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The Homological Dimension of Commutative Group Schemes over a Perfect Field

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In [10] and [7], Serre and Oort have shown that the category of commutative group schemes of finite type over an algebraically closed field has homological dimension one if the field has characteristic zero and two otherwise. We extend their result by relating the homological dimension of this category of group schemes over any perfect field to the cohomological dimension of the Galois group of the field. In particular, if A and B are abelian varieties over a finite field then our result implies that $\mathrm{Ext}^i(A, B) = 0$ for $i > 2$, and this completes the computation of these groups (see [4]).

Also, Oort and Oda [8] have shown that $\mathrm{Ext}^2(A, B) = 0$ if A and B are abelian varieties over an algebraically closed field. We give a short proof of this.

Notation. k is a perfect field and Γ the Galois group (over k) of the algebraic closure $\bar{k}$ of k. All group schemes will be commutative. $\mathscr{G}$ (resp. $\mathscr{F}$) is the category of group schemes of finite type (resp. finite) over k, and $\bar{\mathscr{G}}$ (resp. $\bar{\mathscr{F}}$) the corresponding categories over $\bar{k}$. $\mathscr{M}$ (resp. $\mathscr{N}$) is the category of discrete Γ-modules (resp. discrete finite Γ-modules) and $\mathscr{A}\ell$ the category of abelian groups. If $\mathscr{C}$ is an abelian category, $\mathscr{P}\mathscr{C}$ and $\mathscr{I}\mathscr{C}$ denote the associated pro and ind categories. $\mathscr{I}\mathscr{N}$ will be identified with the category of discrete torsion Γ-modules. The Cartier duality functor $\mathscr{F} \to \mathscr{F}$ extends to give an anti-equivalence between the categories $\mathscr{P}\mathscr{F}$ and $\mathscr{I}\mathscr{F}$. If G is in $\mathscr{G}$, then $\bar{G} = G \otimes_k \bar{k}$. If Z is a commutative group (scheme) and ℓ a fixed prime, then Z_n and $Z^{(n)}$ are the kernel and cokernel respectively of multiplication by ℓ^n on Z, and $Z(\ell) = \varinjlim_n Z_{\ell^n}$ is the ℓ-primary component of Z.

1. HOMOLOGICAL DIMENSION

We refer the reader to [5] and [6] for the basic definitions and results on extensions in an abelian category and its associated ind and pro categories.

PROPOSITION. *For any G and H in $\mathscr{G}$, there is a spectral sequence*

$$H^i(\Gamma, \operatorname{Ext}^j_{\bar{\mathscr{G}}}(\bar{G}, \bar{H})) \Rightarrow \operatorname{Ext}^{i+j}_{\mathscr{G}}(G, H)$$

Proof. Fix an H in $\mathscr{G}$ and consider the functors

$$\alpha : \mathscr{P}\mathscr{G} \to \mathscr{M}, \qquad \alpha(G) = \operatorname{Hom}_{\mathscr{P}\bar{\mathscr{G}}}(\bar{G}, \bar{H}),$$
$$\beta : \mathscr{M} \to \mathscr{A}\ell, \qquad \beta(L) = L^\Gamma = \operatorname{Hom}_\Gamma(\mathbf{Z}, L)$$
$$\gamma : \mathscr{P}\mathscr{G} \to \mathscr{A}\ell, \qquad \gamma(G) = \operatorname{Hom}_{\mathscr{P}\mathscr{G}}(G, H).$$

$\beta\alpha = \gamma$, and the base change functor $G \to \bar{G}$ is exact, and so the required spectral sequence will be the spectral sequence of a composite functor, $(R^i\beta)(R^j\alpha) \Rightarrow R^{i+j}\gamma$, provided we prove (i) the functor $(G \to \bar{G}) : \mathscr{P}\mathscr{G} \to \mathscr{P}\bar{\mathscr{G}}$ preserves projectives (and therefore $R^j\alpha(G) = \operatorname{Ext}^j_{\mathscr{P}\bar{\mathscr{G}}}(\bar{G}, \bar{H})$), and (ii) α takes projectives to β-acyclics.

(i) Let P be projective in $\mathscr{P}\mathscr{G}$. If k' is a finite extension of k, then $P' = P \otimes_k k'$ is projective in $\mathscr{P}\mathscr{G}'$ ($\mathscr{G}'$ = category of group schemes of finite type over k') because the base change functor $\mathscr{P}\mathscr{G} \to \mathscr{P}\mathscr{G}'$ has an exact right adjoint, viz the Weil restriction of scalars functor. Now, if $\varphi : G \to G'$ is an epimorphism in $\mathscr{P}\bar{\mathscr{G}}$, and $\psi : P \to G'$ is any morphism, then G, G', φ, and ψ are all defined over some finite extension of k, and hence ψ lifts to a morphism $\psi' : P \to G$ such that $\varphi\psi' = \psi$.

(ii) We have to show that $H^i(\Gamma, \operatorname{Hom}_{\mathscr{P}\bar{\mathscr{G}}}(\bar{P}, \bar{H})) = 0$ for $i > 0$ and any projective P in $\mathscr{P}\mathscr{G}$. It follows from the structure theorem for group schemes over perfect fields ([9], ([1] 10.6, 15.5)) that it suffices to take P to be the projective envelope of a group scheme G which is (a) finite, (b) an abelian variety, (c) a torus, or (d) the additive group $\mathbf{G}_a$ (c.f. the arguments of ([10] 3.2)).

(a) We shall use the fact that $\mathscr{F}$ is a product of categories $\mathscr{F} = \mathscr{F}_e \times \mathscr{F}_{ce} \times \mathscr{F}_{ce}$ where $\mathscr{F}_e$ consists of finite étale group schemes, $\mathscr{F}_{ce}$ of connected group schemes whose duals are connected, and $\mathscr{F}_{ce}$ of connected group schemes whose dual is étale.

LEMMA. *If P is projective in $\mathscr{P}\mathscr{F}$, then it is also projective in $\mathscr{P}\mathscr{G}$.*

Proof. Let $P = \varinjlim (P_j \xrightarrow{\varphi_{ij}} P_i)$, $P_i \in \mathscr{F}$. It suffices to prove that if $0 \to G' \to G \xrightarrow{\tau} P_i \to 0$ is an exact sequence in $\mathscr{G}$, then there is a j and a morphism $\varphi' : P_j \to G$ such that $\varphi\varphi' = \varphi_{ij}$ ([10] 3.1, Prop 2). If $P_i \in \mathscr{F}_e$, then the reduced subgroup scheme of G maps onto P_i, and so we may assume G reduced to begin with. Then there is a finite subgroup scheme F_0 of $\bar{G}$ mapping onto $\bar{P}_i$ ([10] 4.3), and the union of F_0 with all its conjugates

generates a finite subgroup scheme of $\bar{G}$ which is of the form $\bar{F}$, where F is a subgroup scheme of G mapping onto P_i. Thus, there is a morphism $\varphi' : P_j \to F \subset G$ with $\varphi\varphi' = \varphi_{ij}$. If $P_i \in \mathscr{F}_{ce} \times \mathscr{F}_{cc}$, then the argument in ([7], pII. 7-4) shows that there is again a finite subgroup scheme F of G mapping onto P_i.

Note now that

$$\mathrm{Hom}_{\mathscr{P}\mathscr{F}}(\bar{P}, \bar{H}) = \varinjlim_i \mathrm{Hom}_{\mathscr{F}}(\bar{P}_i, \bar{H}) = \varinjlim_i \varinjlim_j \mathrm{Hom}_{\mathscr{F}}(\bar{P}_i, \bar{H}_j)$$

where the H_j run over the finite subgroup schemes of H. Since cohomology commutes with direct limits, case (a) of the proposition reduces to the statement: if P is projective in $\mathscr{P}\mathscr{F}$, and H is in $\mathscr{F}$, then $H^i(\Gamma, \mathrm{Hom}_{\mathscr{P}\mathscr{F}}(\bar{P}, \bar{H})) = 0$ for all $i > 0$. We prefer to prove the dual statement: if I is injective in $\mathscr{I}\mathscr{F}$, and H is in $\mathscr{F}$, then $H^i(\Gamma, \mathrm{Hom}_{\mathscr{I}\mathscr{F}}(\bar{H}, \bar{I})) = 0$ for all $i > 0$.

Assume first that H and I are in $\mathscr{I}\mathscr{F}_e$ (which may be identified with $\mathscr{I}\mathscr{N}$). From the above results we get that $\bar{I}$ is a divisible torsion group. If $0 \to P_1 \to P_0 \to H \to 0$ is a resolution of H in $\mathscr{M}$ by modules which are free and finitely generated as abelian groups, then

$$0 \to \mathrm{Hom}_{\mathscr{M}\mathscr{C}}(H, I) \to \mathrm{Hom}_{\mathscr{M}\mathscr{C}}(P_0, I) \to \mathrm{Hom}_{\mathscr{M}\mathscr{C}}(P_1, I) \to 0$$

is an injective resolution of $\mathrm{Hom}_{\mathscr{M}\mathscr{C}}(H, I)$ in $\mathscr{I}\mathscr{N}$. It follows that $(R^i\beta')(\mathrm{Hom}_{\mathscr{M}\mathscr{C}}(H, I)) = 0$ for $i > 0$, where β' is the functor β restricted to $\mathscr{I}\mathscr{N}$. However, the right-derived functors of β and β' agree because each can be computed using resolutions by induced modules. Hence $H^i(\Gamma, \mathrm{Hom}_{\mathscr{I}\mathscr{F}}(\bar{H}, \bar{I})) = 0$ for $i > 0$.

Now suppose H and I are in $\mathscr{I}\mathscr{F}_{cc}$. Then $\boldsymbol{\alpha}_p$ is the only simple object in $\mathscr{F}_{cc}$, and so we may take I to be the injective envelope of $\boldsymbol{\alpha}_p$. $\bar{I}$ is the injective envelope of $\bar{\boldsymbol{\alpha}}_p$ ([3], I4), and $\mathrm{Hom}_{\mathscr{I}\mathscr{F}}(\bar{H}, \bar{I})$ is the Dieudonné module of $\bar{H}$. This last is obtained from the Dieudonné module of H (over k) by tensoring with the Witt ring over $\bar{k}$ (loc. cit.) and so is acyclic for β.

Finally, assume H and I are in $\mathscr{I}\mathscr{F}_{ce}$. This case is a consequence of the first case, because $\mathscr{F}_{ce}$ is dual to the subcategory of $\mathscr{F}_e \approx \mathscr{N}$ of objects killed by a power of p ($p = \mathrm{char}\ k$). But this last category is self-dual.

(b) Let P be the projective envelope of an abelian variety A over k. We claim that P is uniquely divisible, i.e. for any positive integer n, the map $n : P \to P$ is an isomorphism. Indeed, $n : P \to P$ must be injective because this is so for any projective object in $\mathscr{P}\mathscr{G}$ (c.f. [10]). A is divisible, and so under $P \to A$, nP maps onto A. From the definition of the projective envelope this implies that $nP = P$.

It follows that $\mathrm{Hom}_{\mathscr{P}\bar{\mathscr{G}}}(\bar{P}, \bar{H})$ is uniquely divisible and hence β-acyclic.

(c) The same argument applies.

(d) If $\mathrm{char}(k) = 0$, then again the same proof works, and so we may assume that $\mathrm{char}(k) = p \neq 0$.

A refinement of the proof of ([10] Sec. 4, Prop 3) shows that if P is projective in $\mathscr{P}\mathscr{G}(p)$ ($\mathscr{G}(p)$ = subcategory of $\mathscr{G}$ of objects which are killed by a power of p) then it is projective in $\mathscr{P}\mathscr{G}$. Thus, the projective envelope P of $\mathbf{G}_a$ is in $\mathscr{P}\mathscr{G}(p)$, and we have to show that if H is in $\mathscr{G}(p)$, then $\mathrm{Hom}_{\mathscr{P}\bar{\mathscr{G}}}(\bar{P}, \bar{H})$ is β-acyclic. There is an exact sequence $0 \to P_0 \to P \to \pi_0(P) \to 0$ with P_0 connected and reduced, and $\pi_0(P)$ is again projective because the functor $\pi_0 : \mathscr{P}\mathscr{G} \to \mathscr{P}\mathscr{F}$ preserves projectives (it has an exact right adjoint, the inclusion functor $\mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{G}$). We know, by case (a), that $\mathrm{Hom}_{\mathscr{P}\bar{\mathscr{G}}}(\overline{\pi_0(P)}, \bar{H})$ is β-acyclic, and so it remains to show that $\mathrm{Hom}_{\mathscr{P}\bar{\mathscr{G}}}(\bar{P}_0, \bar{H})$ is β-acyclic, but this is a simple consequence of the fact that $\mathrm{Ext}^i_{\mathscr{P}\bar{\mathscr{G}}}(\bar{\mathbf{G}}_a, \bar{\mathbf{G}}_a)$ is β-acyclic, all i (c.f. [7] II.14-2).

COROLLARY. $\mathrm{Ext}_{\mathscr{G}}{}^i(G, H)$ *is torsion for all* $i > 1$.

Proof. $\mathrm{Ext}_{\mathscr{G}}{}^j(G, H)$ is torsion for all $j > 1$ ([7], pII. 14-2) and $H^i(\Gamma, \mathrm{Ext}_{\mathscr{G}}{}^j(G, H))$ is torsion for all $i > 0$.

For a fixed prime ℓ, we define the ℓ-homological dimension of $\mathscr{G}$, $\mathrm{hd}_\ell(\mathscr{G})$ to be the greatest integer m such that $\mathrm{Ext}_{\mathscr{G}}{}^m(G, H)(\ell) \neq 0$ for some G and H in $\mathscr{G}$. Clearly, the homological dimension of $\mathscr{G}$, $\mathrm{hd}(\mathscr{G}) \geqslant \max_\ell \mathrm{hd}_\ell(\mathscr{G})$, and after the above corollary, $\mathrm{hd}(\mathscr{G}) \leqslant \max_\ell(\max(1, \mathrm{hd}_\ell(\mathscr{G})))$. However, this implies that there is equality, $\mathrm{hd}(\mathscr{G}) = \max_\ell \mathrm{hd}_\ell(\mathscr{G})$, because the exact sequence $0 \to \mathbf{Z}/\ell\mathbf{Z} \to \mathbf{Z}/\ell^2\mathbf{Z} \to \mathbf{Z}/\ell\mathbf{Z} \to 0$ does not split over any field.

We write $\mathrm{cd}_\ell(\Gamma)$ for the ℓ-cohomological dimension of Γ [11].

THEOREM 1. *With the above notations,*

$$\mathrm{hd}_\ell(\mathscr{G}) = 1 + \mathrm{cd}_\ell(\Gamma), \qquad \ell \neq \mathrm{char}(k),$$
$$= 2, \qquad \ell = \mathrm{char}(k).$$

Proof. It follows easily from the proposition and ([7], pII. 14-2) that $\mathrm{hd}_\ell(\mathscr{G})$ is not greater than the right side.

Conversely, if $\ell = p = \mathrm{char}(k)$, then $\mathrm{Ext}_{\mathscr{G}}{}^2(\boldsymbol{\alpha}_p, \boldsymbol{\alpha}_p) = k^+ \neq 0$ and so $\mathrm{hd}_\ell(G) = 2$.

If $\ell \neq \mathrm{char}(k)$, then from the proposition, there is an injection $H^i(\Gamma, \mathrm{Hom}_{\mathscr{P}\bar{\mathscr{G}}}(\mathbf{Z}_\ell, \bar{M})) \to \mathrm{Ext}^i_{\mathscr{P}\bar{\mathscr{G}}}(\mathbf{Z}_\ell, M)$ for any M in $\mathscr{F}$. There is an M in $\mathscr{F}$, killed by ℓ, such that $H^i(\Gamma, \mathrm{Hom}_{\mathscr{P}\bar{\mathscr{G}}}(\mathbf{Z}_\ell, M)) \neq 0$ for $i = \mathrm{cd}_\ell(\Gamma)$. Then $\mathrm{Ext}^i_{\mathscr{P}\bar{\mathscr{G}}}(\mathbf{Z}_\ell, M) \neq 0$, and there is an injection,

$$\mathrm{Ext}^i_{\mathscr{P}\bar{\mathscr{G}}}(\mathbf{Z}_\ell, M) \to \mathrm{Ext}^{i+1}_{\mathscr{P}\bar{\mathscr{G}}}(\mathbf{Z}/\ell\mathbf{Z}, M).$$

Remark. If $\ell = p = \text{char}(k)$, and k is not p-algebraically closed, i.e. if k has an algebraic extension whose degree is divisible by p, then $\text{cd}_p(\Gamma) = 1$ ([11] pI-21] and so in this case also, $\text{hd}_\ell(G) = \text{cd}_\ell(\Gamma) + 1$.

2. Abelian Varieties

In this section, $k = \bar{k}$ is algebraically closed.

Theorem 2. *If A and B are abelian varieties over k, then* $\text{Ext}^2_{\mathscr{G}}(A, B) = 0$.

Proof. Let $\mathscr{F}_m$ be the subcategory of $\mathscr{F}$ group schemes killed by ℓ^m. Then $\varinjlim_m \text{Ext}^1_{\mathscr{F}_m}(A_m, B_m) \approx \text{Ext}^1_{\mathscr{G}}(A, B)(\ell)$. Indeed, by passing to the direct limit with the sequence

$$0 \to \text{Ext}^1_{\mathscr{G}}(A, B)^{(m)} \to \text{Ext}^1_{\mathscr{G}}(A_m, B) \to \text{Ext}^2_{\mathscr{G}}(A, B)_m \to 0$$

and using that $\text{Ext}^1_{\mathscr{G}}(A, B)$ is torsion, we get an isomorphism

$$\varinjlim_m \text{Ext}^1_{\mathscr{G}}(A_m, B) \approx \text{Ext}^2_{\mathscr{G}}(A, B)(\ell).$$

There are homomorphisms

$$\begin{array}{ccc} \text{Ext}^1_{\mathscr{G}}(A_m, B_m) & \xrightarrow{\pi} & \text{Ext}^1_{\mathscr{G}}(A_m, B) \\ & {\scriptstyle i}\nwarrow \quad \nearrow{\scriptstyle s} & \\ & \text{Ext}^1_{\mathscr{F}_m}(A_m, B_m) & \end{array}$$

where π is the surjection induced by the inclusion $B_m \to B$, i is the obvious injection, and s applied to a short exact sequence replaces any group scheme E in the sequence by E_m. It is easy to see that $s(\pi i) = 1$, and $(\pi i)s = 1$, and so s is an isomorphism.

Since $\text{Ext}^1_{\mathscr{F}_m}(A_m, B_m)$ is obviously zero for $\ell \neq \text{char}(k)$, we may assume that $\ell = p = \text{char}(k)$. We claim (i) the map

$$\text{Ext}^1_{\mathscr{P}\mathscr{F}}(T_pA, T_pB) \to \text{Ext}^1_{\mathscr{F}_m}(A_m, B_m)$$

which replaces each pro-group scheme E in a short exact sequence by $E^{(m)}$, is surjective (T_p has the same meaning as in ([4] p 64)), and (ii) $\text{Ext}^1_{\mathscr{P}\mathscr{F}}(T_pA, T_pB)$ is a torsion group. These statements imply the theorem because (i) gives a surjection

$$\text{Ext}^1_{\mathscr{P}\mathscr{F}}(T_pA, T_pB) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p \to \varinjlim \text{Ext}^1_{\mathscr{F}_m}(A_m, B_m)$$

and (ii) implies that the first group is zero.

Both (i) and (ii) may be proved by using Dieudonné modules. (i) follows easily from the description of $\mathrm{Ext}^1_{\mathscr{F}_m}(A_m, B_m)$ in terms of pairs of semi-linear maps on the Dieudonné modules given in ([4] p 71, (P_1)). For (ii), it suffices to show that $\mathrm{Ext}^1_{\mathscr{D}\mathscr{F}}(T_pA, T_pB) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is zero, and this follows from the classification theorem ([3] p. 35).

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