them (often by use of variational techniques) as system problems characterized by integral equations, attention is directed to the recent excellent text on nonlinear integral equations by Smirnov[110], complementing several recent texts on linear integral equation theory, including the pertinent content in the unparalleled text on approximate methods by Kantorovich and Kryloff[151] which contains a most excellent account of variational methods originated by Ritz, Galerkin, Treffetx, Grammel and others (collocation, least squares, etc.) and yet other methods as discussed in the author's summary of these methods[152]. Finally note may be made of several works especially useful with respect to the details of the theory of parametric oscillation[35,75,112].

Background and basic knowledge so gained will prove most helpful to a rapid assimilation of the more advanced of recent control engineering texts devoted to treatment of nonlinear systems on other than the now-familiar describing-function and phase-plane approaches: either in part, through one or more chapters enfolding content such as an introduction to Lyapunov's second method[2,31,32,37,38,101,136]; or in whole, with content ranging from easy to medium difficult[2,27,34,41,61,73,113,129,130,131,153,154]; to the quite sophisticated texts by Letov[68,69] and Lurje[72,73], to which especial attention is to be directed: in that they provide, at present, the most comprehensive rigorously-based account of nonlinear control systems theory, of such basic nature that knowledge of this content ought now be held, or rapidly gained, by all seriously-interested in study and research in control engineering.

In a complementive sense, and in knowledge that much of the theory and method used to effect analysis of the operating performance of nonlinear control systems were earlier, and are now, used in other branches of technology--especially in nonlinear mechanics in the large and nonlinear vibration theory in particular and in nonlinear electric circuit analysis; that a great deal of the theory and of numerous powerful methods developed for solution of problems therein are only now coming over into control engineering (as, for example, use of Lyapunov's second method in English-language literature); and that much yet remains to be bought over (such as a more general use of abstract-space theory), the control engineer can well take-up study of what has been developed to-date in these associated fields.

A rational program therein is to start with some of the well-written, simple to moderate-difficulty, shorter texts on general treatment[28,46,61,114], then in a specialized area of nonlinear electric systems[11,14,39,54,56,57,108,122] or nonlinear mechanical systems[58-60,87,93,97,102,111,115,119,120,144,147], though some overlap occurs among the two groups. Subsequently, one would take-up the recent exhaustively-detailed accounts enfolded in the second editions of books by Minorsky[85] (this pending work ought be the most complete account in English); by Bulgakov[19] and by Andronov, Witt and Chaikin[4,5], both of which are strong in illustrative numerical examples; by Kauderer[53], which has a particularly-well-detailed account of parametrically-produced oscillations); by Malkin[80], especially strong in the more abstract phases of theory; by Mitropolskv[89], especially concerned with transient phenomena; and by Hayashi[45], which advances perhaps the most detailed account of sub-harmonic oscillatory phenomena in forced nonlinear systems.

As evidenced in a number of the above mentioned texts, a central problem in modern control systems analysis and design comprises investigation and determination of the nature of various aspects of stability; with respect both to specified desired driving inputs and to undesired disturbance inputs. To such end the considerable literature stemming from a lengthy history of work in other connections--as well detailed in the excellent history of the development of the stability of motion by Moisseev(91), can be brought over and used for solution of the problems of control theory. Thus, the algebraic methods initiated in the work of Routh(104) (among the first of modern workers) and of Hurwitz, the topological and trajectorial techniques initiated by Poincare and advanced by Birkhoff(13), and the analytic techniques founded by Liapounov(7) have in recent years been expanded by a host of workers, particularly in the USSR, to the end that there is now a very comprehensive and well-integrated body of available theory which has been set out in well-detailed texts by, especially, Soviet authors(10,21,23,30,55,78,79, 94,100,127).

A particular attention has centered in recent years on Lyapunov's methods. The control engineer desirous of gaining a firm foundation and an extensive knowledge of his approaches, and their enlargement by a host of workers (particularly by his fellow countrymen), now has available to him two recent excellent texts written especially for those interested primarily in control engineering: by Malkin(79) and by Hahn(44). The former is somewhat broader in scope and less concisely written, the latter is concerned principally with Liapounov's second, or direct, method.

To summarize: Malkin's book comprises a thorough, clearly-written treatment of Lyapunov theory, supported by numerous illustrative examples drawn from both the technical and mathematical literature, and enfolding discussion and illumination of valuable work contained in difficult-to-obtain (at least in the U.S.) sources. The context comprises six chapters, divided into three sections of successively increasing complexity and abstractness. The first three chapters enfold general account of the stability problem for equilibrium points, outlines the basic theory of the direct method thereto, and accounts its interlinking (with use of interesting geometric sketches) to Routh-Hurwitz stability criteria as stemming from consideration of the equations of first approximation. Chapters IV and V, constituting the second section, are respectively concerned with certain critical cases of equilibrium motion and with the stability of periodic oscillations (limit cycles) as characterized by linear and nonlinear periodic-coefficient equations. Chapter VI is concerned principally with certain quite advanced aspects of general theory, the theory of first approximation, the theory of stability by first approximation and with certain pertinent critical cases. Other writings by Malkin are enfolded in[77] or complement this text[78].

Hahn's eight-chapter text[144], concerned largely with analytic aspects, comprises an introductory account of basic theory and of sufficient conditions for stability or instability (I, II), exemplification by solution of various rather general problems drawn from technology, particularly control engineering theory (III), generalization and ramification of the basic theory, ranging from the inverse problem to critical cases (IV-VII), and extension to investigation of the stability of systems characterized by partial differential, difference-differential, and difference equations. A lengthy bibliography of some 98 authors, many entailing from two to a dozen or so entries, evidences the degree of the author's avowed purpose of summarizing "all pertinent publications up to and including 1957", thus digesting the most significant work up to some three years ago. The subsequent literature, 1957-1960, is already large in extent: but the particularly interested reader can easily gather it from the pertinent abstracting journals, as desired.

These two exhaustive works are quite formidable in their entirety, though the first several chapters of each can be read without too great analytic demands. Accordingly, one interested in somewhat simpler introductions, yet somewhat more advanced than is to be found in control engineering textbooks to date, might well look over the excellent survey by Antosiewicz[7], read the interesting reports by Lefschetz[66,67] and LaSalle[64], turn, as feasible, to the excellent accounts, well buttressed by numerical examples, in the writings by Hahn[41], (as editor), Reissig[103], and Zubov[127,128]; and, finally, as indeed is necessary to keep abreast in this rapidly developing area of control theory, to a systematic reading of the periodical control literature. The variety of applications and the extent and complexity of use in control theory, which Lyapunov theory has already gained in countries in which little had been written therein only several years ago is manifest in, say, the general programs of the recent IFAC Conference in Moscow and of the present Joint National Automatic Control Conference at MIT, in certain of the survey papers which will appear in the Proceedings of these Conferences, and in the large body of already published papers, on such diverse areas as electrical machines[124], nuclear reactors[142], general nonlinear control systems[150], continuous-time and discrete-time systems[51], pulsed sampled-data systems[143], to cite only a few items which the author has recently read or worked-on with interest.

In recapitulation of the foregoing it is of interest to note the very considerable degree to which the methods affording exact treatment of nonlinear systems engineering, in the various aspects of mechanics, electric circuits and control engineering, were conceived in the USSR and the rapidity with which they were developed and utilized in practice, a veiwpoint well-emphasized in LaSalle and Lefschetz's report[63]. An overall survey of this, in considerable depth, is afforded by the well-detailed review articles by Alekseeva[3], Rytov[105], Mandelstam and others[81,82] on nonlinear mechanics and electric circuits in general and by Hahn and others[40,43,74,107] on control engineering in particular; and in the accounts of the professional work of several of the more distinguished Soviet workers[6,16,36,83,112] whose names occur repeatedly above.

Finally, it may not be inappropriate to close with the rather apt quotation from the recent 48-th Wilbur Wright Memorial Lecture, "Mathematics and Aeronautics" by the distinguished English mathematician Dr. M. J. Lighthill[45] director of the Royal Aircraft Establishment, at Farnborough, England: "The great scientific and engineering advances of the present day are coming from the bringing together of widely different departments of knowledge--for example, the way in which electron microscopy has been used to solve the chemical bonds of genetics or solid state quantum theory to transform electronic circuits": and in such thought to consider the kindred way in which quite abstract theory developed earlier in nonlinear mechanics and electric circuit theory has provided a reservoir that now affords means of solution of pressing modern-day problems in nonlinear control engineering theory and practice--a reservoir, moreover, which is both not fully tapped[22] and is yet filling.[126].

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A PROBLEM IN STABILITY ANALYSIS BY DIRECT MANIPULATION OF THE EQUATION OF MOTION

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A PROBLEM IN STABILITY ANALYSIS BY DIRECT MANIPULATION OF THE EQUATION OF MOTION

INTRODUCTION

Stability analysis by the Lyapunov method is simply the problem of finding a positive definite function, V, whose time derivative, taken in the direction of the motion of a system, is non-positive. If V also becomes infinite with infinite deviation from equilibrium, stability for all possible initial conditions is assured. The failure of a function to satisfy these conditions does not necessarily imply instability.

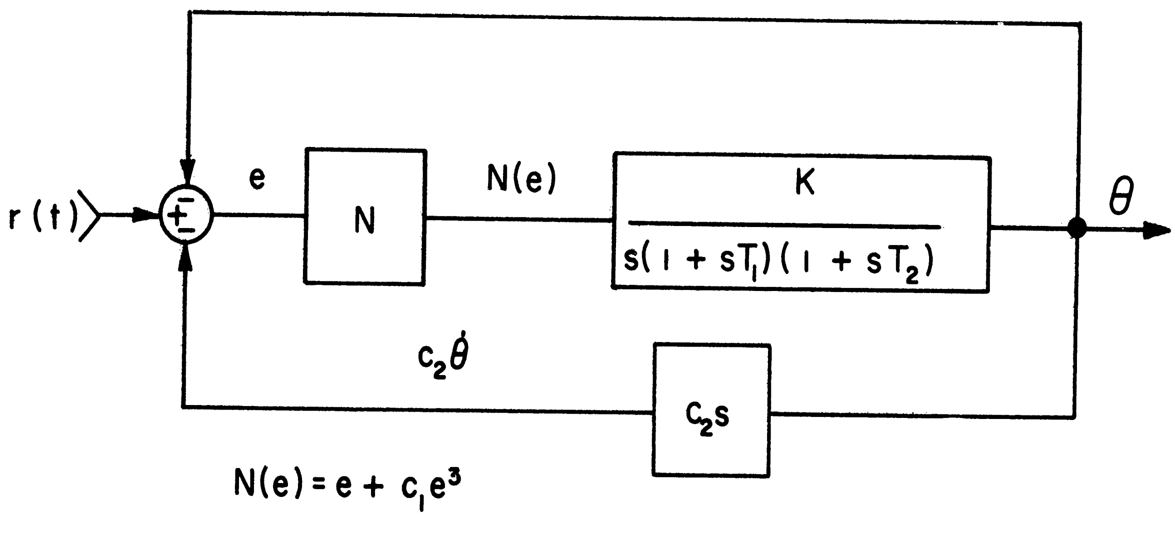
Various formal methods for producing such functions, notably the one of Lur'e, are available. All of these reduce to certain fundamental manipulations of the equation of motion. The nature of the desired operations is not always evident from the equation alone and, therefore, the formal procedures are useful.

The following problem is instructive in that it can be investigated by purely "brute force" techniques without any more knowledge of the Lyapunov method than the statement above. Such an approach is suggested. It can also be solved by the method of Lur'e and by the use of describing functions. The result from describing function analysis is given for comparison.

By carrying through the indicated operations, it is seen that the production of Lyapunov functions is just a matter of integrating parts of the differential equation. The question of which parts to integrate may be answered by trial and error and inspection of the results. Such an inductive process is useful only to gain an understanding of the principles involved. The Lyapunov function achieved in this way was originally produced by a more formal method of matrix integration, soon to be published, and it was observed that the mechanics of the method for this special case reduce to those indicated.

THE PROBLEM

The third order system shown below has a linear transfer function with three poles and no zeros preceded by a nonlinear gain which increases with increasing error. It is compensated for stability by a derivative feedback which modifies the signal to the nonlinear element. It is desired to find the conditions under which this system is stable for all inputs.



It is assumed that K, T_1, T_2, c_1 and c_2 are all positive constants. By making the substitutions;

$$b_1 = \frac{T_1 + T_2}{T_1 T_2}, \quad b_2 = \frac{1}{T_1 T_2}, \quad b_3 = \frac{K}{T_1 T_2}, \quad \text{and } b_4 = \frac{Kc_1}{T_1 T_2},$$

the differential equation of the system becomes;

$$\dddot{\theta} + b_1\ddot{\theta} + \dot{\theta} = b_3 \left[r(t)-\theta-c_2\dot{\theta}\right] + b_4 \left[r(t)-\theta-c_2\dot{\theta}\right]^3.$$

In accordance with definition (10) of the paper "Principal Definitions of Stability", stability of the equilibrium position must exist after a time t_1 if r(t) is fixed at the instantaneous value it assumes at t_1. Therefore, after t_1 the substitution; $y = \theta - r(t)$, reduces the problem to a consideration of the following equation of motion.

$$[D] = \dddot{y} + b_1\ddot{y} + (b_2 + b_3c_2)\dot{y} + b_3y + b_4y^3$$

$$+ 3b_4c_2y^2\dot{y} + 3b_4c_2^2y\dot{y}^2 + b_4c_2^3\dot{y}^3 = 0$$

In order that the system be stable for all input functions r(t), the equilibrium position of this equation must be stable for all possible initial disturbances.

CONDITIONS FOR STABILITY FROM DESCRIBING FUNCTION ANALYSIS

The describing function for the nonlinear gain is $(1 + \frac{3}{4} c_1 e^2)$. If $N(e)$ is replaced by a constant gain of this magnitude, the equation of the system, expressed in operational notation is;

$$[s^3 + b_1 s^2 + b_2 s + c_2(b_3 + \tfrac{3}{4} b_4 e^2) s + (b_3 + \tfrac{3}{4} b_4 e^2)] \Theta$$

$$= (b_3 + \tfrac{3}{4} b_4 e^2) \; R\,(s)$$

The Routh-Hurwitz criterion predicts stability for this when;

$$[b_1 b_2 + (b_1 c_2 - 1)\,(b_3 + \tfrac{3}{4} b_4 e^2)] > 0$$

In terms of the variables in the previous equation, this expression becomes;

$$b_1 b_2 + (b_1 c_2 - 1)\,[b_3 + \tfrac{3}{4} b_4 \,(y_0 + c_2 \dot{y}_0)^2] > 0,$$

where y_0 and $\dot{y}_0$ are interpreted as the peak magnitudes of the fundamental components of the variables y and $\dot{y}$. The condition for stability is placed in this analytic form for comparison with the result derived by the Lyapunov method.

A MEANS FOR FINDING A LYAPUNOV FUNCTION

It is well known that integrals for certain differential equations can be found by multiplying the equation by an appropriate integrating factor. The integrating factors are usually obtained by a combination of inspection and guesswork. A common practice in second order, conservative systems is to obtain an energy integral by multiplying by the derivative of the dependent variable. In higher order systems the integrating factor is more complicated.

As an example of the way this may be applied to the problem, let the equation of motion be divided into linear and nonlinear parts.

$$[L] = \dddot{y} + b_1\ddot{y} + (b_2 + b_3c_2)\dot{y} + b_3y$$

$$[N] = b_4\ (y^3 + 3c_2y^2\dot{y} + 3c_2{}^2y\dot{y}^2 + c_2{}^3\dot{y}^3)$$

First consider the equation $[L] = 0$. It may be observed that the first and third terms can be integrated by multiplying by $\ddot{y}$. Therefore,

$$\ddot{y}[L] = \frac{d}{dt}\left[\frac{\ddot{y}^2}{2} + (b_2 + b_3c_2)\frac{\dot{y}^2}{2}\right] + b_1\ddot{y}^2 + b_3y\ddot{y} = 0.$$

The term in the brackets might be selected as a candidate for a Lyapunov function for the linear equation except that it is only semi-definite. Another integration can also be accomplished by multiplication by $\dot{y}$.

$$\dot{y}[L] = \frac{d}{dt}\left[\frac{b_1\dot{y}^2}{2} + \frac{b_3y^2}{2}\right] + \dddot{y}\ \dot{y} + (b_2 + b_3c_2)\dot{y}^2 = 0$$

Here the bracketed term also has some of the desired characteristics, although it is also semi-definite. The next logical step is to try a combination of the equations above.

$$(b_1\dot{y} + \ddot{y})\,[L] = \frac{d}{dt}\left[\frac{\ddot{y}^2}{2} + (b_1^{\,2} + b_2 + b_3c_2)\frac{\dot{y}^2}{2} + b_3\frac{y^2}{2}\right]$$

$$+ b_1\dddot{y}\,\dot{y} + b_3y\ddot{y} + b_1\ddot{y}^2 + b_1(b_2 + b_3c_2)\dot{y}^2 = 0$$

The term in the bracket is positive definite but since $[L] = 0$, its derivative is $-(b_1\dddot{y}\,\dot{y} + b_3y\ddot{y} + b_1\ddot{y}^2 + b_1(b_2 + b_3c_2)\dot{y}^2)$, which cannot be non-positive. However, the undesirable terms in the derivative can be converted to a more convenient form by an integration by parts, giving;

$$(b_1\dot{y} + \ddot{y})\,[L] = \frac{d}{dt}\left[\frac{\ddot{y}^2}{2} + b_1\ddot{y}\dot{y} + (b_1^{\,2} + b_2 + b_3c_2)\frac{\dot{y}^2}{2} + b_3y\dot{y} + b_1b_2\frac{y^2}{2}\right]$$

$$+ (b_1b_2 + b_1b_3c_2 - b_3)\dot{y}^2 = 0.$$

Now, if the bracketed expression is chosen as a Lyapunov function, its derivative is, $-(b_1b_2 + b_1b_3c_2 - b_3)\dot{y}^2$, which is non-positive when the coefficient in parentheses is positive. From Sylvester's inequalities* it may be found that this is the same condition under which the bracketed expression is positive definite. Thus, a Lyapunov function and its derivative for the equation, $[L] = 0$, are;

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* See page 11 of "The 'Direct Method' of Lyapunov in the Analysis and Design of Discrete-Time Control Systems", J. E. Bertram.

$$V_L = \frac{\ddot{y}^2}{2} + b_1\ddot{y}\dot{y} + (b_1^2 + b_2 + b_3c_2)\frac{\dot{y}^2}{2} + b_3y\dot{y} + b_1b_3\frac{y^2}{2}$$

$$\dot{V}_L = -(b_1b_2 + b_1b_3c_2 - b_3)\,\dot{y}^2.$$

The remainder of the problem consists in applying similar reasoning to the equation, $[D] = 0$. In this case, the same integrating factor may be used for the nonlinear part of the equation as is used above for the linear part.

A SOLUTION

By applying the operations indicated for the linear part of the equation to the nonlinear part, and eliminating the undesirable remaining terms through an integration by parts, it can be deduced that;

$$(b_1\dot{y} + \ddot{y})\,[N] = \frac{d}{dt}\Big[b_4c_2^3\frac{\dot{y}^4}{4} + b_4c_2^2y\dot{y}^3 + \frac{3}{2}c_2b_4y^2\dot{y}^2$$

$$+ b_4y^3\dot{y} + b_1b_4\frac{y^4}{4}\Big]$$

$$+ b_4(b_1c_2-1)(3y^2 + 3c_2\dot{y}y + c_2^2\dot{y}^2)\,\dot{y}^2.$$

That the variable term in parentheses is semi-definite is verified by Sylvester's inequalities. Call the bracketed expression V_N.

This is monotonically increasing in y and $\dot{y}$ as can be verified by observing that

$$\frac{\partial V_N}{\partial y} \text{ and } \frac{\partial V_N}{\partial \dot{y}}$$

have zeroes only for zero values of y and $\dot{y}$. Therefore, V_N is semi-definite.

As a Lyapunov function for the equation, $[D] = 0$, the sum of the functions V_L and V_N may be used. Let $V_N + V_L = V$. Then $(b_1\dot{y} + \ddot{y})[D] = \frac{dV}{dt} - \dot{V} = 0$, where;

$$V = b_1 b_3 \frac{y^2}{2} + b_3 y\dot{y} + (b_1^2 + b_2 + c_2 b_3) \frac{\dot{y}^2}{2} + b_1 \dot{y}\ddot{y} + \frac{\ddot{y}^2}{2}$$

$$+ b_1 b_4 \frac{y^4}{4} + b_4 y^3 \dot{y} + \frac{3}{2} c_2 b_4 y^2 \dot{y}^2 + b_4 c_2^2 y \dot{y}^3 + b_4 c_2^3 \frac{\dot{y}^4}{4}$$

and;

$$\dot{V} = -\left\{ b_1 b_2 + (b_1 c_2 - 1) \left[b_3 + b_4 (3y^2 + 3c_2 y\dot{y} + c_2^2 \dot{y}^2) \right] \right\} \dot{y}^2.$$

$\dot{V}$ is non-positive for;

$$b_1 b_2 + (b_1 c_2 - 1) \left[b_3 + b_4 (3y^2 + 3c_2 y\dot{y} + c_2^2 \dot{y}^2) \right] > 0.$$

This inequality may be compared with the one obtained by the use of describing functions. In both cases stability exists for all inputs when;

$$(b_1 c_2 - 1) > 0.$$

APPLICATION OF LIAPUNOV'S SECOND METHOD TO CONTROL SYSTEMS WITH NONLINEAR GAIN

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I. Introduction

The major difficulty in applying Liapunov's "Second Method" to the analysis of practical control systems is due to the lack of a straightforward procedure of finding a Liapunov's function (i.e. a function of the system variables satisfying Liapunov's stability or instability theorems). However, several Liapunov's functions have been developed that apply to a large group of control systems that can be described by the so-called "first canonic form" of system differential equations. The first canonic form is defined in Section II. As is shown in Section II any autonomous closed-loop system with a single nonlinear gain element can be described by the first canonic form of differential equations. Several Liapunov's functions applicable to systems expressed in the first canonic form of differential equations are outlined in Section III. These functions enable one to establish sufficient conditions for asymptotic stability of such systems.

The systems to which the procedure of stability analysis presented in this paper is applicable can be represented by the block diagram shown in Figure 1.

It is assumed that the input into the system, $r(t)$, is removed at time $t = 0$, i.e.,

$$r(t) = 0 \qquad \text{for all} \qquad t > 0. \tag{1.1}$$

Under the above assumption the block diagram of the system can be simplified as shown in Figure 2.

It will also be assumed that the input-output characteristics of the nonlinear gain element can be described by a continuous, single-valued function;

$$y = f(x); \qquad\qquad f(o) = 0, \tag{1.2}$$

where x is the input into and y the output of the nonlinear element.

II. Canonic Transformations

Consider a closed-loop servo system described by a set of differential equations

$$\frac{dz_i}{dt} = \lambda_i z_i \quad + \quad f(x) \qquad i = 1, 2, \dots n, \tag{2.1a}$$

$$x = \sum_{i=1}^{n} a_1 \, z_1 \tag{2.1b}$$

and

$$\frac{dx}{dt} = \sum_{i=1}^{n} \beta_i \quad z_i \; - \; rf(x). \tag{2.1c}$$

This form of differential equations is called the First Canonic Form (or Lur'e's First Canonic Form) of differential equations (Ref. 1, p. 1357). It may be used to represent a closed-loop system with a single nonlinear gain element, and with the driving function removed at time $t = 0$. The block diagram of such a system is shown in Figure 2.

To show that Equation (2.1) actually represents the system of Figure 2, let

$$D \triangleq \frac{d}{dt} \tag{2.2}$$

Then, from Equation (2.1a) and Equation (2.1b) one finds

$$(D - \lambda_i) \quad z_i = y \qquad i = 1, 2, \dots n \tag{2.3}$$

and

$$x = \sum_{i=1}^{n} a_i \quad z_i, \tag{2.4}$$

where

$$y = f(x) \tag{1.2}$$

represents the nonlinear element characteristics.

Solving Equation (2.3) for z_i and substituting into Equation (2.4) one obtains

$$\frac{x}{y} = \sum_{i=1}^{n} \quad \frac{a_i}{D - \lambda_i} \; . \tag{2.5}$$

Note that the loop transfer function of the system of Figure 2 is

$$G(s) = G_1(s)\; G_2(s) = -\frac{X(s)}{Y(s)} . \qquad (2.6)$$

Consequently, from Equation (2.5) and Equation (2.6) the loop transfer function is

$$G(s) = -\sum_{i=1}^{n} \frac{a_i}{s - \lambda_i} \qquad (2.7)$$

Equation 2.7 indicates that the constants λ_i are the poles of the loop transfer function G(s) at the corresponding poles. Thus the first canonic form of differential equations for the system of Figure 2 (or that of Figure 1 with either constant or zero input) can be obtained from the partial fraction expansion of the loop transfer function G(s). To complete the transformation of system differential equations into the first canonic form one may differentiate Equation 2.1b with respect to time and then substitute Equation 2.1a. This procedure yields

$$\beta_i = a_i \lambda_i \qquad i = 1, 2, \ldots n, \qquad (2.8)$$

and

$$r = -\sum_{i=1}^{n} a_i . \qquad (2.9)$$

Once the numerical values for the coefficients λ_i, a_i, β_i and r have been calculated it may be possible to prove the stability of a system by means of one of the Liapunov's functions discussed in the next section.

III. Liapunov's Function

Lur'e has considered the function

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i\, a_j\, z_i\, z_j}{\lambda_i + \lambda_j} - \int_0^x f(a)\, da \qquad (3.1)$$

as a Liapunov's function for systems described by the first canonic form of differential equations. It can be shown* that this V-function is negative definite** if the following inequality is satisfied:

$$\int_0^x f\,(a)\quad da \geqslant 0. \tag{3.2}$$

provided that the constants a_i are real for corresponding real λ_i's and are in pairs of complex conjugates for corresponding complex conjugate pairs of λ_i's and that Re $\lambda_i < 0$.

The time derivative of this Liapunov's function, in connection with the first canonic form of system differential equations, is

$$\frac{dV}{dt} = r\, f\,(x)^2 + (\sum_{i=1}^{n} a_1 z_i)^2 -$$

$$-f(x) \sum_{i=1}^{n} z_i \left(\beta_i - 2\, a_i \sum_{i=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} \right). \tag{3.3}$$

The time derivative of this Liapunov's function [Equation (3.3)] can be made positive semidefinite***by letting

$$2\, a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} + \beta_i \quad i = 1, 2, \ldots n. \tag{3.4}$$

Lur'e has also shown that by adding to the Liapunov's function of Equation (3.1), the term

$$\emptyset = A_1 z_1^{\,2} + A_2 z_2^{\,2} + \ldots . + A_s z_s^2 + C_1\ z_{s+1}\ z_{s+2}$$

$$+ C_3\ z_{s+3}\ z_{s+4} + \ldots\ C_{s-n-1}\ z_{n-1}\ z_n \tag{3.5}$$

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*Ref. 2, p.46

*i.e., $\dot{V}$ is negative everywhere in the phase space of the variable x, except at the origin where it is equal to zero.

***I.e., $\frac{dV}{dt}$ is non-negative everywhere in the phase space of the system variable x.

where the constants A and C are infinitesimally small negative numbers, the time derivative of the Liapunov's function [Equation (3.3)] can be made positive definite. Consequently the application of Liapunov's stability theorem leads to the following stability theorem known as

Lur'e's Theorem:*

If a system described by Equation (2.1) satisfies the following conditions:

a. There exists at least one solution of a set of stability equations [Equation (3.4)] such that a_i are real for corresponding real λ_i's and are in pairs of complex conjugates for corresponding complex conjugate pairs of λ_i;

b. $\int_0^x f(a)\, da > 0$; $f(o) = 0$;

c. the constant $r > 0$,

d. Re $\lambda_i < 0$ for all $i=1,2, \dots n$;

then the system is globally asymptotically stable. Local asymptotic stability can also be established by means of Lure's theorem, if there is a range of values of x, containing the equilibrium state, over which Equation (3.2) is satisfied.

The preceding stability equation [Equation (3.4)] may frequently reject systems that are actually stable since it puts too many restrictions on the system. Since Lur'e's theorem represents sufficient conditions for asymptotic stability, which may not always be necessary conditions for stability, it is possible to relax the requirements of Lur'e's theorem considerably, thus making it applicable to a greater number of stable systems.

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*Ref. 2, p. 51

By adding to and substracting from Equation (3.3) the quantity

$$2\sqrt{r}\ f(x) \sum_{i=1}^{n} a_i z_i$$

and then selecting as stability equations

$$2\, a_i \left(\sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} - \sqrt{r} \right) = \beta_i \quad i=1,2, \ldots n \tag{3.6}$$

one obtains

$$\frac{dV}{dt} = \left[\sqrt{r}\ f(x) + \sum_{i=1}^{n} a_i\, z_i \right]^2 \tag{3.7}$$

Consequently, Equation (3.6) can also be used as a stability equation in Lur'e's Theorem. In other words the roots a_i of Equation (3.6) can be used instead of the roots a_i of Equation (3.4) to prove that a system is stable by the use of Lur'e's Theorem.

The exact solution of Equation (3.6) is not known for higher order systems ($n > 3$). It is sometimes possible to find approximate values of the roots a_i of Equation (3.6) by rewriting this equation as

$$a_i^2 = \frac{\lambda_i}{2} \left[\beta_i + 2\sqrt{r}\ a_i - 2\, a_i \sum_{j=2}^{n} \frac{a_j}{\lambda_i + \lambda_j} \right]; \ i=1,2,..n \tag{3.8}$$

then assuming the values of a_i and a_j on the right-hand side of Equation (3.8), solving for a_i and repeating the procedure until the change in the values of a_i became negligible. The relationship*

$$\sum_{j=1}^{n} \frac{a_j}{\lambda_j} = \sqrt{\sum_{j=1}^{n} \frac{\beta_j}{\lambda_j} + r} + \sqrt{r} \tag{3.9}$$

can be used to check the answer.

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*Ref. 2, p. 55

Lur'e also considered the function

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i \; a_j}{\lambda_i + \lambda_j} \quad z_i \, z_j \tag{3.10}$$

as a possible Liapunov's function in connection with the first canonic form of differential equations and obtained the stability equation

$$2 \, a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} = a_i ; \; i = 1, 2, \ldots . \; n. \tag{3.11}$$

A system was shown to be asymptotically stable if:

a. the roots a_i of Equation (3.11) satisfy the requirements of Lur'e's stability theorem,

b. Re $\lambda_i < 0$ for all $i = 1, 2, \ldots n$,

c. $x \, f(x) \gtrless 0$ for all $|x| > 0$; $f(o) = 0$.

Various other simplified stability criteria (i.e., other stability equations based on the above two Liapunov's functions as well as other V-functions) have been successfully applied to prove stability of closed-loop systems with a single nonlinear element. References 1 through 4 contain many examples of such simplified stability criteria.

Many stable systems will be rejected by the simplified stability criteria [Equations (3.4, 3.6, and 3.11)] presented in this outline due to rather weak restrictions on nonlinear gain characteristics. A procedure whereby the nonlinear element characteristics are restricted to a much smaller region of its input-output plane is reported in Reference 3.* Such restrictions decrease considerably the number of actually stable closed-loop systems which otherwise would have been rejected by the V-functions of Section 3 of this paper.

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*It will also be included in Technical Report No. 3, to be published in October, 1960 by Purdue University, School of Electrical Engineering under Contract No. AF 29(600)-1933 from the Holloman Air Force Base, New Mexico.

It must be emphasized, however, that the problem of finding Liapunov's functions which would yield necessary conditions for stability of the systems shown in Figure 1 in case of higher order systems $(n > 3)$ has not been solved. Hence, if one set of stability equations rejects a system, it does not mean that the system is definitely unstable, since a different set of stability equations may be used to prove that such a system is stable.

IV. Problems

Problem 1:

Consider a closed loop system shown in Figure 1 with

$$G_1(s) = 1$$

$$G_2(s) = \frac{1}{s(s+1)(s+2)}$$

and the nonlinear element being a saturating amplifier. The input-output characteristics of the amplifier [Figure (3)] satisfy the inequality

$x\, f(x) \gtrdot 0$ for all $|x| > 0$; $f(o) = 0$.

Let the driving function $r(t)$ be removed at time $t = 0$.

a. Arrange the differential equations describing this system for time $t > 0$ into the canonic form

$$\frac{dz_i}{dt} = \lambda_i z_i + f(x) \qquad i = 1,2, \ldots n,$$

and

$$x = \sum_{i=1}^{n} a_i z_i.$$

b. Find also

$$\frac{dx}{dt} = ?$$

Find numeric values for all the coefficients of the canonic equations.

Problem 2:

Consider the closed-loop servo system described by the canonic equations

$$\frac{dz_1}{dt} = -2z_1 + f(x)$$

$$\frac{dz_2}{dt} = -3z_2 + f(x)$$

$$\frac{dz_3}{dt} = -5z_3 + f(x)$$

$$x = 0.333\, z_1 = z_2 + 0.667\, z_3.$$

Find the loop transfer function $g(s)$ if $r(t) = 0$ for all $t > 0$.

Problem 3:

a. Consider the V-function of Equation (3.1) as a possible Liapunov's function for the system of Problem 2. Assume that the nonlinear element is a saturating amplifier with the input-output characteristics as shown in Figure 3. What conclusions can you draw about the stability of the system of Problem 2?

b. Try

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i\, a_j\, z_i\, z_j}{\lambda_1 + \lambda_j}$$

as a possible Liapunov's function for the system of Problem 2. Can you draw any stability conclusion from this V-function and its time derivative $\frac{dV}{dt}$?

Problem 4:

The V-functions of Section III cannot be used as Liapunov's functions for the system of Example 1. This is due to the fact that

$$\lambda_1 = 0.$$

In such cases where one of the poles of the loop transfer function G(s) is at the origin of the s-plane, the function

$$V = \frac{1}{2} \; a_1 z_1^2 + \sum_{i=2}^{n} \sum_{j=2}^{n} \frac{a_i \, a_j}{\lambda_i + \lambda_j} \; z_i \, z_j - \int_0^x f\,(a) \; da$$

may frequently be used as a Liapunov's function.

a. What conclusions can you draw about the stability of the system of Problem 1 from this V-function?

b. The results of part (a) should not be surprising. Can you give some reasons explaining the result of part (a)? (Hint: replace the nonlinear element by a linear amplifier, i.e., let y =kx).

Problem 5:

Consider the system shown in Figure 1 with

$$G(s) = \frac{s + 1}{(s+2)\;(s+3)}$$

and a saturating nonlinear element, the imput-output characteristic of which is shown in Figure 3.

a. Find the region of the nonlinear element input-output characteristic plane [Figure (4)] over which the system is stable.

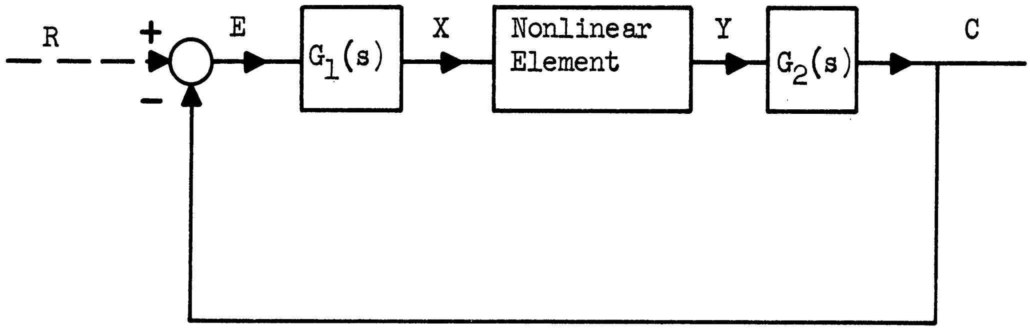

Fig. 1 Block Diagram of a Closed-Loop System with a Single Nonlinear Element

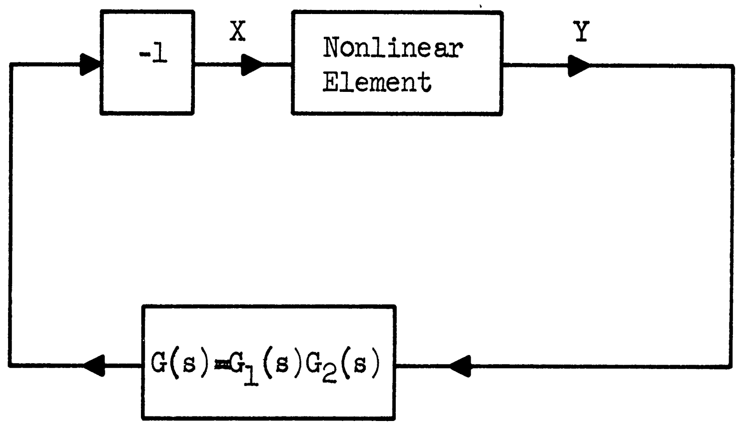

Fig. 2 Simplified Block Diagram of a Closed-Loop System with a Single Nonlinear Element

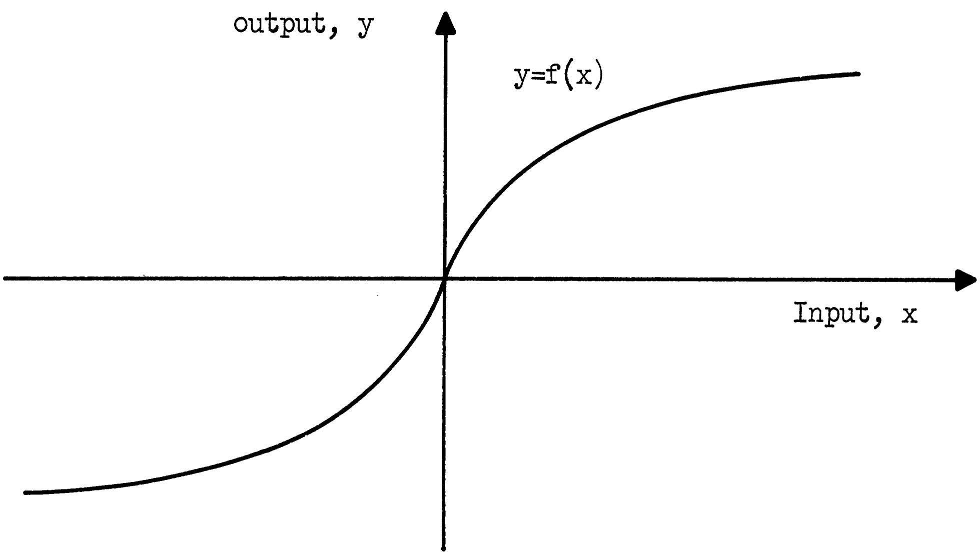

Fig. 3 Input-Output Characteristics of a Saturating Amplifier

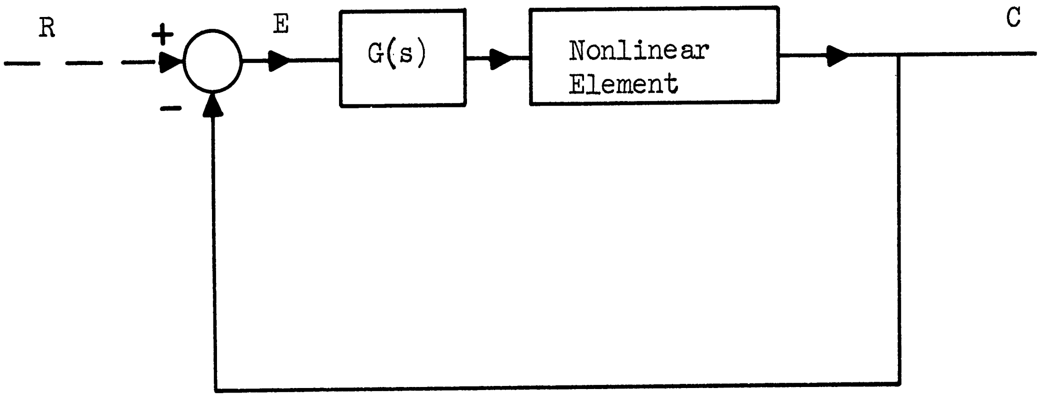

Fig. 4 Block Diagram of the System of Problem 2.

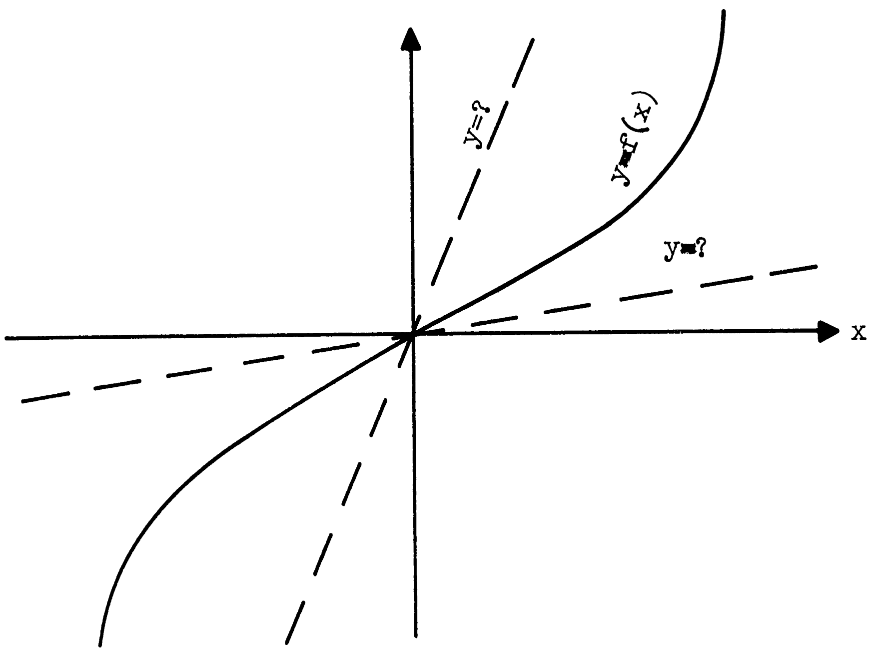

Fig. 5 Allowable region of the amplifier gain for the system of problem 5.

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SOLUTIONS OF THE WORKSHOP PROBLEMS

Problem 1:

a) The loop transfer function of this system is

$$G(s) = \frac{1}{s(s+1)\ (s+2)}$$

Expanding this transfer function into partial fractions one obtains

$$G(s) = \frac{0.5}{s} - \frac{1}{s+1} + \frac{0.5}{s+2}$$

Consequently, from Equation 2.7 and Equation 2.1 the canonic form of differential equations for this system becomes

$$\frac{dz_1}{dt} = f(x),$$

$$\frac{dz_2}{dt} = -\ z_2 + f(x),$$

$$\frac{dz_3}{dt} = -\ 2\ z_3 + f(x),$$

and

$$x = -\ 0.5\ z_1 + z_2 - 0.5\ z_3.$$

b) Differentiation of the last equation with respect to time and substitution of the preceding three equations yields

$$\frac{dx}{dt} = -\ z_2 + z_3.$$

Problem 2:

The canonic equations can be rewritten in operational notation as follows:

$$z_1 = \frac{y}{D+2}$$

$$z_2 = \frac{y}{D+3}$$

$$z_3 = \frac{y}{D+5}$$

and

$$x = 0.333\ z_1 - z_2 + 0.667\ z_3$$

where

$$y = f(x)$$

represents the output of the nonlinear element. Eliminating the canonic variables among the above four equations one obtains

$$x = \left[\frac{0.333}{D+2} - \frac{1}{D+3} + \frac{0.667}{D+5} \right] y,$$

or, from Equation 2.7,

$$G(s) = -\frac{X(s)}{Y(s)} = \frac{-0.333}{s+2} + \frac{1}{s+3} - \frac{0.667}{s+5} .$$

Thus

$$G(s) = \frac{s+1}{(s+2)\ (s+3)\ (s+5)}$$

Problem 3:

a) The V-function of Equation 3.1 implies the use of either Equation 3.4 or Equation 3.6 as the stability equations for the system. Note that, from Equation 2.8, 2.9 and 2.1a

$$\frac{dx}{dt} = -\ 0.667\ z_1 + 3.000\ z_2 - 3.333\ z_3 + 0.000\ y.$$

Hence

$$r = 0$$

and consequently Equation 3.6 is, for this system, identical to Equation 3.4. Substitution of the numerical values in Equation 3.4 yields:

$$0.500\ a_1^2 + 0.400\ a_1 a_2 + 0.286\ a_1 a_3 = 0.667,$$

$$0.400\ a_1 a_2 + 0.333\ a_2^2 + 0.250\ a_2 a_3 = -3.000,$$

$$0.286\ a_1 a_3 + 0.250\ a_2 a_3 + 0.200\ a_3^2 = 3.333.$$

Simultaneous solution of the above three equations yield the constants

$$a_1 = + 3.333,$$

$$a_2 = - 12.000,$$

$$a_3 = + 11.667.$$

Hence the requirements of Lur'e's Theorem are satisfied and thus the system is globally asymptotically stable.

b) The suggested V-function implies the use of Equation 3.11 as the stability equation. Substitution of numerical values into Equation 3.11 yields

$$0.500\ a_1^2 + 0.400\ a_1a_2 + 0.286\ a_1a_3 = - 0.333,$$

$$0.400\ a_1a_2 + 0.333\ a_2^2 + 0.250\ a_2a_3 = 1.000,$$

$$0.286\ a_1a_3 + 0.250\ a_2a_3 + 0.200\ a_2^2 = 0.667.$$

Simultaneous solution of the above equations yields

$$a_1 = 1.340 - j0.513,$$

$$a_2 = - 4.360 + j2.720,$$

$$a_3 = 3.033 - j3.210.$$

Since all λ's of this system are real and all a's are complex, the V-function used in this part rejects this system, even though it was shown in part (a) that the system is actually stable.

Problem 4:

a) The time derivative of the proposed V-function, corresponding to the first canonic form of system differential equations is

$$\frac{dV}{dt} = \left(\sum_{i=2}^{n} a_i z_i \right)^2 + f(x) \left[(a_1 - \beta_1) z_i + \sum_{i=2}^{n} Z_i \right.$$

$$\left. \left(2\, a_i \sum_{i=2}^{n} \frac{a_j}{\lambda_i + \lambda_j} - \beta_i \right) \right] + r \left[f(x) \right]^2 .$$

The stability equations, obtained by setting the term in $f(x)$ equal to zero, are

$$a_1 = \beta_1$$

and

$$2\, a_i \sum_{j=2}^{n} \frac{a_j}{\lambda_i + \lambda_j} = \beta_i$$

Substituting the numerical values (from the solution of Problem 1) one obtains

$$a_1 = 0$$

and

$$- a_2^2 - 0.667\, a_2 a_3 = - 1.000,$$

$$- 0.667\, a_2 a_3 - 0.500\, a_3^2 = 1.000.$$

Hence

$$a_2 = - 1.414 - j1.000,$$

$$a_3 = - 1.414 + j2.000.$$

Since λ_2 and λ_3 are real while a_2 and a_3 are complex, the stability equations reject this system.

b) If the V-function used in this problem could prove that this system is stable, then it should also prove that this system remains stable if the nonlinear amplifier is replaced by a linear amplifier with a positive gain K. That is, it should prove that the system, described by the equations

$$G(s) = \frac{1}{s(s+1)\ (s+2)}$$

and

$$y = kx; \quad 0 \leqslant k \leqslant \infty$$

is also stable. This is obviously impossible, since the linearized system is unstable for sufficiently high values of gain K.

Problem 5:

Expanding the loop transfer function G(s) into partial fractions one obtains

$$G(s) = \frac{-1}{s+2} + \frac{2}{s+3} .$$

Hence, from Equation 2.1, the canonic equations for this system are:

$$\frac{dz_1}{dt} = -\ 2z_1 + f(x)$$

$$\frac{dz_2}{dt} = -\ 3z_2 + f(x)$$

$$x = z_1 - 2\ z_2$$

and

$$\frac{dx}{dt} = -\ 2\ z_1 + 6\ z_2 - f(x)$$

Try Equation 3.10 as a possible Liapunov's function for this system. From the stability equation (Equation 3.11) one obtains

$$- 0.5\ a_1^{\ 2} - 0.400\ a_1 a_2 = 1.000$$

$$- 0.400\ a_1 a_2 - 0.333\ a_2^{\ 2} = - 2.000.$$

Simultaneous solution of the above equations yields

$$a_1 = - 6.448$$

$$a_2 = + 3.552.$$

Consequently, from Lur'e's Theorem, this system will be globally asymptotically stable if

$$x\ f(x) \geqslant 0 \text{ for all } |x| > 0; \quad f(o) = 0.$$

This means that for global asymptotic stability the input-output characteristics of the nonlinear element shall be confined to the 1st and 3rd quadrants of the x-y plant.