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Likelihood Functions for Stochastic Signals in White Noise*

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For a general stochastic signal in white noise absolute continuity is proved and the Radon–Nikodym derivative is given. These results were stated in a previous paper (Duncan 1968). Independent of the absolute continuity result, a modification is proved for the hypothesis with signal present.

1. INTRODUCTION

This paper is a sequel to an earlier paper by the author (Duncan 1968) where likelihood functions were obtained for diffusion process signals. While the general result was noted there, in this paper we explicitly prove the more general result and show that the proof easily follows from the techniques used in the previous work. We shall also indicate in a rigorous mathematical way how the hypotheses may be changed using some results for the decomposition of supermartingales.

2. PROBLEM STATEMENT

We consider the following detection problem.

$$dY_t = Z_t\, dt + dB_t \qquad \text{for signal present} \tag{1}$$

$$= dB_t \qquad \text{for signal not present} \tag{2}$$

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where the process B is an n-dimensional Brownian motion, the process Z is an n-dimensional process independent of B with

$$\iint Z_s^T Z_s \, dP \, ds < \infty \tag{3}$$

and $Y_0 \equiv 0$ and $t \in [0, 1]$.

Since the processes B and Z are independent, the stochastic differential equation for signal present defines one and only one process Y. This detection problem description models a fairly general class of detection problems for a stochastic signal in white noise. It can be shown (Duncan 1969) that condition (3) is close to the most general condition for absolute continuity. The assumption of independence of signal and noise can be dropped if existence and uniqueness properties can be established by other means for the stochastic differential equation for signal present.

3. Main Result

The main result shows that μ_Y is absolutely continuous with respect to μ_B, denoted $\mu_Y \ll \mu_B$, where μ_B and μ_Y are the measures for the processes B and Y respectively. With the absolute continuity of measures the Radon–Nikodym derivative, $d\mu_Y/d\mu_B$, is also obtained in a convenient form.

Theorem 1. *Consider the detection problem described by* (1), (2), *and* (3). *Then* $\mu_Y \ll \mu_B$ *where* μ_B *and* μ_Y *are the measures for the processes* B *and* Y *respectively. The Radon–Nikodym derivative is*

$$\psi_t = \exp\left[\int_0^t \hat{Z}_s^T \, dY_s - \frac{1}{2}\int_0^t \hat{Z}_s^T \hat{Z}_s \, ds\right] \tag{4}$$

where $\hat{Z}_s = E[Z_s \mid Y_u, 0 \leqslant u \leqslant s]$ *with* Y *defined by* (1) *while in* ψ_t, Y *has the* μ_B *distribution.*

Proof. Initially let Z be a bounded uniformly stepwise process i.e., there exists a finite subdivision of [0, 1], $0 = t_0 < t_1 < \cdots < t_n = 1$ and a finite constant M such that

$$Z_t(\omega) = Z_{t_i}(\omega) \qquad t_i \leqslant t < t_{i+1} \qquad i = 0, 1, \ldots, n-1 \tag{5}$$

and $|Z_t(\omega)| < M$. Let μ_{YZ} and μ_{BZ} be the measures for the processes (Y, Z) and (B, Z) respectively. To show that $\mu_{YZ} \ll \mu_{BZ}$ we fix Z, a finite number of random variables, whose values are in a compact subset of a finite dimensional Euclidean space. With Z fixed, Y is a translate of Brownian motion and the Radon–Nikodym derivative is well known. Let $\phi(Z)$ be this Radon–Nikodym derivative. Since ϕ is a continuous function of Z we can use a countable dense set of values of Z to determine the Radon–Nikodym derivative, ϕ, which is

$$\phi_t = \exp\left[\int_0^t Z_s^T \, dY_s - \frac{1}{2}\int_0^t Z_s^T Z_s \, ds\right]. \tag{6}$$

To show that $\mu_Y \ll \mu_B$ we note that μ_{BZ} is a product measure from the independence of the processes B and Z and, therefore, to obtain the measure μ_Y we merely integrate the Radon–Nikodym derivative, ϕ, on the measure μ_Z. Define

$$\psi_t = E_{\mu_Z}\phi_t \tag{7}$$

where E_{μ_Z} denotes integration with respect to the measure μ_Z. Therefore

$$\frac{d\mu_Y}{d\mu_B} = \psi. \tag{8}$$

Applying the formula for stochastic differentials (Itô, 1951) to ϕ_t we obtain

$$\phi_t = 1 + \int_0^t \phi_s Z_s^T \, dB_s\,. \tag{9}$$

A simple verification shows that

$$\int \phi_s^2 Z_s^T Z_s \, ds < \infty \qquad \text{a.s. } \mu_{BZ} \tag{10}$$

so that the stochastic integral in (9) can be defined as a pointwise and L^1 limit of finite sum approximations to the integral. For the finite sums we have

$$\int E_{\mu_Z} \sum_{i=1}^n \phi_{t_i} Z_{t_i}^T (B_{t_{i+1}} - B_{t_i}) \, d\mu_B = \int \sum_{i=1}^n [E_{\mu_Z}\phi_{t_i} Z_{t_i}^T](B_{t_{i+1}} - B_{t_i}) \, d\mu_B \tag{11}$$

Since the limit of the integrand on the right hand side of (11) is well defined we have

$$E_{\mu_Z}\int_0^t \phi_s Z_s^T \, dB_s = \int_0^t E_{\mu_Z}\phi_s Z_s^T \, dB_s \qquad \text{a.s. } \mu_B\,. \tag{12}$$

Thus

$$\psi_t = 1 + \int_0^t E_{\mu_Z}\phi_s Z_s^T \, dB_s \, . \tag{13}$$

Let $\Gamma_t = \ln \psi_t$ and apply the formula for stochastic differentials (Itô, 1951) (which can be easily verified to be valid here) to obtain

$$d\Gamma_t = \frac{E_{\mu_Z}\phi_t Z_t^T \, dB_t}{E_{\mu_Z}\phi_t} - \frac{1}{2}\frac{E_{\mu_Z}\phi_t Z_t^T E_{\mu_Z}\phi_t Z_t \, dt}{[E_{\mu_Z}\phi_t]^2}. \tag{14}$$

Consider the expression

$$\frac{E_{\mu_Z}\phi_t Z_t}{E_{\mu_Z}\phi_t}. \tag{15}$$

Since $\phi = d\mu_{YZ}/d\mu_{BZ}$ the expression (15) is the conditional expectation $E[Z_t \mid Y_u, 0 \leqslant u \leqslant t]$ i.e., (15) has the proper measurability properties for $E[Z_t \mid Y_u, 0 \leqslant u \leqslant t]$ and it calculates the correct probabilities. Thus

$$\hat{Z}_t \triangleq E[Z_t \mid Y_u, 0 \leqslant u \leqslant t] = \frac{E_{\mu_Z}\phi_t Z_t}{E_{\mu_Z}\phi_t} \tag{16}$$

and

$$\psi_t = \exp\left[\int_0^t \hat{Z}_s^T \, dB_s - \frac{1}{2}\int_0^t \hat{Z}_s^T \hat{Z}_s \, ds\right]. \tag{17}$$

For the case of an arbitrary process Z satisfying (3) we can obtain a sequence of bounded uniformly stepwise processes which converge to Z in $L^2(dt\, dP)$. By the Kolmogorov–Doob inequality for the stochastic integral and the usual L^1 bound for the ordinary integral we have that

$$\phi^{(n)} \to \phi \qquad \text{uniformly in } t \text{ a.s. } \mu_{BZ} \, . \tag{18}$$

All that remains to verify is that the absolute continuity has been preserved i.e., that the $\phi^{(n)} \to \phi$ in $L^1(d\mu_{BZ})$. A necessary and sufficient condition for $\phi^{(n)} \to \phi$ in $L^1(d\mu_{BZ})$ is that the sequence $\{\phi^{(n)}\}$ be uniformly integrable. The fact that

$$\sup_n \int \phi^{(n)} \ln \phi^{(n)} \, d\mu_{BZ} < \infty \tag{19}$$

implies uniform integrability of the sequence $\{\phi^{(n)}\}$. The remaining arguments are the same to show

$$\psi_t = \exp\left[\int_0^t \hat{Z}_s^T \, dB_s - \frac{1}{2}\int_0^t \hat{Z}_s^T \hat{Z}_s \, ds\right]. \qquad \blacksquare$$

Similar to the detection of a Markov signal in white noise we can show that ψ_t induces the drift term $\hat{Z}_t$ and our two hypotheses become

$$dY_t = \hat{Z}_t \, dt + dB_t \qquad \text{for signal present,} \tag{20}$$

$$= dB_t \qquad \text{for signal not present.} \tag{21}$$

We shall use another technique to prove the above relationships independent of the absolute continuity properties. This technique will use some results for the decomposition of supermartingales (Meyer 1962, 1963).

We shall initially assume that $Z \leqslant 0$. Let $\mathscr{F}_t = \mathscr{B}(Y_u, 0 \leqslant u \leqslant t)$, the sub-$\sigma$-field generated by the random variables $\{Y_u, 0 \leqslant u \leqslant t\}$. We assume that all the sub-σ-fields are augmented. Recall the fundamental probability space $(\Omega, \mathscr{F}, P)$. The triple $(Y_t, \mathscr{F}_t, P)$ is a supermartingale, i.e., if $t > s$.

$$\begin{aligned} E[Y_t \mid \mathscr{F}_s] &= E[Y_s \mid \mathscr{F}_s] + E\left[\int_s^t Z_u \, du + B_t - B_s \mid \mathscr{F}_s\right] \\ &= Y_s + E\left[\int_s^t Z_u \, du \mid \mathscr{F}_s\right] \\ &\leqslant Y_s . \end{aligned} \tag{22}$$

We shall apply the supermartingale decomposition result of Meyer (1962, 1963) to the supermartingale Y. First we verify that Meyer's hypotheses are satisfied, i.e., that the supermartingale is in class (DL). Since the Brownian motion B is trivially in class (DL) an easy calculation shows that Y is in class (DL). Applying the decomposition we obtain

$$Y_t = X_t - A_t \tag{23}$$

where $(X_t, \mathscr{F}_t)$ is a martingale and A is a natural increasing process. This decomposition is then unique (Meyer, 1963, 1966).

We wish to identify the martingale X and the natural increasing process A in terms of the signal and the noise in (1). We first consider the martingale X. Since the supermartingale Y is continuous and locally square integrable the martingale X is also continuous and locally square integrable. For any locally square integrable martingale X there is a unique increasing process, $\langle X \rangle$, such that $X_t^2 - \langle X \rangle_t$ is locally a square integrable martingale (Kunita–Watanabe 1967).

Consider the sum

$$\sum_{i=1}^{n} (Y_{t_{i+1}} - Y_{t_i})^2 \tag{24}$$

where the partitions $P_n = \{t_1, t_2, ..., t_{n+1}\}$ become dense in the interval [0, 1]. By assuming initially that Z is a bounded uniformly stepwise process and then taking the necessary limit we can show, following K. Itô (1951), that

$$\sum_{i=1}^{n} (Y_{t_{i+1}} - Y_{t_i})^2 \to 1 \qquad \text{a.s.} \tag{25}$$

Since this must also be true when we use the expression for Y given by (23) and since the limit in (25) is not random, considering a sum with the expression (23), we obtain

$$\langle X \rangle \tag{26}$$

using results of Kunita and S. Watanabe (1967). Thus

$$\langle X \rangle_t = t \tag{27}$$

and by a result of P. Lévy (Doob 1953) the martingale X must be Brownian motion.

It only remains then to determine the natural increasing process in terms of Z. We denote following Meyer (1966)

$$p_h Y_t = E[Y_{t+h} \mid \mathscr{F}_t]. \tag{28}$$

Define

$$A_t^h = \int_0^t \frac{Y_s - p_h Y_s}{h} \, ds. \tag{29}$$

Meyer (1966) has shown that if Y is in class (D) then for every stopping time T

$$A_T = \lim_{h \to 0} A_T^h \tag{30}$$

in the sense of the weak topology $\sigma(L^1, L^\infty)$. For the general case where Y is in class (DL) we can truncate the supermartingale via stopping times. The increasing process A is natural and by Meyer's results it is unique.

Proceeding from (1) it is not difficult to compute the natural increasing process A.

$$Y_{t+h} = Y_t + \int_t^{t+h} Z_u \, du + B_{t+h} - B_t \,. \tag{31}$$

Conditioning on $\mathscr{F}_t$ we obtain

$$E[Y_{t+h} \mid \mathscr{F}_t] = Y_t + E\left[\int_t^{t+h} Z_u \, du \mid \mathscr{F}_t\right]. \tag{32}$$

By the properties of absolute continuity we have

$$\lim \frac{1}{h} \int_t^{t+h} Z_u \, du = Z_t \qquad \text{a.s. } P \text{ and for almost all } t. \tag{33}$$

Since

$$A_t^h = \int_0^t \frac{Y_s - p_h Y_s}{h} \, ds \tag{34}$$

it follows that

$$\begin{aligned} A_t &= \lim A_t^h \\ &= \int_0^t E[Z_s \mid \mathscr{F}_s] \, ds = \int_0^t \hat{Z}_s \, ds. \end{aligned} \tag{35}$$

By Doob's optional sampling theorem (Doob, 1953) (35) can be verified for A_T where T is a stopping time.

For an arbitrary signal Z satisfying (3), noting the result for $Z \leqslant 0$, we define the process X as

$$X_t = Y_t - \int_0^t \hat{Z}_s \, ds \tag{36}$$

where

$$\hat{Z}_s = E[Z_s \mid \mathscr{F}_s]. \tag{37}$$

A simple computation shows that $(X_t, \mathscr{F}_t)$ is a martingale with continuous sample paths. Computing its oscillation as we did for Y in (24) and using the fact that $\int \hat{Z}_s \, ds$ is of bounded variation we obtain the same limit as in (25). Thus $(X_t, \mathscr{F}_t)$ is a Brownian motion. Therefore we can write our hypotheses for the detection problem as

$$dY_t = \hat{Z}_t \, dt + dB_t \qquad \text{for signal present} \tag{38}$$

$$= dB_t \qquad \text{for signal not present.} \tag{39}$$

Using the results for the proof of Theorem 1 we could show that the formal limit obtained for the likelihood function for (38) and (39) is (i) a Radon–Nikodym derivative and (ii) the correct measure to associate with Y for signal present.

4. Remarks

With additional assumptions on the signal and noise terms (e.g., Lipschitz continuity) the assumption of independence of signal and noise becomes unnecessary and we can solve the detection problem and obtain a result similar to (4). The independence assumption was merely a simple scheme to establish existence and uniqueness for the process Y with signal present.

The stochastic differential equation description has wide applicability in detection problems and is useful for establishing necessary and sufficient conditions for absolute continuity (Duncan 1969).

Results similar to those obtained in Theorem 1 have been claimed by Kailath (1969), though Kailath's methods seem at best imprecise. The original rigorous derivation was obtained by Duncan (1968) who in fact explained the generalizations.

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