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THE SPLITTING PRINCIPLE FOR GROTHENDIECK RINGS OF SCHEMES

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§0. INTRODUCTION

ONE OF the important theorems in complex topological K-theory concerns the structure of the $K_0(X)$ algebra, $K_0(P(E))$, where E is a complex vector bundle over a topological space X and $P(E)$ is the projective bundle associated to E. Explicitly it states that $K_0(P(E))$ is isomorphic to $K_0(X)[T]/I$ where T is an indeterminate and I is the ideal generated by the polynomial $\sum(-1)^i[\Lambda_i(E)]T^i$. A special case of this theorem is one form of the periodicity theorem.

A closely related theorem is the splitting principle which states that, if E is a complex bundle over a space Y, then there is a space X and a map $p\colon X \to Y$ such that $p^*\colon K_0(Y) \to K_0(X)$ is a monomorphism and the image of E in $K_0(X)$ can be represented as the sum of the images of line bundles over X. The proofs of these theorems and their applications can be found in notes from lectures by Atiyah [1].

It is the purpose of this paper to prove theorems in algebraic K-theory directly analogous to these theorems. In this case Y is the spectrum of a commutative ring, R, and $K_0(Y)$ is the Grothendieck ring of the category of locally free modules of finite type of Y or, equivalently, the category of finitely generated projective modules over R. More generally, Y is a quasi-compact scheme and $K_0(Y)$ is the Grothendieck group of the locally free $\mathcal{O}_Y$-modules of finite type. $P(\mathcal{E})$, $\mathcal{E}$ a locally free $\mathcal{O}_Y$-module, is the projective fiber of $\mathcal{E}$ as defined by Grothendieck in [3].

THEOREM. *Let Y be a quasi-compact scheme. Let $\mathcal{E}$ be any locally free $\mathcal{O}_Y$-module of finite type. Then there is an injective ring homomorphism $\Phi\colon K_0(Y)[T]/I \to K_0(P(\mathcal{E}))$ where T is an indeterminate and I is the ideal of $K_0(Y)[T]$ generated by the polynomial*

$$\sum(-1)^i[\Lambda_i(\mathcal{E})]T^i.$$

If Y is Noetherian, Φ is an isomorphism.

The conclusion of this theorem is analogous to the topological theorem.

COROLLARY. (*The Splitting Principle*). *Let Y be a quasi-compact scheme over an affine scheme. Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be locally free $\mathcal{O}_Y$-modules of finite type. There exists a scheme X and a projective morphism $p\colon X \to Y$ such that:*

(1) *$p^*\colon K_0(Y) \to K_0(X)$ is a monomorphism.*

(2) *Each $p^*(\mathcal{E}_i)$ has a finite filtration whose quotients are locally free of rank less than or equal to one.*

This theorem is proved through the repeated use of the result about $K_0(P(E))$.

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§1. THE PROJECTIVE FIBER AND ITS MODULES

In this section some of the pertinent facts concerning the projective fiber and its construction as done by Grothendieck in [3] are reviewed and some preparatory propositions are proved. Since this paper is concerned only with locally free modules it would be possible to use the classical construction instead of Grothendieck's dual construction but it is simpler to use his work.

Let S be any graded ring of positive degree, then Proj(S) denotes the spectrum of homogeneous prime ideals. Let f be a homogeneous element of S of degree $d \geq 1$. $D_+(f) = D(f) \cap \text{Proj}(S)$, where $D(f)$ is all the prime ideals not containing f. The collection of sets $D_+(f)$ for all f in S form a basis for Proj(S). $S_{(f)}$ denotes the ring consisting of all elements of $S_{(f)}$ of the form x/f^k where x is of degree kd. Since $D_+(f) = \text{Spec}\,(S_{(f)})$, Proj ($S$) has a prescheme structure which can be shown to be a scheme. If S is a graded A-algebra, then Proj (S) is a scheme over Spec (A).

The above construction can be generalized to the situation where Spec (A) is replaced by a prescheme Y and S is replaced by a quasi-coherent graded $\mathcal{O}_Y$-algebra of positive degree $\mathcal{S}$. There is a scheme $X = \text{Proj}\,(\mathcal{S})$ over Y with structure morphism $p\colon X \to Y$ such that for each affine open subset U of Y, $X|_{p^{-1}(U)} = \text{Proj}\,(\Gamma(U, \mathcal{S}))$. If $\mathcal{S}$ is an $\mathcal{O}_Y$-algebra of finite type, then $X = \text{Proj}\,(\mathcal{S})$ is of finite type over Y.

Let Y be a prescheme and $\mathcal{S}$ be a graded $\mathcal{O}_Y$-algebra generated by $\mathcal{S}_1$. There is a functor $\sim$ from the category of graded quasi-coherent $\mathcal{S}$-modules, $\underset{\sim}{S}$ to the category of quasi-coherent $\mathcal{O}_X$-modules, $\underset{\sim}{M}$ which is the usual localization functor.

Definition 1.1. $\mathcal{S}(n)$, $n \in Z$, is the graded $\mathcal{S}$-module given by letting

$$\mathcal{S}(n)_i = \mathcal{S}_{n+i}\, \mathcal{O}_X(n) = \mathcal{S}(n)^{\sim}.$$

If $\mathcal{M} \in \underset{\sim}{M}$, then $\mathcal{M}(n) = \mathcal{M} \otimes \mathcal{O}_X(n)$.

If $\mathcal{S}$ is generated by $\mathcal{S}_1$, then for $n, m \in Z$ $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) = \theta_X(n+m)$ and $\mathcal{O}_X(n) = (\theta_X(1))^n$ [2, Corollary 3.2.7, p. 55].

There is a functor $\Gamma_*\colon \underset{\sim}{M} \to \underset{\sim}{S}$ defined by $\Gamma_*(\mathcal{M}) = \coprod_{n \in Z} p_*(\mathcal{M}(n))$, $\mathcal{M} \in M$.

Definition 1.2. Let M be an $\mathcal{S}$-module.

(1) M is *TF* if there is an N such that $\coprod_{n \geq N} M_n$ is of finite type.

(2) M is *TN* if there is an N such that for $n > N$, $M_n = 0$.

THEOREM 1.3. *Let* $\underset{\sim}{C}$ *be the category of all graded quasi-coherent* $\mathcal{S}$*-modules that are TF. If* $\mathcal{S}_1$ *is of finite type as an* $\mathcal{O}_Y$*-module, then*

(1) *If* $M \in \underset{\sim}{C}$, $M^{\sim}$ *is an* $\mathcal{O}_X$*-module of finite type.*

(2) *If* $M \in \underset{\sim}{C}$, $M^{\sim} = 0$ *if and only if* M *is TN.*

(3) $(\Gamma_*(\mathcal{M}))^{\sim} = \mathcal{M}$ *for all quasi-coherent* $\mathcal{O}_X$*-modules.*

(4) *If* Y *is either quasi-compact scheme or the underlying space of* Y *is noetherian, then, if* $\mathcal{M}$ *is of finite type, there is an* $M \in \underset{\sim}{C}$ *such that* $M^{\sim} = \mathcal{M}$.

The proof of these four statements is in [2, Section 3.4, p. 59].

Let $(Y, \mathcal{A})$ be any ringed space and let $\mathcal{E}$ be any $\mathcal{A}$-module over S. There exists a graded $\mathcal{A}$-algebra $S(\mathcal{E})$ which is called the symmetric $\mathcal{A}$-algebra of $\mathcal{E}$. It satisfies the usual universal property of a symmetric algebra [3, Paragraph 1.7.4, p. 15]. The equations $S(\mathcal{E} \oplus \mathcal{F}) = S(\mathcal{E}) \otimes S(\mathcal{F})$ and $S(\mathcal{E})_y = S(\mathcal{E}_y)$ for all y in Y are valid, as one would expect from the affine case.

Let Y be a prescheme, then, if $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_Y$-module, $S(\mathcal{E})$ is a quasi-coherent $\mathcal{O}_Y$-algebra generated by $S_1(\mathcal{E}) = \mathcal{E}$. $P(\mathcal{E})$ denotes the Y-scheme Proj $(S(\mathcal{E}))$ and is called the projective fiber of $\mathcal{E}$ over Y. If $\mathcal{E}$ is of finite type, then $P(\mathcal{E})$ is of finite type over Y. If $\mathcal{E} = \mathcal{O}_Y{}^k$ then $S(\mathcal{E}) = \mathcal{O}_Y[t_1, \ldots, t_k]$ and $P_Y{}^{k-1}$ denotes $P(\mathcal{E})$. Note that $P_Y{}^0 = Y$.

Definition 1.4. Let Y be any prescheme. $\underset{\sim}{L}_Y$ is the category of locally-free $\mathcal{O}_Y$-modules of finite rank.

For the rest of the paper the following conditions and notation will apply even if they are not explicitly mentioned. Y is a quasi-compact scheme, $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite type, and $X = P(\mathcal{E})$.

In order to prove the following proposition, we need a lemma based on [4, Proposition 2.22, p. 100]

LEMMA 1.5. *Let R be a ring,* $X = P_R{}^{k-1}$, *and* $\mathcal{L}$ *be in* $\underset{\sim}{L}_X$. *Then*

(1) $H^0(X, \mathcal{L}(n))$ *is an R-module of finite type for large n.*

(2) $H^i(X, \mathcal{L}(n)) = 0$ *for* $i > 0$ *and large n.*

Proof. The lemma is true if $\mathcal{L} = \mathcal{O}_k(n)$ by [4, Corollary 2.1.14, p. 99]. Since $H^i(X, \_) = 0$ for $i > k - 1$, we can use an induction argument. Assume that (2) is valid for $i > j$. By [3, Corollary 2.7.9, p. 40] there is a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_x(-a)^m \to \mathcal{L} \to 0.$$

$\mathcal{K}$ is obviously in $\underset{\sim}{L}_x$. Now the long exact cohomology sequence and the induction hypothesis applied to $\mathcal{K}$ yield either (1) or (2), depending on j.

PROPOSITION 1.6. *Let* $\mathcal{L}$ *be in* $\underset{\sim}{L}_X$. $\Gamma_n(\mathcal{L})$ *is a locally-free* $\mathcal{O}_Y$*-module of finite type for large n.*

Proof. It is clear that since these are local properties and since X is quasi-compact, it suffices to consider the case where $\mathcal{E}$ is a free $\mathcal{O}_Y$-module of rank k and Y is an affine scheme. Let R be the ring associated to Y. The construction of $X = P(\mathcal{E})$ in this case is dual to the classical case and $X = P_Y{}^{k-1} = P_R{}^{k-1}$ [3, Corollary 4.2.6, p. 75].

Let $\{U_i\}$ be the standard affine covering of $P_R{}^{k-1}$. Let $C^P = C^P(\{U_i\}, \mathcal{L}(n))$ be the alternating Čech p-cochains relative to $\{U_i\}$, where n is large enough so that Lemma 1.5 is valid. Let $R_{i_0}, \ldots, _{i_p}$ be the affine ring of $U_{i_0} \cap \cdots \cap U_{i_p} = U_{i_0} \cdots _{i_p}$ and $L_{i_0} \cdots _{i_p}$ be the $R_{i0} \cdots _{i_p}$-module corresponding to $\mathscr{L} \mid U_{i0} \cdots _{ip}$. $L_{i0} \cdots _{ip}$ is projective over $R_{i0} \cdots _{ip}$ and therefore over R, since $R_{i0} \cdots i_p$ has a free basis of monomials as an R-module. Since $C^P = \coprod L_i t \cdots _{ip}$, we have C^P is projective over R.

By Lemma 1.5 the sequence $0 \to \Gamma_n(\mathscr{L}) \to C^0 \to \cdots \to C^{k-1} \to 0$ is exact. Therefore, $\Gamma_n(\mathscr{L})$ is a projective R-module.

PROPOSITION 1.7. *If* $0 \to \mathscr{L}' \to \mathscr{L} \to \mathscr{L}'' \to 0$ *is exact in* $\underset{\sim}{L}_X$ *then* $0 \to \Gamma_*(\mathscr{L}') \to \Gamma_*(\mathscr{L}) \to \Gamma_*(\mathscr{L}'') \to 0$ *is TN exact.*

Proof. Since Y is quasi-compact one can assume that $Y = \operatorname{Spec}(R)$ and $X = P_R{}^{k-1}$. The sequence $0 \to \Gamma_n(\mathscr{L}') \to \Gamma_n(\mathscr{L}) \to \Gamma_n(\mathscr{L}'') \to H'(X, \mathscr{L}'(n))$ is exact. By Lemma 1.5 $H'(X_1 \mathscr{L}'(n)) = 0$ for large n and therefore the sequence is TN exact.

§2. THE MORPHISM ϕ

We first define a map $\phi\colon K_0(Y)[T] \to K_0(X)$ which induces Φ on $K_0(Y)[T]/I$. The structural morphism $p\colon X \to Y$ induces a homomorphism of rings from $K_0(Y)$ to $K_0(X)$. This homomorphism of rings is denoted by p^*. Then $\phi(\sum a_i T^i) = \sum p^*(a_i)[\mathscr{O}_X(-i)]$.

PROPOSITION 2.1. *ϕ is a ring homomorphism.*

Proof. Since p^* is a ring homomorphism, it suffices to check that $\phi(T^{i+j}) = \phi(T^i)\phi(T^j)$. This is obvious.

PROPOSITION 2.2. *If Y is a Noetherian scheme then ϕ is onto.*

Proof. Let L be in $\underset{\sim}{L}_X$. Since Y is Noetherian, $\Gamma_*(\mathscr{L})$ is TF by [4, Corollary 2.3.2, p. 104]. Then there exists an N such that for $n \geq N$, $\coprod_{n \geq N} \Gamma_n(\mathscr{L}) = \mathscr{M}$ is a finitely generated graded module over $S(\mathscr{E})$ and by Proposition 1.6 $\Gamma_n(\mathscr{L})$ is a locally free $\mathscr{O}_y$-module of finite type. Since $\mathscr{M}^\sim \cong \mathscr{L}$ by Theorem 1.3, it suffices to show $[\mathscr{M}^\sim]$ is in the image of ϕ. Suppose $\Gamma_i(\mathscr{L})$, $N \leq i \leq N_1$, generate $\mathscr{M}$ as an $S(\mathscr{E})$ module. Let $\mathscr{P}_0 = \coprod_N^{N_1} \Gamma_n(\mathscr{L}) \otimes S(\mathscr{E})(-n)$. $[\mathscr{P}_0{}^\sim]$ is clearly in the image of ϕ and there is a homomorphism of $\mathscr{P}_0$ onto $\mathscr{M}$. It is clear that the kernel is a finitely generated graded module over $S(\mathscr{E})$ and each submodule of a fixed degree is a locally-free $\mathscr{O}_y$-module of finite type. Therefore by repeating the process, one can define $\mathscr{P}_i$, $0 \leq i \leq k$, k greater than the rank of $\mathscr{E}$ such that

$$0 \to \mathscr{N} \to \mathscr{P}_k \to \cdots \to \mathscr{P}_0 \to \mathscr{M} \to 0$$

is exact and $[\mathscr{P}_i{}^\sim]$ is in the image of ϕ.

Let m be the first integer such that $\mathscr{N}_m \neq 0$ and consider the homomorphism from $\mathscr{N}_m \otimes S(\mathscr{E})(-m)$ to $\mathscr{N}$. We claim that this homomorphism is a locally split monomorphism. It suffices to consider the case where $Y = \operatorname{Spec}(R)$ and $\mathscr{E}$ is free of rank $r < k$. In this case we can view $S(\mathscr{E})$ as the polynomial ring $R[t_1, \ldots, t_r]$ which we will write as S. $\mathscr{M}$ corresponds to the S-module, M. M is projective as an R-module and the P_i are projective as graded S-modules. By [5, Section 5, p. 8] M has projective dimension $\leq r$ as an ungraded S-module, and, therefore, projective dimension $\leq r$ as a graded S-module by [2, Section 6, p. 76]. Thus N is projective as a graded S-module. Let $Q_i(N) = N_i/D_i(N)$ where

$D_i(N) = \sum_{j<i} S_{i-j} N_j$. By [2, Section 6, Lemma 2, p. 77] we have N is isomorphic to $\coprod_{mi=}^{\infty} Q_i(N) \otimes S(-i)$. $Q_m(N) = N_m$ and therefore $N_m \otimes S(-m) \to N$ is a split monomorphism. Returning to the general situation we have that the cokernel $\mathcal{N}'$ of $\mathcal{N}_m \otimes S(-m) \to \mathcal{N}$ satisfies the same local conditions as $\mathcal{N}$ and in addition has $\mathcal{N}_n' = 0$ for $n < l$ where $l > m$. The homomorphism $\mathcal{N}_l' \otimes S(-l) \to \mathcal{N}'$ is again a locally split monomorphism. One proceeds in this manner on successive cokernels until the resulting cokernel is zero, which happens after a finite number of steps since $\mathcal{N}$ is TF. By induction, starting with the zero cokernel, it is seen that the images of all the cokernels in $K_0(X)$ and, therefore, $[\mathcal{N}^\sim]$ is in the image of ϕ. Hence, $[\mathcal{M}^\sim] = \sum (-1)^i [\mathcal{P}_i^\sim] + (-1)^{k+1}[\mathcal{N}^\sim]$ is also in the image of ϕ.

Proposition 2.3. *Let $\mathcal{E}$ be any locally-free $\mathcal{P}_Y$-module. Suppose that k is the maximum rank of any fiber. There exists the following TN exact sequence of $S(\mathcal{E})$ modules.*

$$0 \to \Lambda_k(\mathcal{E}) \otimes S(\mathcal{E})(-k) \to \cdots \to \Lambda_1(\mathcal{E}) \otimes S(\mathcal{E})(-1) \to \Lambda_0(\mathcal{E}) \otimes (\mathcal{E}) \to 0.$$

We first note the obvious corollary.

Corollary 2.4. $\phi\,(\sum_{i=1}^{i=k} (-1)^i [\Lambda_i(\mathcal{E})] T^i) = 0$.

The above sequence is obtained by the construction of a differential graded algebra and showing that the differential is exact. Let $\mathcal{E}$ be as in the proposition. $\mathcal{S}(\mathcal{E})$ is the graded algebra defined by letting $\mathcal{S}_{2n}(\mathcal{E}) = S_n(\mathcal{E})$ and $\mathcal{S}_{2n+1}(\mathcal{E}) = 0$. $\Lambda(\mathcal{E})$ is the standard exterior algebra of $\mathcal{E}$ over $\mathcal{O}_Y$. $D(\mathcal{E})$ is the graded $\mathcal{O}_Y$-algebra $\Lambda(\mathcal{E}) \otimes \mathcal{S}(\mathcal{E})$. Since $\Lambda_1(\mathcal{E})$ and $S_1(\mathcal{E}) = \mathcal{S}_2(\mathcal{E})$ are canonically identified with $\mathcal{E}$ and since $D_1(\mathcal{E}) = \Lambda_1(\mathcal{E})$ there is a map of $D_1(\mathcal{E})$ to $D_2(\mathcal{E})$. This is ∂_1. ∂_i is defined to be zero on $\Lambda_0(\mathcal{E}) \otimes \mathcal{S}_i(\mathcal{E})$ for all i. These maps can then be extended to a differential on the algebra, since the elements of $D_i(\mathcal{E})$ can be expressed as the sums of products of elements of lower degree where by induction the differential has been defined.

The rules $\partial(d_1 + d_2) = \partial(d_1) + \partial(d_2)$ and $\partial(d_1 d_2) = \partial(d_1)d_2 + (-1)^{\deg} d_1\, \partial(d_2)$ where d_1 and d_2 are homogeneous elements of $D(\mathcal{E})$ therefore proscribe ∂ on elements of $D_i(\mathcal{E})$.

In order to show that ∂ is exact it suffices to check the local case, i.e. when $\mathcal{E}$ is a free module over a ring R of rank r. In this case, however, $D(\mathcal{E})$ is the tensor product of r copies of $D(R)$, and it suffices to show that $D(R)$ has a contracting homotopy. $\Lambda(R)$ is $R[x]/(x^2)$ and $S(R)$ is $R[y]$. Therefore $D_{2i}(R)$ is the free R-module with basis $1 \otimes y^i$, $D_{2i+1}(R)$ is the free R-module with basis $x \otimes y^i$, and $\partial(x \otimes y^i) = 1 \otimes y^{i+1}$ and $\partial(1 \otimes y^i) = 0$. The contracting homotopy is obvious.

Returning to the original situation we have that ∂ restricts to a homomorphism from $\Lambda_q(\mathcal{E}) \otimes S_p(\mathcal{E})$ to $\Lambda_{q-1}(\mathcal{E}) \otimes S_{p+1}(\mathcal{E})$ such that the sequence

$$0 \to \Lambda_n(\mathcal{E}) \otimes S_0(\mathcal{E}) \to \cdots \to \Lambda_0(\mathcal{E}) \otimes S_n(\mathcal{E}) \to 0$$

is exact for $n > 0$. Therefore ∂ defines a homomorphism of degree zero

$$\Lambda_q(\mathcal{E}) \otimes S(\mathcal{E})(-q) \text{ to } \Lambda_{q-1}(\mathcal{E}) \otimes S(\mathcal{E})(-q+1)$$

such that the sequence

$$0 \to \Lambda_k(\mathcal{E}) \otimes S(\mathcal{E})(-k) \to \cdots \to \Lambda_0(\mathcal{E}) \otimes S(\mathcal{E}) \to 0$$

is TN exact, since $\Lambda_n(\mathcal{E}) = 0$ for $n > k$.

§3. $K_0(P(\mathscr{E}))$ AND THE SPLITTING PRINCIPLE

We define $\Phi: K_0(Y)[T]/I \to K_0(X)$ to be the homomorphism induced by ϕ. In the original proof of these theorems the inverse of Φ was constructed using the concept of a modified Hilbert characteristic polynomial. That proof required a stronger assumption on Y. I am indebted to the referee for the following much simpler argument which yields a stronger result.

Definition 3.1. Let t be an indeterminate. Define $P = \{\sum_{-\infty}^{\infty} a_n t^n \mid a_n \in K_0(Y)$ and for $n < 0$ $a_n = 0$ except for a finite number of values of $n\}$ and $Q = \{\sum_{-\infty}^{\infty} a_n t^n \mid a_n \in K_0(Y)$ and $a_n = 0$ except for a finite number of values of $n\}$.

The elements of P, $\sum [S_n(\mathscr{E})]t^n$ and $\sum (-1)^n [\Lambda_n(\mathscr{E})] t^n$, are denoted by $\sigma(\mathscr{E}, t)$ and $\lambda(\mathscr{E}, t)$, respectively.

Definition 3.2. χ' is defined to be the function $\underset{\sim}{L}_X$ to P/Q defined by

$$\chi'(\mathscr{L}) = \sum_{n \geq n_{\mathscr{L}}} [\Gamma_n(\mathscr{L})]t^n \bmod Q,$$

where $n_{\mathscr{L}}$ is large enough so that $\Gamma_n(\mathscr{L})$ is locally free of finite type as an $\mathscr{O}_Y$-module, as in Proposition 1.6.

PROPOSITION 3.3. *χ' defines a function, $\chi: K_0(X) \to P/Q$.*

$$\chi[\mathscr{O}_x(i)] \equiv t^{-i}\sigma(\mathscr{E}, t) \ mod\ Q.$$

Proof. The fact that χ induces a function on $K_0(X)$ follows immediately from Proposition 1.7 and the definition of Q. By [4, Proposition 2.1.15, p. 99], $\Gamma_n(\mathscr{O}_X(i)) \cong S_{n+i}(\mathscr{E})$. Therefore $\chi[\mathscr{O}_x(i)] \equiv \sum [S_{n+i}(\mathscr{E})]t_i^n \equiv t^{-i}\sigma(\mathscr{E}, t) \bmod Q$.

LEMMA 3.4. $\sigma(\mathscr{E}, t)\, \lambda(\mathscr{E}, t) = 1$.

Proof. This equation follows immediately from the long exact sequence that arose in the proof of Proposition 2.1.

THEOREM 3.5. *Let Y be a quasi-compact scheme. Let $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of finite type. Then there is an injective ring homomorphism $\Phi: K_0(Y)[T]/I \to K_0(P(\mathscr{E}))$ where T is indeterminate and I is the ideal of $K_0(Y)[T]$ generated by the polynomial*

$$\sum (-1)^i [\Lambda^i(\mathscr{E})] T^i.$$

If Y is Noetherian, Φ is an isomorphism.

Proof. Corollary 2.4 shows I is contained in the kernel of ϕ, so ϕ induces a homomorphism Φ on $K_0(Y)[T]/I$. If Y is Noetherian, Φ is onto by Proposition 2.2. It remains to show that Φ is a monomorphism.

Y is the finite union of disjoint open sets on each of which $\mathscr{E}$ has constant rank. Hence, we can assume that $\mathscr{E}$ has constant rank k; and, therefore, $[\Lambda_k(\mathscr{E})]$ is of constant rank 1 and a unit in $K_0(Y)$. Any element of $K^0(Y)[T]/I$ can be written as the image of a polynomial of degree less than k since $[\Lambda_k(\mathscr{E})]$ is a unit. Let $\sum_{i=0}^{k-1} a_i T^i \in K^0(Y)[T]$ such that Φ sends its image mod I to zero in $K_0(X)$, i.e. $\sum_{i=0}^{k-1} a_i[\mathscr{O}_x(-i)] = 0$. Applying χ we get $\sum_{i=0}^{k-1} a_i t^i \sigma(\mathscr{E}, t) \equiv 0$ mod Q. Multiplying by $\lambda(\mathscr{E}, t)$ and using Lemma 3.4, we get $\sum_{i=0}^{k=1} a_i t^i \in \lambda(\mathscr{E}, t)Q$. If

$\sum_{i=0}^{k-1} a_i t^i = 0$ in P we are finished. If not then $\sum_{i=0}^{k-1} a_i t^i = \lambda(\mathscr{E}, t) \sum_p^q c_i t^i$ where $c_p \neq 0$, $c_q \neq 0$, and $p \leq q$. The minimal degree of a non-zero coefficient of the product is p and the maximal degree is $q + e$, so $0 \leq p$ and $q + e \leq e - 1$. This implies $q < p$ which is a contradiction. Therefore $a_i = 0$ for all i.

The splitting principle follows from this theorem with the help of the following lemma.

LEMMA 3.6. *Let Y be a prescheme, $\mathscr{E}$ be any quasi-coherent $\mathscr{O}_Y$-module, $X = P(\mathscr{E})$, and $p: X \to Y$ be the structure morphism. Then there is a homomorphism $\alpha_1{}^{\#}: p^*(\mathscr{E}) \to \mathscr{O}_X(1)$ which is surjective.*

Proof. $\alpha_1{}^{\#}$ come functorially from the homomorphism $\mathscr{E} \otimes S(\mathscr{E}) \to S(\mathscr{E})(1)$. Since $\mathscr{E}$ generates $S(\mathscr{E})$ the homomorphism is surjective [3, Proposition 4.1.6, p. 72].

COROLLARY 3.7. *Let Y be a quasi-compact scheme over an affine scheme. Let $\mathscr{E}_1, \ldots, \mathscr{E}_n$ be locally free $\mathscr{O}_Y$-modules of finite type. There exists a scheme X and a projective morphism $p: X \to Y$ such that:*

(1) *$p^*: K_0(Y) \to K_0(X)$ is a monomorphism.*

(2) *Each $p^*(\mathscr{E}_i)$ has a finite filtration whose quotients are locally free of rank less than or equal to one.*

Proof. Let X_1 be $P(\mathscr{E}_1)$ and $p_1: X_1 \to Y$ be the structure morphism. By Theorem 3.5 $p_1{}^*$ is a monomorphism. Let $\mathscr{E}_1{}^1$ be the kernel of $\alpha_1{}^{\#}$, the quotient is $\mathscr{O}_{X_1}(1)$. Let $X_2 = P(\mathscr{E}_1{}^1)$ and $p_2: X_2 \to X_1$. Since X_1 is quasi-compact, we can again apply Theorem 3.5. $p_2{}^*(\mathscr{O}_{X_1}(1))$ still has rank 1. One can continue this process until $p_k{}^* \cdots p_1{}^*(\mathscr{E}_1)$ has a filtration of the proscribed form and then do the other modules. Since the composite of monomorphisms is monic, one constructs an X such that both (1) and (2) are satisfied.

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