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Problems on Beams.

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I find in *Weisbach's Mechanics and Engineering*, vol. i. p. 208, that when a beam is loaded over part of its length, the maximum moment—and hence the *dangerous section*—is *assumed* to be at the centre of the load. I find by investigation that this assumption is not correct.

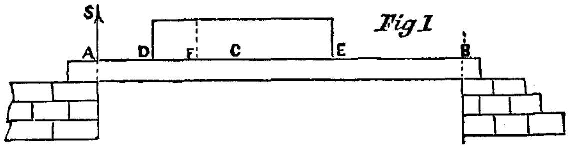


Fig 1

Let $l =$ AB $=$ the length of the beam.

$2a =$ DE $=$ length of the load, which is uniformly distributed over DE.

C be the centre of the load.

$l_1 =$ AC $\qquad l_2 =$ CB.

$w =$ the load on a unit of length, say 1 inch.

$x =$ AF $=$ the distance to any section.

S $=$ the re-action of the support.

Then

$$\text{AD} = l_1 - a \qquad \text{DF} = x - \text{AD} = x - l_1 + a.$$

Load on $\text{DF} = w(x - l_1 + a)$.

" $\quad \text{DE} = 2wa$.

By the principle of moments we have

$$\text{S}l = 2wa . l_2 \quad \therefore \quad \text{S} = 2wa\frac{l_2}{l}.$$

Observing that the lever arm of the load on DF is $\frac{1}{2}$ DF, we find that the moment of strain on the section at F is

$$\text{S}x - \frac{w}{2}(x - l_1 + a)^2,$$

$$\text{or,} \quad \frac{2wal_2}{l}x - \frac{w}{2}(x - l_1 + a)^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which we wish to make a maximum. Differentiate and place equal zero, and we have

$$\frac{2wal_2}{l} - w(x - l_1 + a) = 0.$$

We have also $l_1 + l_2 = l \quad \therefore \quad x = a\left(1 - \frac{2l_1}{l}\right) + l_1 \quad . \quad . \quad (2)$

Hence if $l_1 < \frac{1}{2}l \quad ; \quad x > l_1$

$l_1 = \frac{1}{2}l \qquad x = l_1$

$l_1 > \frac{1}{2}l \qquad x < l_1$

So that the maximum strain is at the centre of the loading only when the centre of the loading is over the middle of the beam; and *in all other cases it is nearer the middle of the beam than the centre of the load is.*

To find the maximum strain, substitute the value of x, Eq. (2), in expression (1).

Weisbach, vol. i. p. 209, gives the following example:

"What load may a hollow cast-iron beam sustain. if its outer depth and breadth be 8 inches and 4 inches, and inner depth and breadth 6 inches and 2 inches; and if, further, the middle of the load, uniformly distributed over 3 feet in length, is distant from one support 4, and the other 2 feet?"

If $\frac{1}{6}R = 1000$ lbs., we shall have, for the strength of the *dangerous section*, from the well known formula,

$$\frac{1}{6}R\,\frac{bd^3 - b_1d_1^{\,3}}{d} = 1000\,\frac{4 \times 512 - 2 \times 216}{8} = 202000.$$

On the hypothesis that the *dangerous section* is at the centre of the loading, we make $x = l_1$ in (1), which reduces it to $\frac{2wal_1l_2}{l} - \frac{wa^2}{2}$ which must equal 202000, or by substituting l_1, l_2 and a we have $23wa = 202000$.

$$\therefore \quad 2wa = 17565 \text{ lbs. as found by Weisbach.}$$

But from Eq. (2) we find that the *dangerous section* is

$$x = \tfrac{3}{2}\left(1 - \tfrac{2}{3}\right) + 2 = 2\tfrac{1}{2} \text{ ft.} = 30 \text{ inches from A.}$$

This substituted in (1) gives

$$432w = 202000$$

$$\therefore \quad w = \frac{202000}{432}.$$

$$\therefore \quad 2aw = 36w = \frac{202000 \times 36}{432} = 16833 \text{ lbs.}$$

Hence Weisbach's hypothesis gives too much by $17565 - 16833 = 732$ lbs., which is only a little more than 4 per cent. too much; a quantity which is fully provided for in the small co-efficient for rupture.

From Eq. (2) we see that the distance between the centre of the loading and the section of maximum strain is

$$x - l_1 = a\left(1 - \frac{2l_1}{l}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let $AD = y = l_1 - a \quad \therefore \quad a = l_1 - y$, which substituted in (3) gives

$$x - l_1 = (l_1 - y)\left(1 - \frac{2l_1}{l}\right)$$

which is evidently a maximum for $y = 0$, hence Eq. (3) is a maximum so far as the distance AD is concerned, when it (AD) is zero; or when one end of the load is over the support. For this condition $a = l_1$, which reduces (3) to

$$x - l_1 = l_1\left(1 - \frac{2l_1}{l}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now, consider l_1 as the variable, and we find that (4) is a maximum for $l_1 = \frac{1}{4}l$; or, $2l_1 = \frac{1}{2}l$; or the load must extend to the middle of the beam. For these conditions make $a = l_1 = \frac{1}{4}l$ in (3), (2), and (1), and we have

$$x - l_1 = \tfrac{1}{8}l$$
$$x = \tfrac{3}{8}l.$$

The maximum moment Eq. (1) is $= \frac{9}{128} wl^2 = w\frac{1}{2}l \times \frac{9}{64}l = \frac{9}{64} \text{W}l$, in which W is the total load on one-half the beam.

The strain at the middle of the loading, when the load extends from one end to the middle of the beam, is $\frac{1}{8}\text{W}l$; *hence*, the *maximum strain* is $\frac{9}{64}\text{W}l \div \frac{1}{8}\text{W}l = 1\frac{1}{8}$ *times that at the middle of the loading, when the load extends from one support to the middle of the beam.*

Of the Beam fixed at both ends, and a load midway between the fixed points.

Much has been said upon this case, on account of the discrepancy between theory and the results of experiment. Those who are familiar with the analytical solution of the problem, know that analysis shows that the beam is equally liable to break at the middle and at the ends, and that the moment of the strain at each of these three points is $\frac{1}{8}\text{P}l$, in which

P = the load at the middle.

l = the distance between the fixed points.

If the beam be *supported* at its ends, the strain would be $\frac{1}{4}\text{P}l$; hence the strains are as 2 to 1.

Barlow, in his Treatise on the Strength of Materials, pp. 35 and 138, says their strengths are to each other as 3 to 2.

On pages 125 and 126 of the same work, he explains how the experiments were made.

I do not propose to solve the problem, but make these remarks for those who are familiar with it. Barlow, in the work referred to, attempts a solution in which he deduces the ratio already given, viz: 3 to 2, but I think an error has crept into his reasoning, which I will proceed to point out. He says (Art. 53) "it is shown that it requires *four* times the weight to produce the same deflection in the beam supported at each end, as is requisite to produce the same quantity in a beam of half the length."

Fig 2

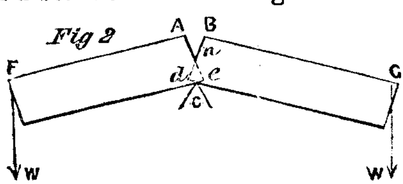
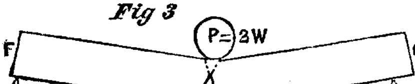


Fig 3



By referring to Article 35, I find that he arrives at this conclusion by assuming that in the deflection the fibres are nowhere elongated or compressed, except those directly over the support C, fig. (2); or under the weight P, fig. (3). With this view, his conclusion is correct: but the view is erroneous. For all the fibres from A to G, on one side, are elongated; and, on the other, compressed. During