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ON THE NEW QUASIPARTICLE FACTORIZATION OF THE j-SHELL

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Abstract: The new Elliott-Evans classification scheme by means of a quasiparticle factorization of the j-shell of both protons and neutrons is developed further. The needed reduced matrix elements follow from the equivalence between quasiparticle isospins and quasiparticle quasispins. The transformation to states of good particle number n is simplified by imbedding the quasiparticle quasispins in the five-dimensional quasispin group R(5). This leads to a factoring of the transformation coefficients. One factor is independent of J and other subgroup labels of Sp$(2j+1)$ and carries the dependence on the subgroup labels of R(5). Simple recursion formulae are derived from which this factor can be calculated in complete generality. The second factor carries the dependence on the subgroup labels of Sp$(2j+1)$ and must be calculated for each j. Since it is independent of n and T it is sufficient to calculate this factor for particular (most convenient) values of n and T. A calculation of the coefficients is illustrated with $j = \frac{5}{2}$ for which complete tables are given. An extension of the quasiparticle factorization technique to the nuclear LST scheme is discussed.

1. Introduction

The quasiparticle formalism recently developed by Armstrong and Judd [1]) for the atomic l-shell has led to a more complete classification scheme of l^n configurations of identical electrons. One of the great advantages of this new scheme is that it leads to a calculation of many-particle matrix elements without the need for fractional parentage coefficients, with the use of only a few reduced matrix elements and standard techniques of Racah algebra. A generalization to nuclear j^n configurations of both protons and neutrons, involving an analogous factorization into quasiparticle spaces, has recently been given by Elliott and Evans [2]). This leads to a complete classification scheme for nuclear shells with $j \leqq \frac{7}{2}$. However, although the total angular momentum J and isospin T are good quantum numbers in this new scheme, nucleon number is in general not a good quantum number. It is the purpose of this contribution to show how, with a slight modification of point of view, expressions for the reduced matrix elements needed for the Ellictt-Evans scheme follow at once, and the transformation to states of good particle number is simplified by making use of the symmetries of the states for the coupled quasiparticle spaces. A further simplification is achieved by imbedding the quasiparticle isospins (quasiparticle quasispins) in the five-dimensional quasispin group R(5). This leads to a factoring of the transfor-

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mation coefficients. One factor is independent of J and other subgroup labels of $\mathrm{Sp}(2j+1)$ and carries the dependence on T and other subgroup labels of R(5). Simple recursion formulae are derived from which this factor can be calculated in complete generality. The second factor is independent of T and n (nucleon number) so that it is sufficient to calculate this factor for states of a particular n and T. Knowledge of states with $n = 2j+1$ and $2j+2$ (half-full shell and half-full shell plus one) is therefore sufficient for all but a few states of a j-shell.

A brief review of the Elliott-Evans classification scheme is given in sect. 2. Expressions for the reduced matrix elements of the quasiparticle operators are given in sect. 3. Although these follow at once from the observation (already made by Elliott and Evans) that quasiparticle isospin operators are equivalent to quasiparticle quasispin operators, a derivation is given in some detail since the phase factors for the reduced matrix elements require some care. The transformation coefficients to states of good particle number are discussed in sect. 4. A complete calculation of these coefficients is illustrated with $j = \frac{5}{2}$, and tables for this case are given in an appendix. Tables for $j = \frac{7}{2}$ are somewhat bulky and will be given elsewhere. The technique used can be applied to shells with $j > \frac{7}{2}$. Applications to the calculation of matrix elements of one-and two-body operators are given in sect. 5. Finally, an extension of the quasiparticle factorization technique to the nuclear LST scheme is discussed briefly in sect. 6.

2. The Elliott-Evans classification scheme

Elliott and Evans introduce the quasiparticle operators

$$\lambda^\dagger_{mm_t} = \frac{1}{\sqrt{2}}\left(a^\dagger_{mm_t}+(-1)^{j-m+\frac{1}{2}-m_t}a_{-m-m_t}\right) = (-1)^{j+m+\frac{1}{2}+m_t}\lambda_{-m-m_t},$$

$$\mu^\dagger_{mm_t} = \frac{1}{\sqrt{2}}\left(a^\dagger_{mm_t}-(-1)^{j-m+\frac{1}{2}-m_t}a_{-m-m_t}\right) = -(-1)^{j+m+\frac{1}{2}+m_t}\mu_{-m-m_t}, \tag{1}$$

where $a^\dagger_{mm_t}$, a_{mm_t} are nucleon creation and annihilation operators and where m and m_t are magnetic substates in j- and t-space. In order to obtain a set of $4j+2$ *independent* quasiparticle creation operators and $4j+2$ *independent* quasiparticle annihilation operators Elliott and Evans restrict the m quantum number to be positive. We find it more convenient to restrict the isospin quantum number m_t instead, such that m can range from $-j$ to $+j$ for both quasiparticles, and define

$$\lambda^\dagger_m = \frac{1}{\sqrt{2}}\left(a^\dagger_{m+\frac{1}{2}}-(-1)^{j-m}a_{-m-\frac{1}{2}}\right), \qquad \mu^\dagger_m = \frac{1}{\sqrt{2}}\left(a^\dagger_{m-\frac{1}{2}}-(-1)^{j-m}a_{-m+\frac{1}{2}}\right), \qquad m = -j, \ldots, +j. \tag{2}$$

Now $\lambda^\dagger(\lambda)$ and $\mu^\dagger(\mu)$ are Fermion operators *mathematically* equivalent to identical nucleon creation (annihilation) operators. The $2j+1$ operators $\lambda_m^\dagger$, and $\lambda_m = (\lambda_m^\dagger)^\dagger$, satisfy the usual anticommutation rules. In addition any λ-operator anticommutes with any μ-operator, that is these are distinguishable quasiparticles. [To avoid confusion in notation it should be noted that the single index quasiparticle operators of eq. (2) are different from the double index quasiparticle operators of eq. (1). In fact, the single index operator $\lambda_m^\dagger$ is more closely related to the double index operator $\mu_{mm_t}^\dagger$. Henceforth only the notation of eq. (2) will be used. See also the remarks following eq. (7).]

TABLE 1

Generators, groups, representations

$$\begin{bmatrix} (a^\dagger a^\dagger)^{JT} \\ (aa)^{JT} \\ (a^\dagger a)^{JT} \end{bmatrix} \supset \begin{bmatrix} [(\lambda^\dagger\lambda^\dagger)^{J_\lambda}; (\lambda\lambda)^{J_\lambda}; (\lambda^\dagger\lambda)^{J_\lambda}] \\ \times[(\mu^\dagger\mu^\dagger)^{J_\mu}; (\mu\mu)^{J_\mu}; (\mu^\dagger\mu)^{J_\mu}] \end{bmatrix} \supset \begin{bmatrix} (\lambda^\dagger\lambda)^{J_\lambda \mathrm{odd}} \times \begin{bmatrix} (\lambda^\dagger\lambda^\dagger)^0; (\lambda\lambda)^0; \\ \frac{1}{2}(\lambda^\dagger\lambda)^0 + \frac{1}{2}(\lambda\lambda^\dagger)^0; \end{bmatrix} \\ \times(\mu^\dagger\mu)^{J_\mu \mathrm{odd}} \times \begin{bmatrix} (\mu^\dagger\mu^\dagger)^0; (\mu\mu)^0; \\ \frac{1}{2}(\mu^\dagger\mu)^0 + \frac{1}{2}(\mu\mu^\dagger)^0; \end{bmatrix} \end{bmatrix}$$

$$R(8j+4) \supset \begin{bmatrix} R_\lambda(4j+2) \\ \times R_\mu(4j+2) \end{bmatrix} \supset \begin{bmatrix} [\mathrm{Sp}_\lambda(2j+1)\times \mathrm{SU}_\lambda(2)] \\ \times[\mathrm{Sp}_\mu(2j+1)\times \mathrm{SU}_\mu(2)] \end{bmatrix}$$

	λ-space:	
$(\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\ldots\pm\frac{1}{2})$	$(\frac{1}{2}\,\frac{1}{2}\ldots\pm\frac{1}{2})$	$(1^{v_\lambda}0^{j+\frac{1}{2}-v_\lambda})\times(T_\lambda = \frac{1}{2}(j+\frac{1}{2}-v_\lambda)$
	similarly for μ-space	$v_\lambda \to J_\lambda$ [a])

[a]) Allowed J_λ follow from v_λ by the rules valid for identical particles; see, e.g., table 2 of ref. [2]).

The group theoretical basis of the quasiparticle classification scheme is shown in table 1. The full set of operators $a^\dagger a^\dagger$, aa, $a^\dagger a$ generate the group $\mathrm{R}(8j+4)$. The only irreducible representations of this group which are realized are $(\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\ldots\pm\frac{1}{2})$ for n even (odd), respectively; to be denoted by $\Delta_\pm$. The set of operators $\lambda^\dagger\lambda^\dagger$, $\lambda\lambda$, $\lambda^\dagger\lambda$ generate a subgroup $\mathrm{R}(4j+2)$; similarly for $\mu^\dagger\mu^\dagger$, $\mu\mu$, $\mu^\dagger\mu$. Unlike the operators $(a^\dagger a)^{JT}$ which are double spherical tensor operators in J- and T-space, the operators $(\lambda^\dagger\lambda)^{J_\lambda}$ are constructed by means of a single j-space vector coupling coefficient. Since the λ and μ operators are *mathematically* equivalent to a set of identical nucleon operators, their subgroup chain is the conventional one for identical particles shown here in terms of the direct product of the symplectic group in $2j+1$ dimensions with an identical particle quasispin group. The irreducible representations of $\mathrm{Sp}_\lambda(2j+1)\times \mathrm{SU}_\lambda(2)$ are specified by the quasispin quantum number $\frac{1}{2}(j+\frac{1}{2}-v_\lambda)$ related to an identical particle seniority number v_λ (not to be confused with any real seniority). The quasispin group $\mathrm{SU}_\lambda(2)$ is generated by the three operators $(\lambda^\dagger\lambda^\dagger)^0$, $(\lambda\lambda)^0$, $\frac{1}{2}[(\lambda^\dagger\lambda)^0+(\lambda\lambda^\dagger)^0]$, all of rank zero in λ-space. These are the λ-particle isospin operators

$\boldsymbol{T}_\lambda$ of Elliott and Evans. More specifically

$$\begin{aligned}
T_{\lambda+} &= \sum_{m>0}(-1)^{j-m}\lambda_m^\dagger\lambda_{-m}^\dagger = \mathscr{S}_{\lambda+}, & T_{\mu+} &= \sum_{m>0}(-1)^{j-m}\mu_{-m}\mu_m = \mathscr{S}_{\mu-},\\
T_{\lambda-} &= \sum_{m>0}(-1)^{j-m}\lambda_{-m}\lambda_m = \mathscr{S}_{\lambda-}, & T_{\mu-} &= \sum_{m>0}(-1)^{j-m}\mu_m^\dagger\mu_{-m}^\dagger = \mathscr{S}_{\mu+}, \qquad (3)\\
T_{\lambda 0} &= \tfrac{1}{4}\sum_m(\lambda_m^\dagger\lambda_m-\lambda_m\lambda_m^\dagger) = \mathscr{S}_{\lambda 0}, & T_{\mu 0} &= -\tfrac{1}{4}\sum_m(\mu_m^\dagger\mu_m-\mu_m\mu_m^\dagger) = -\mathscr{S}_{\mu 0},
\end{aligned}$$

with

$$\boldsymbol{T}_\lambda+\boldsymbol{T}_\mu = \boldsymbol{T}, \tag{4}$$

where $\boldsymbol{T}$ is the total isospin operator in conventional form. (In eqs. (3) the quasiparticle isospin operators have also been expressed in terms of quasispin operators $\mathscr{S}$ in standard form.) The symplectic group generators with $J_\lambda(J_\mu) = 1$ are (except for normalization factors) the angular momentum operators of λ- and μ-space. More specifically

$$\begin{aligned}
J_{\lambda_0} &= \sum_m m\lambda_m^\dagger\lambda_m, & J_{\mu 0} &= \sum_m m\mu_m^\dagger\mu_m,\\
J_{\lambda+} &= \sum_m[(j-m)(j+m+1)]^{\frac{1}{2}}\lambda_{m+1}^\dagger\lambda_m, & J_{\mu+} &= \sum_m[(j-m)(j+m+1)]^{\frac{1}{2}}\mu_{m+1}^\dagger\mu_m,\\
J_{\lambda-} &= (J_{\lambda+})^\dagger, & J_{\mu-} &= (J_{\mu+})^\dagger, \qquad (5)
\end{aligned}$$

with

$$\boldsymbol{J}_\lambda+\boldsymbol{J}_\mu = \boldsymbol{J}, \tag{6}$$

where $\boldsymbol{J}$ is the total angular momentum operator.

The spherical tensor character of the operators $\lambda_m^\dagger(\lambda_m)$, and $\mu_m^\dagger(\mu_m)$ in both j- and t-spaces follows from their commutation relations with the operators $\boldsymbol{T}_\lambda$, $\boldsymbol{J}_\lambda$, and $\boldsymbol{T}_\mu$, $\boldsymbol{J}_\mu$. In terms of double spherical tensor operators $A^{j\,\frac{1}{2}}_{m\,m_t}$ and $M^{j\,\frac{1}{2}}_{m\,m_t}$, the relations are

$$\begin{aligned}
\lambda_m^\dagger &= A^{j\frac{1}{2}}_{m+\frac{1}{2}}, & (-1)^{j-m}\lambda_{-m} &= A^{j\frac{1}{2}}_{m-\frac{1}{2}},\\
\mu_m^\dagger &= M^{j\frac{1}{2}}_{m-\frac{1}{2}}, & (-1)^{j-m}\mu_{-m} &= M^{j\frac{1}{2}}_{m+\frac{1}{2}}.
\end{aligned} \tag{7}$$

(These double tensor operators are immediately related to those of Elliott and Evans by: $A^{j\,\frac{1}{2}}_{m\,m_t} = \mu^\dagger_{m\,m_t}$, $M^{j\,\frac{1}{2}}_{m\,m_t} = \lambda^\dagger_{m\,m_t}$. It is therefore clear that the final wave functions $|(J_\lambda J_\mu)J,\ (T_\lambda T_\mu)T\rangle$ are identical with those of Elliott and Evans, except for a trivial interchange of λ and μ.)

Coupled tensor operators are formed from these in the usual way.

For example

$$[A\times M]^{JT}_{M_JM_T} = \sum_{mm_t}\langle jmjm'|JM_J\rangle\langle\tfrac{1}{2}m_t\tfrac{1}{2}m_t'|TM_T\rangle A^{j\frac{1}{2}}_{mm_t}M^{j\frac{1}{2}}_{m'm'_t}. \tag{8}$$

The single-nucleon creation and annihilation operators are expressed in terms of the tensors Λ and M by

$$a^{\dagger}_{m\pm\frac{1}{2}} = \frac{1}{\sqrt{2}}(\Lambda^{j\frac{1}{2}}_{m\pm\frac{1}{2}}+M^{j\frac{1}{2}}_{m\pm\frac{1}{2}}),$$
$$(-1)^{j-m}a_{-m\mp\frac{1}{2}} = \mp\frac{1}{\sqrt{2}}(\Lambda^{j}_{m\pm\frac{1}{2}}-M^{j\frac{1}{2}}_{m\pm\frac{1}{2}}). \tag{9}$$

With these relations any physical operator can be expressed in terms of double spherical tensors built from Λ- and M-operators. The matrix element of any physical operator is then reduced by standard Racah algebra to a few reduced matrix elements of the Λ- and M-operators.

For $j \leqq \frac{7}{2}$ the multiplicity of the set of J-values (J_λ or J_μ) associated with each quasispin quantum number (v_λ or v_μ) is never greater than 1. The $|(J_\lambda J_\mu)J, (T_\lambda T_\mu)T\rangle$ basis therefore furnishes a complete classification scheme for these shells. Note that the irreducible representations $(\frac{1}{2}\frac{1}{2}\ldots+\frac{1}{2})$ and $(\frac{1}{2}\frac{1}{2}\ldots-\frac{1}{2})$ of R_λ $(4j+2)$ or $R_\mu(4j+2)$ are specified automatically since they contain the integral and $\frac{1}{2}-$ integral values of T_λ or T_μ, respectively. The irreducible representation label Δ of $R(8j+4)$ will sometimes have to be designated explicitly since it is common to both the λ- and μ-spaces. The irreducible representations Δ_+ and Δ_- of $R(8j+4)$, (n = even and odd), contain only the states with $2T_\lambda+2T_\mu$ = even and odd, respectively, so that they are designated if both T_λ and T_μ are specified.

3. The reduced matrix elements

The matrix element of an operator Λ in the coupled $|(J_\lambda J_\mu)J, (T_\lambda T_\mu)T\rangle$ basis is reduced to a double-barred matrix element $\langle\Delta' J'_\lambda T'_\lambda\|\Lambda\|\Delta J_\lambda T_\lambda\rangle$ for the λ-space by standard formulae of Racah algebra; similarly for M. Note that the $R(8j+4)$ labels Δ must appear in the expression for the reduced matrix element for the separated λ-space, since Δ is a quantum number common to both λ and μ-spaces. [For the analogous property for electron states, see Cunningham and Wybourne [3]).]

The value of the λ-space reduced matrix element follows at once from the relation between the λ-space isospin and quasispin quantum numbers

$$T_\lambda = \tfrac{1}{2}(j+\tfrac{1}{2}-v_\lambda), \qquad M_{T_\lambda} = \tfrac{1}{2}(n_\lambda-j-\tfrac{1}{2}), \tag{10a}$$

where n_λ is the number of λ quasiparticles, (the eigenvalue of $\Sigma_m \lambda^\dagger_m \lambda_m$). From eq. (7) and the standard definition of double-barred spherical tensor matrix elements on the one hand, and eq. (10a) on the other

$$\begin{aligned}&\langle\Delta' J'_\lambda M'_{J_\lambda} T'_\lambda M'_{T_\lambda}|\lambda^\dagger_m|\Delta J_\lambda M_{J_\lambda} T_\lambda M_{T_\lambda}\rangle \\ &\quad= \frac{\langle J_\lambda M_{J_\lambda} jm|J'_\lambda M'_{J_\lambda}\rangle\langle T_\lambda M_{T_\lambda}\frac{1}{2}\frac{1}{2}|T'_\lambda(M_{T_\lambda}+\frac{1}{2})\rangle\langle\Delta' J'_\lambda T'_\lambda\|\Lambda\|\Delta J_\lambda T_\lambda\rangle}{[(2J'_\lambda+1)(2T'_\lambda+1)]^{\frac{1}{2}}} \\ &\quad= \langle j^{n_\lambda+1}\Delta' v'_\lambda J'_\lambda M'_{J_\lambda}|\lambda^\dagger_m|j^{n_\lambda}\Delta v_\lambda J_\lambda M_{J_\lambda}\rangle.\end{aligned} \tag{11}$$

Using quasispin techniques to factor out the dependence on n_λ

$$\langle j^{n_\lambda+1}\Delta' v'_\lambda J'_\lambda M'_{J_\lambda}|\lambda^\dagger_m|j^{n_\lambda}\Delta v_\lambda J_\lambda M_{J_\lambda}\rangle$$
$$= \frac{\langle J_\lambda M_{J_\lambda} jm|J'_\lambda M'_{J_\lambda}\rangle}{[2J'_\lambda+1]^{\frac{1}{2}}} \frac{\langle T_\lambda M_{T_\lambda} \frac{1}{2}\frac{1}{2}|T'_\lambda(M_{T_\lambda}+\frac{1}{2})\rangle}{\langle T_\lambda - T_\lambda \frac{1}{2}\frac{1}{2}|T'_\lambda(-T_\lambda+\frac{1}{2})\rangle} \langle j^{v_\lambda+1}\Delta' v'_\lambda J'_\lambda||\lambda^\dagger||j^{v_\lambda}\Delta v_\lambda J_\lambda\rangle, \quad (12)$$

so that

$$\langle \Delta' J'_\lambda T'_\lambda||\Lambda||\Delta J_\lambda T_\lambda\rangle = \frac{[2T'_\lambda+1]^{\frac{1}{2}}\langle j^{v_\lambda+1}\Delta' v'_\lambda J'_\lambda||\lambda^\dagger||j^{v_\lambda}\Delta v_\lambda J_\lambda\rangle}{\langle T_\lambda - T_\lambda \frac{1}{2}\frac{1}{2}|T'_\lambda(-T_\lambda+\frac{1}{2})\rangle}. \quad (13a)$$

Another useful formula is obtained by starting with the operator $(-1)^{j-m}\lambda_{-m}$. Again, from eqs. (7) and (10a), and using hermitean conjugation to convert the matrix element of λ_{-m} to one for $\lambda^\dagger_{-m}$ one obtains

$$\langle \Delta' J'_\lambda T'_\lambda||\Lambda||\Delta J_\lambda T_\lambda\rangle = (-1)^{J'_\lambda - J_\lambda - j} \frac{[2T'_\lambda+1]^{\frac{1}{2}}\langle j^{v'_\lambda+1}\Delta v_\lambda J_\lambda||\lambda^\dagger||j^{v'_\lambda}\Delta' v'_\lambda J'_\lambda\rangle^*}{\langle T_\lambda - T_\lambda \frac{1}{2} -\frac{1}{2}|T'_\lambda(-T_\lambda-\frac{1}{2})\rangle}. \quad (14a)$$

Note the reversal of the primed and unprimed quantum numbers (including Δ) in the double-barred matrix element of $\lambda^\dagger$. The derivation for the reduced matrix element $\langle \Delta' J'_\mu T'_\mu\|M\|\Delta J_\mu T_\mu\rangle$ differs in one respect. The relation between n_μ, the number of μ quasiparticles, and M_{T_μ} differs in sign from the corresponding relation for λ-space; [see eq. (3)]:

$$T_\mu = \tfrac{1}{2}(j+\tfrac{1}{2}-v_\mu), \qquad M_{T_\mu} = \tfrac{1}{2}(j+\tfrac{1}{2}-n_\mu). \quad (10b)$$

As a result the analogues of eqs. (13a) and (14a) become

$$\langle \Delta' J'_\mu T'_\mu||M||\Delta J_\mu T_\mu\rangle = \frac{[2T'_\mu+1]^{\frac{1}{2}}\langle j^{v_\mu+1}\Delta' v'_\mu J'_\mu||\mu^\dagger||j^{v_\mu}\Delta v_\mu J_\mu\rangle}{\langle T_\mu T_\mu \frac{1}{2} -\frac{1}{2}|T'_\mu(T_\mu-\frac{1}{2})\rangle}, \quad (13b)$$

$$\langle \Delta' J'_\mu T'_\mu||M||\Delta J_\mu T_\mu\rangle = (-1)^{J'_\mu - J_\mu - j} \frac{[2T'_\mu+1]^{\frac{1}{2}}\langle j^{v'_\mu+1}\Delta v_\mu J_\mu||\mu^\dagger||j^{v'_\mu}\Delta' v'_\mu J'_\mu\rangle^*}{\langle T_\mu T_\mu \frac{1}{2}\frac{1}{2}|T'_\mu(T_\mu+\frac{1}{2})\rangle}. \quad (14b)$$

Since the operators $\lambda^\dagger$ (or $\mu^\dagger$) are mathematically equivalent to identical nucleon creation operators, the double-barred matrix elements of $\lambda^\dagger$ (or $\mu^\dagger$) in eqs. (13) and (14) are, except for their dependence on Δ, given by identical particle reduced matrix elements. The dependence on Δ involves only a phase. This phase dependence must be different for the λ and μ quasiparticles. This can be seen by expressing operators such as $\boldsymbol{J}_\lambda$ ($\boldsymbol{J}_\mu$) in terms of the coupled tensor operators defined by eq. (8). For example

$$(J_\lambda)_q = -\tfrac{1}{6}[j(j+1)(2j+1)]^{\frac{1}{2}}[\Lambda\times\Lambda]^{10}_{q0},$$

while

$$(J_\mu)_q = +\tfrac{1}{6}[j(j+1)(2j+1)]^{\frac{1}{2}}[M\times M]^{10}_{q0}, \quad (15)$$

with a similar difference in sign for operators $(\boldsymbol{T}_\lambda)_q$, $(\boldsymbol{T}_\mu)_q$. The Δ-dependence of the phases can be determined by calculating matrix elements of the operators (15), for example. Although there is some arbitrariness in the possible choices, results consistent with the standard phase conventions of spherical tensor calculus, can be stated as follows:

(i) Matrix elements for μ-space are independent of Δ.

(ii) Matrix elements for λ-space are given by $\langle \Delta' \ldots \| \lambda^\dagger \| \Delta \ldots \rangle = (-1)^{\phi(\Delta)} \langle \ldots \| \lambda^\dagger \| \ldots \rangle$,
where the Δ-independent double-barred matrix elements are those for identical particles, and where $\phi(\Delta) = 2T_\lambda + 2T_\mu$; that is, $\phi(\Delta) =$ even (odd) for $n =$ even (odd); $n =$ real nucleon number for the state on the right-hand side of the matrix element. Note also that $\phi(\Delta') = 2T'_\lambda + 2T'_\mu = 2T_\lambda + 2T_\mu \pm 1$. [Eqs. (13a) require $\phi(\Delta)$; eqs. (14a) require $\phi(\Delta')$.]

The most convenient form for the reduced matrix elements of Λ then follow from eqs. (13a) if $T'_\lambda = T'_\lambda - \frac{1}{2}$, and eqs. (14a) if $T'_\lambda = T_\lambda + \frac{1}{2}$; similarly for M. The results are

Case 1.

$T'_\lambda = T_\lambda - \frac{1}{2}, \quad v'_\lambda = v_\lambda + 1, \quad T_\lambda = \frac{1}{2}(j + \frac{1}{2} - v_\lambda)$:

$$\langle \Delta' J'_\lambda T'_\lambda \| \Lambda \| \Delta J_\lambda T_\lambda \rangle = (-1)^{2T_\lambda + 2T_\mu + 1} [2T_\lambda + 1]^{\frac{1}{2}} \langle j^{v_\lambda + 1} v_\lambda + 1 J'_\lambda \| \lambda^\dagger \| j^{v_\lambda} v_\lambda J_\lambda \rangle,$$

$T'_\mu = T_\mu - \frac{1}{2}, \quad v'_\mu = v_\mu + 1, \quad T_\mu = \frac{1}{2}(j + \frac{1}{2} - v_\mu)$:

$$\langle \Delta' J'_\mu T'_\mu \| M \| \Delta J_\mu T_\mu \rangle = [2T_\mu + 1]^{\frac{1}{2}} \langle j^{v_\mu + 1} v_\mu + 1 J'_\mu \| \mu^\dagger \| j^{v_\mu} v_\mu J_\mu \rangle. \tag{16}$$

Case 2.

$T'_\lambda = T_\lambda + \frac{1}{2}, \quad v'_\lambda = v_\lambda - 1$:

$$\langle \Delta' J'_\lambda T'_\lambda \| \Lambda \| \Delta J_\lambda T_\lambda \rangle = (-1)^{2T_\lambda + 2T_\mu + 1 + J'_\lambda - J_\lambda - j} [2T_\lambda + 2]^{\frac{1}{2}} \langle j^{v_\lambda} v_\lambda J_\lambda \| \lambda^\dagger \| j^{v_\lambda - 1} v_\lambda - 1 J'_\lambda \rangle,$$

$T'_\mu = T_\mu + \frac{1}{2}, \quad v'_\mu = v_\mu - 1$:

$$\langle \Delta' J'_\mu T'_\mu \| M \| \Delta J_\mu T_\mu \rangle = (-1)^{J'_\mu - J_\mu - j} [2T_\mu + 2]^{\frac{1}{2}} \langle j^{v_\mu} v_\mu J_\mu \| \mu^\dagger \| j^{v_\mu - 1} v_\mu - 1 J'_\mu \rangle.$$

Double-barred matrix elements of $\lambda^\dagger$ (or $\mu^\dagger$) are standard double-barred matrix elements for *identical* nucleon configurations, related to identical nucleon cfp's in the usual way

$$\langle j^{v+1} v + 1 J' \| \lambda^\dagger \| j^v v J \rangle = (-1)^v [(v+1)(2J'+1)]^{\frac{1}{2}} \langle j^v(vJ); jJ' | \} j^{v+1} v + 1 J' \rangle. \tag{17}$$

Only identical nucleon cfp's with $n = v$, $n+1 = v+1$ are needed. For the $j = \frac{5}{2}$ shell therefore only four nontrivial matrix elements are needed for any calculation. For $j = \frac{7}{2}$ the number of nontrivial (but well-known) such matrix elements is 30.

4. States of good particle number

Although the calculation of matrix elements in the $|(J_\lambda J_\mu)J, (T_\lambda T_\mu)T\rangle$ basis is extremely simple, the particle number is in general not diagonal in this basis. In order to realize the full power of the quasiparticle factorization it will therefore be useful to make a transformation to states of good particle number. (An alternate approach might involve the simultaneous diagonalization of the number operator and the Hamiltonian, for example.) The transformation to states of good particle number is simplified (i) by making use of the symmetries of the states for the coupled λ and μ spaces, and (ii) by imbedding the λ and μ quasispin groups in the five-dimensional quasispin (seniority) group R(5).

4.1. SYMMETRIES

It is useful to define coupled state vectors $|(J_\lambda J_\mu)J, (T_\lambda T_\mu)T\rangle$, either symmetric (s), or antisymmetric (a), to an interchange of λ and μ quantum numbers:

$$|(ab)J, (cd)T\rangle_{\substack{(s)\\(a)}} \equiv \frac{1}{\sqrt{2}}\{|(ab)J, (cd)T\rangle \pm (-1)^{J-a-b+T-c-d}|(ba)J, (dc)T\rangle\}. \quad (18a)$$

Note that λ quantum numbers always precede μ quantum numbers in the order of the coupling; thus $J_\lambda = a$ in the first term on the r.h.s. of (18a), while $J_\lambda = b$ in the second term. In our notation the magnetic quantum numbers M_J and M_T have been omitted for brevity but are quietly understood. In the special case when both $J_\lambda = J_\mu (= a)$, and $T_\lambda = T_\mu (= c)$, the *normalized* symmetrized state vector is

$$|(aa)J, (cc)T\rangle \quad \text{with } J-2a+T-2c = \begin{matrix}\text{even}\\ \text{odd}\end{matrix} \quad \text{for} \begin{matrix}\text{(s)}\\ \text{(a)}\end{matrix} \text{ states.} \quad (18b)$$

States of a given nucleon number must then be either (s) or (a) states according to the rules:

$$\begin{aligned} &1.\ \text{For} \quad \begin{matrix} n = 2j+1\pm 4k \\ n = 2j \pm 4k \end{matrix} \quad : \quad \text{(s) states only.} \\ &2.\ \text{For} \quad \begin{matrix} n = 2j+3\pm 4k \\ n = 2j+2 \pm 4k \end{matrix} \quad : \quad \text{(a) states only.} \end{aligned} \quad (19)$$

where k = integer. That is, for both even and odd nucleon numbers, states of a given symmetry correspond to n values which differ only by multiples of 4. The corresponding property for electron states has already been noted by Armstrong and Judd [1]). The derivation of the symmetry rules (19) follows directly from the explicit construction of state vectors to be given below. Since states with $n = 2j+1$ are of central importance, an additional symmetry property valid for these states is very useful. For $n = 2j+1$ the quantum numbers $(T_\lambda T_\mu)$ are *either both* integral *or both* half-integral, a property related to conjugation symmety as applied to the half-full shell.

4.2. FIVE-DIMENSIONAL QUASISPIN GROUP

Since operators changing only nucleon number (without change in J, T, or seniority quantum numbers) can easily be constructed in terms of generators of the five-dimensional quasispin group, it is useful to imbed the λ and μ quasispin groups in the seniority group R(5): $R(5) \supset [SU_\lambda(2)\times SU_\mu(2)]$. The ten generators of R(5) are composed of the operators $\boldsymbol{T}_\lambda$, $\boldsymbol{T}_\mu$, $[\Lambda \times M]^{0\,T=1}_{0\,M_T}$ together with the number operator

$$N_{\text{op.}} = (2j+1)+[2(2j+1)]^{\frac{1}{2}}[\Lambda \times M]^{00}_{00}. \tag{20}$$

Irreducible representations of R(5) are specified by the real nucleon seniority v and reduced isospin t, and are given by the two labels, (highest weights), $(\omega_1\omega_2)$ where

$$\omega_1 = j+\tfrac{1}{2}-\tfrac{1}{2}v, \qquad \omega_2 = t. \tag{21}$$

The group chain $R(5) \supset SU(2)\times SU(2)$: (alternately $Sp(4) \supset SU(2)\times SU(2)$), has been studied in detail. The possible $(T_\lambda T_\mu)$ values imbedded in a given irreducible representation $(\omega_1\omega_2)$ are given by the rules [see ref. [4]), e.g.]:

With
$$\begin{aligned}(T_\lambda)_{\max} &= \tfrac{1}{2}(\omega_1+\omega_2),\\ (T_\mu)_{\max} &= \tfrac{1}{2}(\omega_1-\omega_2),\end{aligned} \qquad \text{(highest weight values)} \tag{22a}$$

the possible $(T_\lambda T_\mu)$ values are given by

$$\begin{aligned}T_\lambda &= (T_\lambda)_{\max}-\tfrac{1}{2}k-\tfrac{1}{2}m, \qquad & 0 \leqq k \leqq 2\omega_2,\\ T_\mu &= (T_\mu)_{\max}+\tfrac{1}{2}k-\tfrac{1}{2}m, \qquad & 0 \leqq m \leqq (\omega_1-\omega_2).\end{aligned} \tag{22b}$$

The set of possible $(T_\lambda T_\mu)$ values is thus severely restricted. For states of high v in particular (the richest from the point of view of total number of states), the possible $(T_\lambda T_\mu)$ values are restricted to a few or even a single pair of small values.

4.3. CALCULATION OF TRANSFORMATION COEFFICIENTS

The imbedding in R(5) leads to a factoring of the transformation coefficients, where one factor depending only on the R(5) quantum numbers, including n and T, can easily be calculated in general by simple recursion formulae which follow from the known matrix elements of the generators of R(5). The second factor carries the dependence on the subgroup labels of $Sp(2j+1)$. The transformation coefficients to states of good particle number are defined in terms of matrices c and d by

$$|(\omega_1 t)n\beta T;\alpha J\rangle = \sum_{T_\lambda T_\mu} c^{(\omega_1 t)T}_{n\beta;\,T_\lambda T_\mu} \sum_{J_\lambda J_\mu} d^{T_\lambda T_\mu,\,J}_{(\omega_1 t)\alpha,\,J_\lambda J_\mu} |(J_\lambda J_\mu)J,(T_\lambda T_\mu)T\rangle_{(\sigma)}, \tag{23}$$

where $n\beta T$ are R(5) subgroup labels; β is the R(5) quantum number needed when states of a given nT occur in $(\omega_1 t)$ with a multiplicity greater than one, and will be defined according to ref. [5]). Similarly αJ are $Sp(2j+1)$ subgroup labels; α is the multiplicity label needed for states of a fixed J and v, t. The state vectors $|(J_\lambda J_\mu)J$, $(T_\lambda T_\mu)T\rangle_{(\sigma)}$ with symmetry subscript (σ) are normalized state vectors defined according to eqs. (18), where $(\sigma) = $ (s) or (a) for fixed n according to the rules (19). The c-coefficients are independent of the quantum numbers $\alpha J J_\lambda J_\mu$ of the sympletic group. They form orthogonal matrices, whose rows are labelled by n and β (or in place of n, the more natural R(5) quantum number $H_1 = \frac{1}{2}n-j-\frac{1}{2}$), while columns are labeled by $T_\lambda T_\mu$. In addition the c-coefficients are functions of $(\omega_1 t)$ and total isospin T. The d coefficients on the other hand are independent of the R(5) quantum numbers $n\beta T$. They form orthogonal matrices whose rows are labelled by $(\omega_1 t)\alpha$, while columns are labelled by $J_\lambda J_\mu$, for each possible value of J and $T_\lambda T_\mu$.

Since the c-coefficients follow from properties of R(5) they can be calculated most easily. It is useful to define the R(5) generators in terms of the total isospin operators $\boldsymbol{T}$, and $N_{\text{op.}}$, and the pair creation and annihilation operators:

$$\begin{aligned} A^\dagger(M_T) &= \tfrac{1}{2}\sum_{mm_t} \langle \tfrac{1}{2}m_t\, \tfrac{1}{2}m_t' | 1M_T\rangle (-1)^{j-m} a^\dagger_{mm_t} a^\dagger_{-mm't}, \\ A(M_T) &= (A^\dagger(M_T))^\dagger. \end{aligned} \tag{24}$$

These can be expressed in terms of the λ, μ-space operators $\boldsymbol{T}_\lambda$, $\boldsymbol{T}_\mu$, Λ, M by

$$\begin{aligned} A^\dagger(q) &= \frac{1}{\sqrt{2}}(-(T_\lambda)_q+(T_\mu)_q)+\tfrac{1}{2}[2j+1]^{\frac{1}{2}}[\Lambda\times M]^{01}_{0q}, \\ (-1)^q A(-q) &= \frac{1}{\sqrt{2}}(-(T_\lambda)_q+(T_\mu)_q)-\tfrac{1}{2}[2j+1]^{\frac{1}{2}}[\Lambda\times M]^{01}_{0q}, \end{aligned} \tag{25}$$

(where isospin operators T_q are standard spherical tensors, e.g. $T_{\pm 1} = \mp\sqrt{\frac{1}{2}}(T_\pm)$. These lead to the useful relations

$$A^\dagger(1)-A(-1) = (T_\lambda)_+-(T_\mu)_+, \tag{26a}$$

$$-A^\dagger(-1)+A(1) = (T_\lambda)_--(T_\mu)_-. \tag{26b}$$

Operators which leave the quantum numbers α, J, v, t, T invariant but change nucleon number (by ± 2 units) can easily be constructed in terms of the R(5) generators [5,6]. E.g.

$$\begin{aligned} O_{\Delta n=+2} &= \frac{1}{\sqrt{2}}\sum_q (-1)^q A^\dagger(q) T_{-q}, \\ O_{\Delta n=-2} &= \frac{1}{\sqrt{2}}\sum_q A(q) T_q. \end{aligned} \tag{27}$$

Expressed in terms of $\boldsymbol{T}_\lambda$, $\boldsymbol{T}_\mu$, Λ and M, these operators are

$$\begin{aligned} O_{\Delta n=+2} &= -\tfrac{1}{2}\boldsymbol{T}_\lambda^2+\tfrac{1}{2}\boldsymbol{T}_\mu^2+[\tfrac{1}{8}(2j+1)]^{\frac{1}{2}}\sum_q (-1)^q[\Lambda\times M]^{01}_{0q}T_{-q}, \\ O_{\Delta n=-2} &= -\tfrac{1}{2}\boldsymbol{T}_\lambda^2+\tfrac{1}{2}\boldsymbol{T}_\mu^2-[\tfrac{1}{8}(2j+1)]^{\frac{1}{2}}\sum_q (-1)^q[\Lambda\times M]^{01}_{0q}T_{-q}, \end{aligned} \tag{28}$$

giving the simple step-operator relation

$$O_{\Delta n=+2}+O_{\Delta n=-2} = -(T_\lambda^2-T_\mu^2). \tag{29}$$

Since the operators of eqs. (26) and (29) are made up only of λ- and μ-space isospin operators which do not change the quantum numbers T_λ and T_μ, they lead to recursion formulae involving only the coefficients $c^{(\omega_1 t)T}_{n\beta;T_\lambda T_\mu}$ with fixed $T_\lambda T_\mu$. The technique is illustrated with the operator (26a) acting on a state with $M_T = T$:

$$\begin{aligned}
&[A^\dagger(1)-A(-1)]|(\omega_1 t)\beta H_1 = \tfrac{1}{2}n-j-\tfrac{1}{2}, TM_T = T; \alpha JM_J\rangle \\
&\quad = \sum_{\beta'} |(\omega_1 t)\beta' H_1+1T+1T+1; \alpha JM_J\rangle \\
&\qquad\qquad \times \langle(\omega_1 t)\beta' H_1+1T+1T+1|A^\dagger(1)|(\omega_1 t)\beta H_1\, TT\rangle \\
&\quad - \sum_{\beta'} |(\omega_1 t)\beta' H_1-1T+1T+1; \alpha JM_J\rangle \\
&\qquad\qquad \times \langle(\omega_1 t)\beta' H_1-1T+1T+1|A(-1)|(\omega_1 t)\beta H_1\, TT\rangle
\end{aligned} \tag{30}$$

$$\begin{aligned}
&\quad = \sum_{T_\lambda T_\mu} c^{(\omega_1 t)T}_{H_1\beta;T_\lambda T_\mu}\{\langle(J_\lambda J_\mu)JM_J, (T_\lambda T_\mu)T+1T+1|T_{\lambda+}|(J_\lambda J_\mu)JM_J, (T_\lambda T_\mu)TT\rangle \\
&\quad -\langle(J_\lambda J_\mu)JM_J, (T_\lambda T_\mu)T+1T+1|T_{\mu+}|(J_\lambda J_\mu)JM_J, (T_\lambda T_\mu)TT\rangle\} \\
&\quad \times\Big[\sum_{J_\lambda J_\mu} d^{T_\lambda T_\mu, J}_{(\omega_1 t)\alpha; J_\lambda J_\mu}|(J_\lambda J_\mu)JM_J, (T_\lambda T_\mu)T+1M_T = T+1\rangle_{(\sigma)}\Big].
\end{aligned} \tag{31}$$

By expanding the state vectors

$$|(\omega_1 t)H_1\pm 1\beta' T+1T+1\rangle = \sum_{T_\lambda T_\mu} c^{(\omega_1 t)T+1}_{H_1\pm 1\beta';T_\lambda T_\mu}\Big[\sum_{J_\lambda J_\mu} d^{T_\lambda T_\mu, J}_{(\omega_1 t)\alpha; J_\lambda J_\mu}|(J_\lambda J_\mu)JM_J, (T_\lambda T_\mu)T+1T+1\rangle_{(\sigma)}\Big],$$

the above leads via the orthonormality of

$$\Big[\sum_{J_\lambda J_\mu} d^{T_\lambda T_\mu, J}_{(\omega_1 t)\alpha; J_\lambda J_\mu}|(J_\lambda J_\mu)JM_J, (T_\lambda T_\mu)TM_T\rangle_{(\sigma)}\Big]$$

to a recursion formula for the $c^{(\omega_1 t)T}_{H_1\beta;T_\lambda T_\mu}$ in terms of the matrix elements of the operators $A^\dagger$, A. These are known [5-7] for all irreducible representations $(\omega_1 t)$ needed for $j \leqq \frac{9}{2}$. They can be expressed in terms of the R(5) Casimir invariant and reduced R(5) Wigner coefficients which are tabulated in refs. [5-7] as general functions of H_1 and T. The final form of the recursion formula is then

Recursion formula I:

$$\begin{aligned}
&\sum_{\beta'} c^{(\omega_1 t)T+1}_{H_1+1\beta';T_\lambda T_\mu}[\omega_1(\omega_1+3)+t(t+1)]^{\frac{1}{2}}\langle(\omega_1 t)\beta H_1\, T;(11)+11||(\omega_1 t)\beta' H_1+1T+1\rangle_1 \\
&+\sum_{\beta'} c^{(\omega_1 t)T+1}_{H_1-1\beta';T_\lambda T_\mu}[\omega_1(\omega_1+3)+t(t+1)]^{\frac{1}{2}}\langle(\omega_1 t)\beta H_1\, T;(11)-11||(\omega_1 t)\beta' H_1-1T+1\rangle_1 \\
&= -c^{(\omega_1 t)T}_{H_1\beta;T_\lambda T_\mu}\left[\frac{2(T_\lambda+T_\mu+T+2)(T_\lambda+T_\mu-T)(T_\lambda-T_\mu+T+1)(T_\mu-T_\lambda+T+1)}{(T+1)(2T+3)}\right]^{\frac{1}{2}}.
\end{aligned} \tag{32}$$

Similarly, eq. (29) leads to

Recursion formula II:

$$\sum_{\beta'} c^{(\omega_1 t)T}_{H_1+1\beta';\,T_\lambda T_\mu}[\{\omega_1(\omega_1+3)+t(t+1)\}T(T+1)]^{\frac{1}{2}}$$
$$\times\langle(\omega_1\, t)\beta H_1\, T;(11)+11\|(\omega_1\, t)\beta' H_1+1T\rangle_1$$
$$+\sum_{\beta'} c^{(\omega_1 t)T}_{H_1-1\beta';\,T_\lambda T_\mu}[\{\omega_1(\omega_1+3)+t(t+1)\}T(T+1)]^{\frac{1}{2}}$$
$$\times\langle(\omega_1\, t)\beta H_1\, T;(11)-11\|(\omega_1\, t)\beta' H_1-1T\rangle_1$$
$$= -\sqrt{2}\, c^{(\omega_1 t)T}_{H_1\beta;\,T_\lambda T_\mu}[T_\lambda(T_\lambda+1)-T_\mu(T_\mu+1)]. \tag{33}$$

In the recursion formulae the double-barred coefficients are reduced R(5) Wigner coefficients of the type tabulated in refs. [5-7]). Sums over β are rarely needed, so that the two recursion formulae are simple three-term recursion formulae in almost all cases of interest.

For example, for irreducible representations $(\omega_1 t) = (\omega_1 0)$, recursion formula I becomes

$$c^{(\omega_1 0)T+1}_{H_1+1;\,T_\lambda T_\mu}[(\omega_1-H_1-T)(\omega_1+3+H_1+T)]^{\frac{1}{2}}$$
$$+c^{(\omega_1 0)T+1}_{H_1-1;\,T_\lambda T_\mu}[(\omega_1+H_1-T)(\omega_1+3-H_1+T)]^{\frac{1}{2}}$$
$$= -2c^{(\omega_1 0)T}_{H_1;\,T_\lambda T_\mu}[(T_\lambda+T_\mu+T+2)(T_\lambda+T_\mu-T)]^{\frac{1}{2}}. \tag{34}$$

As a second example, for irreducible representations $(\omega_1 t) = (tt)$, recursion formula II becomes

$$(t+1)\{c^{(tt)T}_{H_1+1;\,T_\lambda T_\mu}[(T+H_1+1)(T-H_1)]^{\frac{1}{2}}+c^{(tt)T}_{H_1-1;\,T_\lambda T_\mu}[(T-H_1+1)(T+H_1)]^{\frac{1}{2}}\}$$
$$= -2c^{(tt)T}_{H_1;\,T_\lambda T_\mu}[T_\lambda(T_\lambda+1)-T_\mu(T_\mu+1)]. \tag{35}$$

Since only the c-coefficients carry an explicit n-dependence, the symmetry properties embodied in the rules of eq. (19) can be seen explicitly from these recursion formulae. From recursion formula II, applied to an unsymmetrized state, for example, it can be seen that the c-coefficients $c^{(\omega_1 t)T}_{\beta n\pm 2;\,T_\lambda T_\mu=ab}$ and $c^{(\omega_1 t)T}_{\beta n\pm 2;\,T_\lambda T_\mu=ba}$ will have the opposite sign, if the c-coefficients $c^{(\omega_1 t)T}_{\beta n;\,T_\lambda T_\mu=ab}$ and $c^{(\omega_1 t)T}_{\beta n;\,T_\lambda T_\mu=ba}$ have the same sign, and vice versa. Recursion formula I indicates that a change of *both* $\Delta n = \pm 2$ *and* $\Delta T = +1$ leads to no change in the relative phase of the coefficients with $T_\lambda T_\mu = ab$ and ba, respectively. However, now the phase factor $(-1)^{T-a-b}$ which is part of the definition of the symmetrized states (19) will change as T is replaced by $T+1$, so that it is again seen that a step $\Delta n = \pm 2$ induces an overall change in symmetry. States with $n = 2j+1$, $T = j+\frac{1}{2}$ have $T_\lambda = T_\mu = \frac{1}{2}(j+\frac{1}{2})$; $J_\lambda = J_\mu = J = 0$; hence they are (s) states, eq. (18b). States with $n = 2j+2$; $T = j$ can be seen to be (a) states by explicit construction, (based on the phase conventions of sect. 3).

In principle, the d-factors of the transformation coefficients can be calculated by similar techniques. If the $J = 0$, $T = 1$ pair creation and annihilation operators of eq. (24) are replaced by pair creation and annihilation operators coupled to $J = 1$,

$T = 0$, equations analogous to eqs. (26) to (29) can be derived in which T_λ, T_μ, T are replaced by J_λ, J_μ, J, leading to recursion formulae in the d-coefficients in place of the c-coefficients. In practice these are not very useful since matrix elements of operators $[a^\dagger \times a^\dagger]^{J=1,\ T=0}$ are not known to any degree of generality and are complicated functions of j and the symplectic subgroup labels, in particular for states of high seniority where the multiplicities designated by α may be very large.

An alternate procedure has therefore been used to calculate the d coefficients. Certain special states are automatically states of good particle number in the $|(J_\lambda J_\mu)J, (T_\lambda T_\mu)T\rangle$ basis. The most trivial example is the state with $n = 2j+1$, $J = 0$, $T = T_{\text{max.}} = j+\frac{1}{2}$; $J_\lambda = J_\mu = 0$, $T_\lambda = T_\mu = \frac{1}{2}(j+\frac{1}{2})$. Starting with this state it is possible to construct all states with $n = 2j+2$, $n = 2j+1$ and lower T by successive application of the single nucleon creation and annihilation operators $a^\dagger$ and a. In this process $a^\dagger(a)$ are expressed in terms of Λ, M by eqs. (9), and matrix elements of Λ and M in the $|(J_\lambda J_\mu)J, (T_\lambda T_\mu)T\rangle$ basis are reduced by standard Racah algebra to the double-barred matrix elements of sect. 3. In general, the $a^\dagger$ (or a) operator, when acting on an initial state of fixed seniority v and reduced isospin t, will yield states with $v' = v\pm 1$, $t' = t\pm\frac{1}{2}$. It is however, possible to choose n and T to obtain a final state with unique v', t', or at most a combination of states v', t' of which only a single set of values is as yet unknown. If states of a relatively large value of T are known, it is also possible to act with the operators of eqs. (25) to construct states with isospin T–1 and obtain coefficients d with $T_\lambda + T_\mu = T-1$ from the known d-coefficients with $T_\lambda + T_\mu \geqq T$. Additional simplification comes from the orthonormality of the coefficients $d^{T_\lambda T_\mu, J}_{(\omega_1 t)\alpha; J_\lambda J_\mu}$. As a relatively complicated example consider the d-matrices for the $j = \frac{7}{2}$ shell with $(T_\lambda T_\mu) = (1\frac{1}{2})$. For $J = \frac{5}{2}$, e.g., this d-matrix is a 10×10 matrix with (J_λ, J_μ) values of $(2, \frac{3}{2})$, $(2, \frac{5}{2})$, $(2, \frac{9}{2})$, $(4, \frac{3}{2})$, $(4, \frac{5}{2})$, $(4, \frac{9}{2})$, $(4, \frac{11}{2})$, $(6, \frac{9}{2})$, $(6, \frac{11}{2})$, and $(6, \frac{15}{2})$. The $(\omega_1, t; \alpha)$ values are: $(\frac{5}{2}, \frac{1}{2}; \alpha = 1$ and $2)$; $(\frac{5}{2}, \frac{3}{2}; \alpha = 1)$; $(\frac{3}{2}, \frac{3}{2}; \alpha = 1$ and $2)$; $(\omega_1 t; \alpha) = (\frac{3}{2}, \frac{1}{2}; \alpha = 1, \ldots 5)$. However if the five states with $(\omega_1, t) = (\frac{5}{2}, \frac{1}{2})$, $(\frac{5}{2}, \frac{3}{2})$, and $(\frac{3}{2}, \frac{3}{2})$ are known, the remaining five rows of the d-matrix corresponding to the (ω_1, t) value $(\frac{3}{2}, \frac{1}{2})$ follow from orthonormality alone. Since there is complete arbitrariness in the labeling $\alpha = 1, \ldots, 5$ of these states, (due to the incompleteness of the classification scheme $\mathrm{Sp}(2j+1) \supset \mathrm{R}(3)$), any arbitrary orthogonalization process will serve with equal generality to fix the remaining five states of this example.

The full set of c- and d-coefficients needed for $j \leqq \frac{5}{2}$ are tabulated in an appendix. For $j = \frac{7}{2}$ the tables of d-coefficients require considerable space and will be published elsewhere.

5. Matrix elements of one-and two-body operators

The application of the quasiparticle technique to the calculation of matrix elements of physical operators is very straightforward. As a first step, operators $a^\dagger$ and a are expressed in terms of the double spherical tensors Λ, M by means of eqs. (9). Secondly, an operator built from Λ-tensors coupled to one built from M-tensors is reduced via standard formulae of Racah algebra to double-barred matrix elements in the separate λ and μ spaces. These can be expressed in terms of the relatively small number of reduced matrix elements of type $\langle \Delta' J'_\lambda T'_\lambda || A || \Delta J_\lambda T_\lambda \rangle$.

Any one-body operator can be expanded in terms of

$$[a^\dagger \times a]^{JT}_{M_J M_T} = \sum_{mm_t} \langle jmj-m'|JM_J\rangle \langle \tfrac{1}{2}m_t \tfrac{1}{2}-m'_t|TM_T\rangle a^\dagger_{mm_t} a_{m'm'_t} (-1)^{j-m'+\frac{1}{2}-m'_t}. \quad (36)$$

These can be expressed as combinations of A- and M-tensors by

$$[a^\dagger \times a]^{JT}_{M_J M_T} = -\tfrac{1}{2}[1-(-1)^{J+T}][A\times A]^{JT}_{M_J M_T} + \tfrac{1}{2}[1-(-1)^{J+T}][M\times M]^{JT}_{M_J M_T} + \tfrac{1}{2}[1+(-1)^{J+T}][A\times M]^{JT}_{M_J M_T} + \delta_{J0}\delta_{T0}[\tfrac{1}{2}(2j+1)]^{\frac{1}{2}}. \quad (37)$$

The technique of calculating matrix elements will be illustrated in some detail for the most general rotationally invariant, charge-independent two-body interaction acting within a single j-shell. In terms of the two-particle matrix elements

$$V_{JT} = \langle j^2 JM_J\, TM_T|V|j^2 JM_J\, TM_T\rangle,$$

and coupled tensors, defined according to eq. (8), this can be put in the form

$$\begin{aligned} H_{2\text{-body}} = & \left[\frac{1}{2(2j+1)}(n-2j-1)+\tfrac{1}{8}\right] \sum_{JT}(2J+1)(2T+1)V_{JT} \\ & -\tfrac{1}{8}\sum_{JT} V_{JT}[(2J+1)(2T+1)]^{\frac{1}{2}} \\ & \times \Bigg\{ [[A\times A]^{JT}\times[A\times A]^{JT}]^{00}_{00} + [[M\times M]^{JT}\times[M\times M]^{JT}]^{00}_{00} \\ & + \sum_{J_0T_0} [[A\times A]^{J_0T_0}\times[M\times M]^{J_0T_0}]^{00}_{00} \\ & \times \left(2\delta_{JJ_0}\delta_{TT_0} + 4[(2J+1)(2J_0+1)(2T+1)(2T_0+1)]^{\frac{1}{2}} \begin{Bmatrix} j & j & J \\ j & j & J_0 \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & T \\ \frac{1}{2} & \frac{1}{2} & T_0 \end{Bmatrix} \right) \Bigg\}, \end{aligned} \quad (38)$$

with J_0+T_0 = odd and $J+T$ = odd. Except for a trivial constant term and pure A- and M-terms, whose matrix elements are evaluated like those for configurations of identical nucleons (e.g. neutrons only), there is only a single term coupling the two spaces. This coupling terms involves only A-pairs and M-pairs. Its matrix elements are evaluated by standard Racard techniques

$$\begin{aligned} & \langle \Delta(J'_\lambda J'_\mu)JM_J, (T'_\lambda T'_\mu)TM_T | [[A\times A]^{J_0T_0}\times[M\times M]^{J_0T_0}]^{00}_{00} | \Delta(J_\lambda J_\mu)JM_J, (T_\lambda T_\mu)TM_T\rangle \\ & = (-1)^{J_0+T_0+J+J_\lambda+J'_\mu+T+T_\lambda+T'_\mu}[(2J_0+1)(2T_0+1)]^{-\frac{1}{2}} \begin{Bmatrix} J_\lambda & J_\mu & J \\ J'_\mu & J'_\lambda & J_0 \end{Bmatrix} \begin{Bmatrix} T_\lambda & T_\mu & T \\ T'_\mu & T'_\lambda & T_0 \end{Bmatrix} \\ & \times \langle \Delta J'_\lambda T'_\lambda || [A\times A]^{J_0T_0} || \Delta J_\lambda T_\lambda\rangle \langle \Delta J'_\mu T'_\mu || [M\times M]^{J_0T_0} || \Delta J_\mu T_\mu\rangle, \end{aligned} \quad (39)$$

where

$$\langle \Delta J'_\lambda T'_\lambda \| [\Lambda \times \Lambda]^{J_0 T_0} \| \Delta J_\lambda T_\lambda \rangle = \sum_{J''_\lambda T''_\lambda} \langle \Delta J'_\lambda T'_\lambda \| \Lambda \| \Delta'' J''_\lambda T''_\lambda \rangle \langle \Delta'' J''_\lambda T''_\lambda \| \Lambda \| \Delta J_\lambda T_\lambda \rangle$$

$$\times [(2J_0+1)(2T_0+1)]^{\frac{1}{2}} (-1)^{J_\lambda + J'_\lambda + T_\lambda + T'_\lambda + J_0 + T_0} \begin{Bmatrix} J_\lambda & j & J''_\lambda \\ j & J'_\lambda & J_0 \end{Bmatrix} \begin{Bmatrix} T_\lambda & \frac{1}{2} & T''_\lambda \\ \frac{1}{2} & T'_\lambda & T_0 \end{Bmatrix}, \quad (40)$$

with $\Delta' = \Delta$; while $\Delta'' = \Delta_-$ for $\Delta = \Delta_+$, and vice versa. Consequently

$$\langle \Delta cd \| [M \times M]^{J_0 T_0} \| \Delta ab \rangle = -\langle \Delta cd \| [\Lambda \times \Lambda]^{J_0 T_0} \| \Delta ab \rangle. \quad (41)$$

In conclusion, we will finally examine the question: does this new method of calculating matrix elements for the nuclear j-shell of both protons and neutrons have real advantages over the conventional techniques involving c.f.p. expansions. In some ways the new λ, μ-quasiparticle classification of the nuclear j-shell bears a resemblance to the old classification scheme in terms of separate neutron and proton configurations. Both lead to the same set of reduced matrix elements, (quite small in number). In the latter scheme J and nucleon number are automatically good quantum numbers while T is not. In the new scheme J and T are automatically good quantum numbers, but nucleon number is not. However, the new classification scheme differs in one vital respect. By furnishing a complete classification, the new scheme leads to a calculation of matrix elements by straightforward Racah algebra without additional normalization or projection factors, once the transformation to a basis of good nucleon number has been effected. In particular, the calculation of matrix elements of two-body operators is essentially no more complicated than that for one-body operators. On the other hand, the number of transformation coefficients needed to construct states of good particle number may become quite large. For the $j = \frac{7}{2}$ shell, e.g., the number of d coefficients, in particular, is somewhat large. By contrast, the *total* number of cfp's needed to calculate matrix elements of two-body operators for the full $j = \frac{7}{2}$ shell is overwhelming. Even though the five-dimensional quasispin group can be used with both techniques to factor out the n, T dependence of the coefficients, (the c.f.p.'s on the one hand, the transformation coefficients to states of good particle number on the other), this factoring of the n, T dependence of the coefficients is again considerably simpler with the new technique. In summary therefore the new technique does seem to lead to a real simplification in the calculation of matrix elements for the nuclear j-shell.

6. The LST scheme

Exactly as for the j-shell, the space of real particles of the nuclear l-shell can be divided into two subspaces of anticommuting quasiparticles:

$$\lambda^\dagger_{mm_s} = \frac{1}{\sqrt{2}} (a^\dagger_{mm_s+\frac{1}{2}} - (-1)^{l-m+\frac{1}{2}-m_s} a_{-m-m_s-\frac{1}{2}}); \qquad \lambda_{mm_s} = (\lambda^\dagger_{mm_s})^\dagger,$$

$$\mu^\dagger_{mm_s} = \frac{1}{\sqrt{2}} (a^\dagger_{mm_s-\frac{1}{2}} - (-1)^{l-m+\frac{1}{2}-m_s} a_{-m-m_s+\frac{1}{2}}); \qquad \mu_{mm_s} = (\mu^\dagger_{mm_s})^\dagger, \quad (42)$$

$$m = -l, \ldots, +l; \qquad m_s = -\tfrac{1}{2}, +\tfrac{1}{2},$$

TABLE 2

The *LST* quasiparticle factorization. Generators, groups, representations

$$\begin{bmatrix} (a^{\dagger\dagger})^{LST} \\ (aa)^{LST} \\ (a^{\dagger}a)^{LST} \end{bmatrix} \supset \begin{bmatrix} [(\lambda^\dagger\lambda^\dagger)^{L_\lambda S_\lambda}; (\lambda\lambda)^{L_\lambda S_\lambda}; (\lambda^\dagger\lambda)^{L_\lambda S_\lambda}] \supset [(\lambda^\dagger\lambda)^{L_\lambda = \mathrm{odd},0} \times (\lambda^\dagger\lambda)^{01}] \times \begin{bmatrix} (\lambda^\dagger\lambda^\dagger)^{00}; (\lambda\lambda)^{00}; \\ \frac{1}{2}(\lambda^\dagger\lambda)^{00} + \frac{1}{2}(\lambda\lambda^\dagger)^{00}; \end{bmatrix} \\ \times [(\mu^\dagger\mu^\dagger)^{L_\mu S_\mu}; (\mu\mu)^{L_\mu S_\mu}; (\mu^\dagger\mu)^{L_\mu S_\mu}] \supset \text{same in } \mu \end{bmatrix}$$

$$R(16l+8) \supset \begin{bmatrix} R_\lambda(8l+4) & \supset [[[R_\lambda(2l+1) \supset R_{L_\lambda}(3)] \times SU_{S_\lambda}(2)] \times SU_{T_\lambda}(2)] \\ \times R_\mu(8l+4) & \supset \text{same in } \mu \end{bmatrix}$$

$$(\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2} \ldots \pm\tfrac{1}{2}) \supset \begin{matrix} \lambda \text{ space: } (\tfrac{1}{2}\ \tfrac{1}{2} \ldots \pm\tfrac{1}{2}) & \supset [[[(2^{a_\lambda}1^{b_\lambda}0^{l-a_\lambda-b_\lambda}) \supset L_\lambda] \times S_\lambda] \times T_\lambda] \\ (\text{same in } \mu \text{ space}) & \end{matrix}$$

where $a^{\dagger}_{mm_sm_t}$, $(a_{mm_sm_t})$ are real nucleon creation, (annihilation) operators, and m, m_s, m_t are the third components of the orbital angular momentum, spin, and isospin vectors, respectively.

The group theoretical basis of this quasiparticle factorization is shown in table 2. The full set of operators $[a^{\dagger}a^{\dagger}]^{LST}$, $[aa]^{LST}$, $[a^{\dagger}a]^{LST}$ are known to be generators of a group $\mathrm{R}(16l+8)$. As for the nuclear j-shell, we have chosen to restrict the isospin quantum number m_t to select a set of *independent* λ and μ quasiparticle operators. As a result the quasiparticle operators $\lambda^{\dagger}_{mm_s}(\lambda_{mm_s})$, similarly $\mu^{\dagger}_{mm_s}(\mu_{mm_s})$, are *mathematically* equivalent to real atomic electron creation and annihilation operators. The group $\mathrm{R}(16l+8)$ thus factors into two commuting subgroups, each $\mathrm{R}(8l+4)$, one created by the operators $(\lambda^{\dagger}\lambda^{\dagger})^{L_\lambda S_\lambda}$, $(\lambda\lambda)^{L_\lambda S_\lambda}$, $(\lambda^{\dagger}\lambda)^{L_\lambda S_\lambda}$, and the other by the analogous μ-operators. Further subgroups are provided [8] by the extraction of the operators $(\lambda^{\dagger}\lambda)^{L_\lambda S_\lambda}$ with $L_\lambda+S_\lambda$ = odd, which generate a group $\mathrm{Sp}(4l+2)$, and which commute with the operators $(\lambda^{\dagger}\lambda^{\dagger})^{00}$, $(\lambda\lambda)^{00}$, $\frac{1}{2}(\lambda^{\dagger}\lambda)^{00}+(\lambda^{\dagger}\lambda)^{00})$. These in turn constitute the three components of a quasispin operator in the λ-space and are to be denoted by $\boldsymbol{T}_\lambda$ in a notation appropriate to the quasi-particle factorization technique. The further subgroup chains in the λ- and μ-spaces are mathematically equivalent to the conventional classification scheme of atomic electrons. The operators $(\lambda^{\dagger}\lambda)^{L_\lambda=\mathrm{odd},\, S_\lambda=0}$ generate a seniority group $\mathrm{R}_\lambda(2l+1)$. They contain the angular momentum operators $(\lambda^{\dagger}\lambda)^{10}$ and commute with the quasiparticle spin operators $(\lambda^{\dagger}\lambda)^{01}$, and are to be denoted by $\boldsymbol{L}_\lambda$ and $\boldsymbol{S}_\lambda$, respectively. With appropriate normalization and phase factors these quasiparticle angular momentum operators for the λ- and μ-spaces satisfy the relations

$$\begin{aligned} \boldsymbol{L}_\lambda+\boldsymbol{L}_\mu &= \boldsymbol{L}, \\ \boldsymbol{S}_\lambda+\boldsymbol{S}_\mu &= \boldsymbol{S}, \\ \boldsymbol{T}_\lambda+\boldsymbol{T}_\mu &= \boldsymbol{T}, \end{aligned} \tag{43}$$

where $\boldsymbol{L}$, $\boldsymbol{S}$, $\boldsymbol{T}$, are the usual total orbital angular momentum, spin and isospin operators for real nucleons.

Making further use of the mathematical equivalence between the λ operators and real atomic electron operators, it can be seen that (i) the quasispin group generated by $\boldsymbol{T}_\lambda$ has irreducible representations characterized by a seniority quantum number v_λ where

$$T_\lambda = \tfrac{1}{2}(2l+1-v_\lambda), \tag{44}$$

(ii) the irreducible representations of the group $\mathrm{R}_\lambda(2l+1)$ are labeled by $(2^{a_\lambda}1^{b_\lambda}0^{c_\lambda})$ with $a_\lambda+b_\lambda+c_\lambda = l$,

(iii) the possible values of v_λ and S_λ consistent with a specific irreducible representation of $\mathrm{R}_\lambda(2l+1)$ are given by the usual rules of atomic spectroscopy and can also be derived by quasispin techniques. The result can be stated by the theorem:

To every irreducible representation $(2^{a_\lambda}1^{b_\lambda}0^{c_\lambda})$ of $\mathrm{R}_\lambda(2l+1)$ there corresponds a pair of irreducible representations of the direct product group $[\mathrm{SU}_{\mathrm{S}_\lambda}(2)\times\mathrm{SU}_{\mathrm{T}_\lambda}(2)]$

with

$$\begin{pmatrix} S_\lambda, T_\lambda \\ T_\lambda, S_\lambda \end{pmatrix} = (\tfrac{1}{2}(2l+1)-a_\lambda-\tfrac{1}{2}b_\lambda, \tfrac{1}{2}b_\lambda). \tag{45}$$

Analogous results hold for the μ-space. This result is equivalent to a relation by Racah, eq. (20) of ref. [9]).

The quasiparticle factorization again leads to a more complete classification scheme in terms of the above quantum numbers. In particular, the state vectors

$$|(L_\lambda L_\mu)LM_L, (S_\lambda S_\mu)SM_S, (T_\lambda T_\mu)TM_T\rangle,$$

are the basis for a *complete* classification scheme for nuclear shells with $l \leqq 2$. In this basis total L, S, T are good quantum numbers; but neither the particle number n nor quantum numbers such as the Wigner supermultiplet quantum numbers are preserved in this new scheme. Although states of good particle number could in principle be constructed using the techniques employed for the j-shell as a guide, none of the details have been worked out since the nuclear LST scheme is useful mainly in those nuclei in which the Wigner supermultiplet quantum numbers are approximately good. No attempts have been made to regain both Wigner supermultiplet numbers and good particle number since this appears to be a difficult task. As a result the λ, μ-quasiparticle factorization technique may be of little practical value in the nuclear l^n configuration.

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Appendix

TABLES OF TRANSFORMATION COEFFICIENTS

The transformation to states of good particle number is made through the transformation coefficients $c^{(\omega_1 t)T}_{H_1\beta; T_\lambda T_\mu}$ and $d^{T_\lambda T_\mu J}_{(\omega_1 t)\alpha; J_\lambda J_\mu}$ defined by eq. (23). The c-coefficients are functions of the R(5) quantum numbers $(\omega_1 t)$, $H_1 = \frac{1}{2}n-j-\frac{1}{2}$, and β (when needed), and are thus valid for all j. The c-coefficients for representations $(\omega_1 t)$ needed for $j \leqq \frac{5}{2}$ are listed in tables 3. For $j \leqq \frac{3}{2}$ no d-coefficients are needed; that is all the d, can be chosen as $+1$, subject to the usual arbitrariness in overall phases. The d-coefficients for $j = \frac{5}{2}$ are listed in tables 4. Whenever a d-matrix for a particular set of values $T_\lambda T_\mu$, J is 1×1 and can be chosen as $+1$, it is not listed specifically in the tables. Some d-matrices are unit matrices of dimension greater than one. These are tabulated explicitly when they are needed for identification of the label α. As an example, the matrix $\boldsymbol{d}^{1\frac{1}{2},\frac{7}{2}}$ is a 2×2 unit matrix since the states with $(\omega_1 t) = (\frac{3}{2}, \frac{1}{2})$, i.e., $v = 3, t = \frac{1}{2}$; $J = \frac{7}{2}$, $T = \frac{3}{2}$; are automatically states of good n. Specifically, the two states

$$|(J_\lambda J_\mu)J, (T_\lambda T_\mu)T\rangle_{(\sigma)} = |\tfrac{5}{2}, 2)\tfrac{7}{2}, (1, \tfrac{1}{2})\tfrac{3}{2}\rangle_{(a)} \text{ and } |(\tfrac{5}{2}, 4)\tfrac{7}{2}, (1, \tfrac{1}{2})\tfrac{3}{2}\rangle_{(a)}$$

TABLE 3

Tables of the transformation coefficients $c^{(\omega_1 t)T}_{H_1\beta;\, T_\lambda T_\mu}$

Possible $T_{\lambda,\mu}$–$J_{\lambda,\mu}$ values for $j = \frac{5}{2}$

$T_{\lambda,\mu}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
$J_{\lambda,\mu}$	$\frac{3}{2}$ $\frac{9}{2}$	2 4	$\frac{5}{2}$	0

Tables of c-coefficients

$(\omega, t) = (00)$

T	0
	$(T_\lambda T_\mu)$
H_1	(00)
0	$+1$

$(\omega, t) = (\frac{1}{2}\frac{1}{2})$

T	$\frac{1}{2}$
	$(T_\lambda T_\mu)$
H_1	$(0\frac{1}{2})$
$\pm\frac{1}{2}$	$+1$

$(\omega, t) = (10)$

T	0		1
	$(T_\lambda T_\mu)$		
H_1	(00)	$(\frac{1}{2}\frac{1}{2})$	$(\frac{1}{2}\frac{1}{2})$
$+1$	$-[\frac{1}{2}]^{\frac{1}{2}}$	$-[\frac{1}{2}]^{\frac{1}{2}}$	
0			$+1$
-1	$+[\frac{1}{2}]^{\frac{1}{2}}$	$-[\frac{1}{2}]^{\frac{1}{2}}$	

$(\omega, t) = (11)$

T	0	1	
	$(T_\lambda T_\mu)$		
H_1	$(\frac{1}{2}\frac{1}{2})$	(10)	$(\frac{1}{2}\frac{1}{2})$
$+1$		$-[\frac{1}{2}]^{\frac{1}{2}}$	$+[\frac{1}{2}]^{\frac{1}{2}}$
0	$+1$	$+1$	0
-1		$-[\frac{1}{2}]^{\frac{1}{2}}$	$-[\frac{1}{2}]^{\frac{1}{2}}$

$(\omega, t) = (\frac{3}{2}\frac{1}{2})$

T	$\frac{1}{2}$		$\frac{3}{2}$
	$(T_\lambda T_\mu)$		
H_1	$(0\frac{1}{2})$	$(1\frac{1}{2})$	$(1\frac{1}{2})$
$+\frac{3}{2}$	$-[\frac{5}{8}]^{\frac{1}{2}}$	$-[\frac{3}{8}]^{\frac{1}{2}}$	
$+\frac{1}{2}$	$-[\frac{3}{8}]^{\frac{1}{2}}$	$+[\frac{5}{8}]^{\frac{1}{2}}$	$+1$
$-\frac{1}{2}$	$+[\frac{3}{8}]^{\frac{1}{2}}$	$-[\frac{5}{8}]^{\frac{1}{2}}$	-1
$-\frac{3}{2}$	$+[\frac{5}{8}]^{\frac{1}{2}}$	$+[\frac{3}{8}]^{\frac{1}{2}}$	

$(\omega, t) = (\frac{3}{2}\frac{3}{2})$

T	$\frac{1}{2}$	$\frac{3}{2}$	
	$(T_\lambda T_\mu)$		
H_1	$(1\frac{1}{2})$	$(0\frac{3}{2})$	$(1\frac{1}{2})$
$+\frac{3}{2}$		$+\frac{1}{2}$	$+[\frac{3}{4}]^{\frac{1}{2}}$
$+\frac{1}{2}$	$+1$	$+[\frac{3}{4}]^{\frac{1}{2}}$	$-\frac{1}{2}$
$-\frac{1}{2}$	-1	$+[\frac{3}{4}]^{\frac{1}{2}}$	$-\frac{1}{2}$
$-\frac{3}{2}$		$+\frac{1}{2}$	$+[\frac{3}{4}]^{\frac{1}{2}}$

$(\omega, t) = (20)$

T	0			1		2
	$(T_\lambda\, T_\mu)$					
H_1	(00)	$(\frac{1}{2}\frac{1}{2})$	(11)	$(\frac{1}{2}\frac{1}{2})$	(11)	(11)
$+2$	$+[\frac{5}{16}]^{\frac{1}{2}}$	$+[\frac{1}{2}]^{\frac{1}{2}}$	$+[\frac{3}{16}]^{\frac{1}{2}}$			
$+1$				$-[\frac{1}{2}]^{\frac{1}{2}}$	$-[\frac{1}{2}]^{\frac{1}{2}}$	
0	$-[\frac{3}{8}]^{\frac{1}{2}}$	0	$+[\frac{5}{8}]^{\frac{1}{2}}$			$+1$
-1				$+[\frac{1}{2}]^{\frac{1}{2}}$	$-[\frac{1}{2}]^{\frac{1}{2}}$	
-2	$+[\frac{5}{16}]^{\frac{1}{2}}$	$-[\frac{1}{2}]^{\frac{1}{2}}$	$+[\frac{3}{16}]^{\frac{1}{2}}$			

$(\omega, t) = (21)$

T		0		1				2	
		$(T_\lambda\, T_\mu)$							
		$(\frac{1}{2}\,\frac{1}{2})$	(11)	$(\frac{1}{2}\,\frac{1}{2})$	$(\frac{3}{2}\,\frac{1}{2})$	(10)	(11)	$(\frac{3}{2}\,\frac{1}{2})$	(11)
$+2$				$+[\frac{1}{3}]^{\frac{1}{2}}$	$+[\frac{1}{6}]^{\frac{1}{2}}$	$+[\frac{3}{8}]^{\frac{1}{2}}$	$+[\frac{1}{8}]^{\frac{1}{2}}$		
$+1$		$+[\frac{1}{2}]^{\frac{1}{2}}$	$+[\frac{1}{2}]^{\frac{1}{2}}$	0	$-[\frac{1}{2}]^{\frac{1}{2}}$	$-[\frac{1}{2}]^{\frac{1}{2}}$	0	$-[\frac{1}{2}]^{\frac{1}{2}}$	$-[\frac{1}{2}]^{\frac{1}{2}}$
0	β_1			$-[\frac{1}{3}]^{\frac{1}{2}}$	$+[\frac{2}{3}]^{\frac{1}{2}}$	0	0	$+1$	0
	β_2			0	0	$+\frac{1}{2}$	$-[\frac{3}{4}]^{\frac{1}{2}}$		
-1		$-[\frac{1}{2}]^{\frac{1}{2}}$	$+[\frac{1}{2}]^{\frac{1}{2}}$	0	$-[\frac{1}{2}]^{\frac{1}{2}}$	$+[\frac{1}{2}]^{\frac{1}{2}}$	0	$-[\frac{1}{2}]^{\frac{1}{2}}$	$+[\frac{1}{2}]^{\frac{1}{2}}$
-2				$+[\frac{1}{3}]^{\frac{1}{2}}$	$+[\frac{1}{6}]^{\frac{1}{2}}$	$-[\frac{3}{8}]^{\frac{1}{2}}$	$-[\frac{1}{8}]^{\frac{1}{2}}$		

$(\omega, t) = (\frac{5}{2}\frac{1}{2})$

T	$\frac{1}{2}$			$\frac{3}{2}$		$\frac{5}{2}$
	$(T_\lambda\, T_\mu)$					
H_1	$(0\frac{1}{2})$	$(1\frac{1}{2})$	$(1\frac{3}{2})$	$(1\frac{1}{2})$	$(1\frac{3}{2})$	$(1\frac{3}{2})$
$+\frac{5}{2}$	$+[\frac{7}{16}]^{\frac{1}{2}}$	$-[\frac{7}{16}]^{\frac{1}{2}}$	$+[\frac{1}{8}]^{\frac{1}{2}}$			
$+\frac{3}{2}$	$+[\frac{3}{16}]^{\frac{1}{2}}$	$+[\frac{25}{48}]^{\frac{1}{2}}$	$+[\frac{7}{24}]^{\frac{1}{2}}$	$+[\frac{7}{12}]^{\frac{1}{2}}$	$-[\frac{5}{12}]^{\frac{1}{2}}$	
$+\frac{1}{2}$	$-[\frac{3}{8}]^{\frac{1}{2}}$	$-[\frac{1}{24}]^{\frac{1}{2}}$	$+[\frac{7}{12}]^{\frac{1}{2}}$	$-[\frac{5}{12}]^{\frac{1}{2}}$	$-[\frac{7}{12}]^{\frac{1}{2}}$	$+1$
$-\frac{1}{2}$	$-[\frac{3}{8}]^{\frac{1}{2}}$	$-[\frac{1}{24}]^{\frac{1}{2}}$	$+[\frac{7}{12}]^{\frac{1}{2}}$	$-[\frac{5}{12}]^{\frac{1}{2}}$	$-[\frac{7}{12}]^{\frac{1}{2}}$	$+1$
$-\frac{3}{2}$	$+[\frac{3}{16}]^{\frac{1}{2}}$	$+[\frac{25}{48}]^{\frac{1}{2}}$	$+[\frac{7}{24}]^{\frac{1}{2}}$	$+[\frac{7}{12}]^{\frac{1}{2}}$	$-[\frac{5}{12}]^{\frac{1}{2}}$	
$-\frac{5}{2}$	$+[\frac{7}{16}]^{\frac{1}{2}}$	$-[\frac{7}{16}]^{\frac{1}{2}}$	$+[\frac{1}{8}]^{\frac{1}{2}}$			

$(\omega, t) = (30)$

T	0				1			2		3
	$(T_\lambda T_\mu)$									
H_1	(00)	$(\frac{1}{2}\frac{1}{2})$	(11)	$(\frac{3}{2}\frac{3}{2})$	$(\frac{1}{2}\frac{1}{2})$	(11)	$(\frac{3}{2}\frac{3}{2})$	(11)	$(\frac{3}{2}\frac{3}{2})$	$(\frac{3}{2}\frac{3}{2})$
$+3$	$-[\frac{7}{32}]^{\frac{1}{2}}$	$-[\frac{7}{16}]^{\frac{1}{2}}$	$-[\frac{9}{32}]^{\frac{1}{2}}$	$-\frac{1}{4}$						
$+2$					$+[\frac{7}{24}]^{\frac{1}{2}}$	$+[\frac{1}{2}]^{\frac{1}{2}}$	$+[\frac{5}{24}]^{\frac{1}{2}}$			
$+1$	$+[\frac{9}{32}]^{\frac{1}{2}}$	$+\frac{1}{4}$	$-[\frac{7}{32}]^{\frac{1}{2}}$	$-[\frac{7}{16}]^{\frac{1}{2}}$				$-[\frac{1}{2}]^{\frac{1}{2}}$	$-[\frac{1}{2}]^{\frac{1}{2}}$	
0					$-[\frac{5}{12}]^{\frac{1}{2}}$	0	$+[\frac{7}{12}]^{\frac{1}{2}}$			$+1$
-1	$-[\frac{9}{32}]^{\frac{1}{2}}$	$+\frac{1}{4}$	$+[\frac{7}{32}]^{\frac{1}{2}}$	$-[\frac{7}{16}]^{\frac{1}{2}}$				$+[\frac{1}{2}]^{\frac{1}{2}}$	$-[\frac{1}{2}]^{\frac{1}{2}}$	
-2					$+[\frac{7}{24}]^{\frac{1}{2}}$	$-[\frac{1}{2}]^{\frac{1}{2}}$	$+[\frac{5}{24}]^{\frac{1}{2}}$			
-3	$+[\frac{7}{32}]^{\frac{1}{2}}$	$-[\frac{7}{16}]^{\frac{1}{2}}$	$+[\frac{9}{32}]^{\frac{1}{2}}$	$-\frac{1}{4}$						

TABLE 4

Tables of the transformation coefficients $d^{T_\lambda T_\mu, J}_{(\omega_1 t)\alpha;\, J_\lambda J_\mu}$ for $j = \frac{5}{2}$

	$J = 0\ (T_\lambda T_\mu) = (00)$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	$(\frac{3}{2}\frac{3}{2})$	$(\frac{9}{2}\frac{9}{2})$
(10)	$-[\frac{5}{7}]^{\frac{1}{2}}$	$+[\frac{2}{7}]^{\frac{1}{2}}$
(30)	$+[\frac{2}{7}]^{\frac{1}{2}}$	$+[\frac{5}{7}]^{\frac{1}{2}}$

	$J = 0\ (T_\lambda T_\mu) = (\frac{1}{2}\frac{1}{2})$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	(22)	(44)
(10)	$-[\frac{9}{14}]^{\frac{1}{2}}$	$+[\frac{5}{14}]^{\frac{1}{2}}$
(30)	$+[\frac{5}{14}]^{\frac{1}{2}}$	$+[\frac{9}{14}]^{\frac{1}{2}}$

	$J = 1\ (T_\lambda T_\mu) = (00)$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	$(\frac{3}{2}\frac{3}{2})$	$(\frac{9}{2}\frac{9}{2})$
(00)	$+[\frac{33}{35}]^{\frac{1}{2}}$	$-[\frac{2}{35}]^{\frac{1}{2}}$
(20)	$+[\frac{2}{35}]^{\frac{1}{2}}$	$+[\frac{33}{35}]^{\frac{1}{2}}$

	$J = 1\ (T_\lambda T_\mu) = (\frac{1}{2}\frac{1}{2})$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	(22)	(44)
(11)	$-[\frac{6}{7}]^{\frac{1}{2}}$	$+[\frac{1}{7}]^{\frac{1}{2}}$
(20)	$+[\frac{1}{7}]^{\frac{1}{2}}$	$+[\frac{6}{7}]^{\frac{1}{2}}$

	$J = 2\ (T_\lambda T_\mu) = (00)$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)_\alpha$	$(\frac{3}{2}\frac{3}{2})$	$(\frac{9}{2}\frac{9}{2})$
$(10)_1$	$+1$	0
$(10)_2$	0	$+1$

	$J = 2\ (T_\lambda T_\mu) = (10)$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	$(\frac{5}{2}\frac{3}{2})$	$(\frac{5}{2}\frac{9}{2})$
(11)	$-[\frac{3}{7}]^{\frac{1}{2}}$	$-[\frac{4}{7}]^{\frac{1}{2}}$
(21)	$-[\frac{4}{7}]^{\frac{1}{2}}$	$+[\frac{3}{7}]^{\frac{1}{2}}$

	$J = 2\ (T_\lambda T_\mu) = (\frac{1}{2}\frac{1}{2})$		
	$(J_\lambda J_\mu)$		
$(\omega_1 t)_\alpha$	(22)	(44)	(24)
$(10)_1$	$-\frac{15}{7}[\frac{1}{7}]^{\frac{1}{2}}$	$+\frac{1}{7}[\frac{22}{7}]^{\frac{1}{2}}$	$-\frac{4}{7}[\frac{6}{7}]^{\frac{1}{2}}$
$(10)_2$	$+\frac{3}{14}[\frac{33}{7}]^{\frac{1}{2}}$	$+\frac{25}{14}[\frac{3}{14}]^{\frac{1}{2}}$	$-\frac{5}{14}[\frac{11}{14}]^{\frac{1}{2}}$
(21)	$-\frac{5}{14}$	$+\frac{3}{14}[\frac{11}{2}]^{\frac{1}{2}}$	$+\frac{9}{14}[\frac{3}{2}]^{\frac{1}{2}}$

	$J = 3\ (T_\lambda T_\mu) = (10)$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)_\alpha$	$(\frac{5}{2}\frac{3}{2})$	$(\frac{5}{2}\frac{9}{2})$
$(11)_1$	$+1$	0
$(11)_2$	0	$+1$

	$J = 3\ (T_\lambda T_\mu) = (00)$		
	$(J_\lambda J_\mu)$		
$(\omega_1 t)_\alpha$	$(\frac{3}{2}\frac{3}{2})$	$(\frac{9}{2}\frac{9}{2})$	$(\frac{3}{2}\frac{9}{2})$
$(00)_1$	$+\frac{32}{7}[\frac{1}{67}]^{\frac{1}{2}}$	$+\frac{4}{7}[\frac{11 \cdot 13}{3 \cdot 67}]^{\frac{1}{2}}$	$-\frac{1}{7}[\frac{67}{3}]^{\frac{1}{2}}$
$(00)_2$	$+[\frac{11 \cdot 13}{5 \cdot 67}]^{\frac{1}{2}}$	$-8[\frac{3}{5 \cdot 67}]^{\frac{1}{2}}$	0
(20)	$-\frac{8}{7}[\frac{1}{5}]^{\frac{1}{2}}$	$-\frac{1}{7}[\frac{11 \cdot 13}{15}]^{\frac{1}{2}}$	$-\frac{4}{7}[\frac{5}{3}]^{\frac{1}{2}}$

	$J = 3\ (T_\lambda T_\mu) = (\frac{1}{2}\frac{1}{2})$		
	$(J_\lambda J_\mu)$		
$(\omega_1 t)_\alpha$	(22)	(44)	(24)
$(11)_1$	$+\frac{13}{7}[\frac{1}{7}]^{\frac{1}{2}}$	$+\frac{3}{7}[\frac{11}{7}]^{\frac{1}{2}}$	$+\frac{5}{7}[\frac{3}{7}]^{\frac{1}{2}}$
$(11)_2$	$+\frac{5}{14}[\frac{33}{14}]^{\frac{1}{2}}$	$-\frac{25}{14}[\frac{3}{14}]^{\frac{1}{2}}$	$+\frac{1}{7}[\frac{11}{14}]^{\frac{1}{2}}$
(20)	$-\frac{9}{14}[\frac{1}{2}]^{\frac{1}{2}}$	$-\frac{1}{14}[\frac{11}{2}]^{\frac{1}{2}}$	$+\frac{5}{7}[\frac{3}{2}]^{\frac{1}{2}}$

	$J = 4\ (T_\lambda T_\mu) = (00)$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)_\alpha$	$(\frac{9}{2}\frac{9}{2})$	$(\frac{3}{2}\frac{9}{2})$
$(10)_1$	+1	0
$(10)_2$	0	+1

	$J = 4\ (T_\lambda T_\mu) = (10)$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	$(\frac{5}{2}\frac{3}{2})$	$(\frac{5}{2}\frac{9}{2})$
(11)	$+[\frac{55}{63}]^{\frac{1}{2}}$	$+[\frac{8}{63}]^{\frac{1}{2}}$
(21)	$+[\frac{8}{63}]^{\frac{1}{2}}$	$-[\frac{55}{63}]^{\frac{1}{2}}$

	$J = 4\ (T_\lambda T_\mu) = (\frac{1}{2}\frac{1}{2})$		
	$(J_\lambda J_\mu)$		
$(\omega_1 t)_\alpha$	(22)	(44)	(24)
$(10)_1$	$+\frac{1}{28}[\frac{11\cdot 13}{7}]^{\frac{1}{2}}$	$-\frac{17}{28}[\frac{5}{7}]^{\frac{1}{2}}$	$-\frac{5}{14}[\frac{39}{7}]^{\frac{1}{2}}$
$(10)_2$	$+\frac{5}{7}[\frac{11}{7}]^{\frac{1}{2}}$	$+\frac{1}{7}[\frac{65}{7}]^{\frac{1}{2}}$	$-\frac{1}{7}[\frac{3}{7}]^{\frac{1}{2}}$
(21)	$+\frac{3}{28}[15]^{\frac{1}{2}}$	$-\frac{1}{28}[33\cdot 13]^{\frac{1}{2}}$	$+\frac{1}{14}[55]^{\frac{1}{2}}$

	$J = 5\ (T_\lambda T_\mu) = (00)$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	$(\frac{9}{2}\frac{9}{2})$	$(\frac{3}{2}\frac{9}{2})$
(00)	$+[\frac{8}{21}]^{\frac{1}{2}}$	$+[\frac{13}{21}]^{\frac{1}{2}}$
(20)	$+[\frac{13}{21}]^{\frac{1}{2}}$	$-[\frac{8}{21}]^{\frac{1}{2}}$

	$J = 5\ (T_\lambda T_\mu) = (\frac{1}{2}\frac{1}{2})$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	(44)	(24)
(11)	$-[\frac{15}{28}]^{\frac{1}{2}}$	$+[\frac{13}{28}]^{\frac{1}{2}}$
(20)	$-[\frac{13}{28}]^{\frac{1}{2}}$	$-[\frac{15}{28}]^{\frac{1}{2}}$

	$J = 6\ (T_\lambda T_\mu) = (00)$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)_\alpha$	$(\frac{9}{2}\frac{9}{2})$	$(\frac{3}{2}\frac{9}{2})$
$(10)_1$	$+1$	0
$(10)_2$	0	$+1$

	$J = 6\ (T_\lambda T_\mu) = (\frac{1}{2}\frac{1}{2})$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)_\alpha$	(44)	(24)
$(10)_1$	$-[\frac{24}{49}]^{\frac{1}{2}}$	$-[\frac{25}{49}]^{\frac{1}{2}}$
$(10)_2$	$+[\frac{25}{49}]^{\frac{1}{2}}$	$-[\frac{24}{49}]^{\frac{1}{2}}$

	$J = \frac{1}{2}\ (T_\lambda T_\mu) = (0\frac{1}{2})$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	$(\frac{3}{2}2)$	$(\frac{9}{2}4)$
$(\frac{1}{2}\frac{1}{2})$	$+[\frac{3}{35}]^{\frac{1}{2}}$	$+[\frac{32}{35}]^{\frac{1}{2}}$
$(\frac{3}{2}\frac{1}{2})$	$+[\frac{32}{35}]^{\frac{1}{2}}$	$-[\frac{3}{35}]^{\frac{1}{2}}$

	$J = \frac{3}{2}\ (T_\lambda T_\mu) = (0\frac{1}{2})$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	$(\frac{3}{2}2)$	$(\frac{9}{2}4)$
$(\frac{1}{2}\frac{1}{2})$	$+[\frac{33}{35}]^{\frac{1}{2}}$	$-[\frac{2}{35}]^{\frac{1}{2}}$
$(\frac{3}{2}\frac{1}{2})$	$+[\frac{2}{35}]^{\frac{1}{2}}$	$+[\frac{33}{35}]^{\frac{1}{2}}$

	$J = \frac{3}{2}\ (T_\lambda T_\mu) = (1\frac{1}{2})$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	$(\frac{5}{2}2)$	$(\frac{5}{2}4)$
$(\frac{3}{2}\frac{1}{2})$	$+[\frac{2}{7}]^{\frac{1}{2}}$	$+[\frac{5}{7}]^{\frac{1}{2}}$
$(\frac{3}{2}\frac{3}{2})$	$-[\frac{5}{7}]^{\frac{1}{2}}$	$+[\frac{2}{7}]^{\frac{1}{2}}$

	$J = \frac{5}{2}\ (T_\lambda T_\mu) = (0\frac{1}{2})$			
	$(J_\lambda J_\mu)$			
$(\omega_1 t)_\alpha$	$(\frac{3}{2}2)$	$(\frac{3}{2}4)$	$(\frac{9}{2}2)$	$(\frac{9}{2}4)$
$(\frac{1}{2}\frac{1}{2})_1$	$+5[\frac{6}{317}]^{\frac{1}{2}}$	$+3[\frac{15}{317}]^{\frac{1}{2}}$	$+[\frac{32}{317}]^{\frac{1}{2}}$	0
$(\frac{1}{2}\frac{1}{2})_2$	$+\frac{23}{7}[\frac{33}{5 \cdot 317}]^{\frac{1}{2}}$	$-\frac{11}{7}[\frac{22}{3 \cdot 317}]^{\frac{1}{2}}$	$-\frac{9}{7}[\frac{55}{317}]^{\frac{1}{2}}$	$-\frac{1}{7}[\frac{317}{15}]^{\frac{1}{2}}$
$(\frac{3}{2}\frac{1}{2})$	$+\frac{2}{7}[\frac{6}{5}]^{\frac{1}{2}}$	$-\frac{8}{7}[\frac{1}{3}]^{\frac{1}{2}}$	$+\frac{3}{7}[\frac{5}{2}]^{\frac{1}{2}}$	$-\frac{1}{7}[\frac{11}{30}]^{\frac{1}{2}}$
$(\frac{5}{2}\frac{1}{2})$	$-[\frac{10}{49}]^{\frac{1}{2}}$	$+\frac{2}{7}$	$+[\frac{15}{98}]^{\frac{1}{2}}$	$-[\frac{55}{98}]^{\frac{1}{2}}$

	$J = \frac{5}{2}\ (T_\lambda T_\mu) = (1\,\frac{1}{2})$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	$(\frac{5}{2}\,2)$	$(\frac{5}{2}\,4)$
$(\frac{3}{2}\,\frac{1}{2})$	$-[\frac{9}{14}]^{\frac{1}{2}}$	$+[\frac{5}{14}]^{\frac{1}{2}}$
$(\frac{5}{2}\,\frac{1}{2})$	$+[\frac{5}{14}]^{\frac{1}{2}}$	$+[\frac{9}{14}]^{\frac{1}{2}}$

	$J = \frac{7}{2}\ (T_\lambda T_\mu) = (0\,\frac{1}{2})$			
	$(J_\lambda J_\mu)$			
$(\omega_1 t)_\alpha$	$(\frac{3}{2}\,2)$	$(\frac{3}{2}\,4)$	$(\frac{9}{2}\,2)$	$(\frac{9}{2}\,4)$
$(\frac{1}{2}\,\frac{1}{2})_1$	$+\frac{3}{14}[\frac{11\cdot 13}{10}]^{\frac{1}{2}}$	$-\frac{1}{14}[\frac{11\cdot 13}{30}]^{\frac{1}{2}}$	$+\frac{1}{7}[\frac{6\cdot 13}{5}]^{\frac{1}{2}}$	$-\frac{1}{7}[\frac{1}{30}]^{\frac{1}{2}}$
$(\frac{1}{2}\,\frac{1}{2})_2$	$+\frac{9}{14}[\frac{1}{10}]^{\frac{1}{2}}$	$-\frac{59}{14}[\frac{1}{30}]^{\frac{1}{2}}$	$-\frac{1}{7}[\frac{66}{5}]^{\frac{1}{2}}$	$-\frac{1}{7}[\frac{11\cdot 13}{30}]^{\frac{1}{2}}$
$(\frac{3}{2}\,\frac{1}{2})_1$	$-\frac{9}{7}[\frac{3}{35}]^{\frac{1}{2}}$	$-\frac{5}{7}[\frac{5}{7}]^{\frac{1}{2}}$	$+\frac{1}{7}[\frac{55}{7}]^{\frac{1}{2}}$	$+\frac{2}{7}[\frac{11\cdot 13}{35}]^{\frac{1}{2}}$
$(\frac{3}{2}\,\frac{1}{2})_2$	$+\frac{1}{7}[\frac{55}{7}]^{\frac{1}{2}}$	$+\frac{1}{7}[\frac{33}{35}]^{\frac{1}{2}}$	$-\frac{12}{7}[\frac{3}{35}]$	$+\frac{1}{7}[\frac{15\cdot 13}{7}]^{\frac{1}{2}}$

	$J = \frac{7}{2}\ (T_\lambda T_\mu) = (1\,\frac{1}{2})$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)_\alpha$	$(\frac{5}{2}\,2)$	$(\frac{5}{2}\,4)$
$(\frac{3}{2}\,\frac{1}{2})_1$	$+1$	0
$(\frac{3}{2}\,\frac{1}{2})_2$	0	$+1$

	$J = \frac{9}{2}\ (T_\lambda T_\mu) = (0\,\frac{1}{2})$		
	$(J_\lambda J_\mu)$		
$(\omega_1 t)_\alpha$	$(\frac{3}{2}\,4)$	$(\frac{9}{2}\,2)$	$(\frac{9}{2}\,4)$
$(\frac{1}{2}\,\frac{1}{2})_1$	$+[\frac{19}{35}]^{\frac{1}{2}}$	$-[\frac{40}{7\cdot 19}]^{\frac{1}{2}}$	$-[\frac{8\cdot 13}{35\cdot 19}]^{\frac{1}{2}}$
$(\frac{1}{2}\,\frac{1}{2})_2$	0	$+[\frac{13}{38}]^{\frac{1}{2}}$	$-[\frac{25}{38}]^{\frac{1}{2}}$
$(\frac{3}{2}\,\frac{1}{2})$	$+[\frac{16}{35}]^{\frac{1}{2}}$	$+[\frac{5}{14}]^{\frac{1}{2}}$	$+[\frac{13}{70}]^{\frac{1}{2}}$

	$J = \frac{9}{2}\ (T_\lambda T_\mu) = (1\,\frac{1}{2})$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	$(\frac{5}{2}\,2)$	$(\frac{5}{2}\,4)$
$(\frac{3}{2}\,\frac{1}{2})$	$-[\frac{11}{14}]^{\frac{1}{2}}$	$-[\frac{3}{14}]^{\frac{1}{2}}$
$(\frac{3}{2}\,\frac{3}{2})$	$-[\frac{3}{14}]^{\frac{1}{2}}$	$+[\frac{11}{14}]^{\frac{1}{2}}$

	$J = \frac{11}{2}\ (T_\lambda T_\mu) = (0\frac{1}{2})$		
	$(J_\lambda J_\mu)$		
$(\omega_1 t)_\alpha$	$(\frac{3}{2}4)$	$(\frac{9}{2}2)$	$(\frac{9}{2}4)$
$(\frac{1}{2}\frac{1}{2})_1$	$-\left[\frac{79}{15\cdot 7}\right]^{\frac{1}{2}}$	$+\left[\frac{36\cdot 13}{35\cdot 79}\right]^{\frac{1}{2}}$	$-\left[\frac{10\cdot 13}{21\cdot 79}\right]^{\frac{1}{2}}$
$(\frac{1}{2}\frac{1}{2})_2$	0	$+\left[\frac{25}{79}\right]^{\frac{1}{2}}$	$+\left[\frac{54}{79}\right]^{\frac{1}{2}}$
$(\frac{3}{2}\frac{1}{2})$	$+\left[\frac{26}{15\cdot 7}\right]^{\frac{1}{2}}$	$+\left[\frac{18}{35}\right]^{\frac{1}{2}}$	$-\left[\frac{5}{21}\right]^{\frac{1}{2}}$

	$J = \frac{13}{2}\ (T_\lambda T_\mu) = (0\frac{1}{2})$	
	$(J_\lambda J_\mu)$	
$(\omega_1 t)$	$(\frac{9}{2}2)$	$(\frac{9}{2}4)$
$(\frac{1}{2}\frac{1}{2})$	$+\left[\frac{6}{7}\right]^{\frac{1}{2}}$	$-\left[\frac{1}{7}\right]^{\frac{1}{2}}$
$(\frac{3}{2}\frac{1}{2}$	$+\left[\frac{1}{7}\right]^{\frac{1}{2}}$	$+\left[\frac{6}{7}\right]^{\frac{1}{2}}$

are automatically states with $n = 7$, and in this case no further d-transformation is needed. However, the state with $(J_\lambda, J_\mu) = (\frac{5}{2}, 2)$ serves to define the label $\alpha = 1$, while the state with $(J_\lambda, J_\mu) = (\frac{5}{2}, 4)$ serves to define the label $\alpha = 2$. The matrix $\boldsymbol{d}^{0\frac{1}{2},\frac{7}{2}}$ must be labelled according to this same prescription since it is needed along with $\boldsymbol{d}^{1\frac{1}{2},\frac{7}{2}}$ in the construction of states with $(\omega_1 t) = (\frac{3}{2}, \frac{1}{2})$; and $J = \frac{7}{2}$, but $T = \frac{1}{2}$. All but a few of the 1×1 d-matrices for $j = \frac{5}{2}$ can be chosen as $+1$; where it is understood that the order of the quantum numbers $(T_\lambda T_\mu)$ for the symmetrized states, with subscript (σ), is that listed in the tables of c-coefficients, with the further prescription that the order $(J_\lambda J_\mu)$ is chosen as (24) and $(\frac{3}{2}, \frac{9}{2})$ for $(T_\lambda T_\mu) = (\frac{1}{2}\frac{1}{2})$ and (00), respectively, the only two cases where the order of $T_\lambda T_\mu$ does not automatically specify the order of $J_\lambda J_\mu$. The only negative coefficients are:

$$d^{T_\lambda T_\mu, J}_{(\omega_1 t);\, J_\lambda J_\mu} = d^{00,\, 3}_{(10);\, \frac{3}{2}\frac{3}{2}} = d^{10,\, 6}_{(11);\, \frac{5}{2}\frac{9}{2}} = -1.$$

For $j = \frac{7}{2}$ the tables of d-coefficients require considerable space and will be published elsewhere, along with the few additional tables of c-coefficients needed for irreducible representations $(\omega_1 t)$ which occur for $j = \frac{7}{2}$ but not for $j < \frac{7}{2}$.

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