COUNTING TOPOLOGICAL MANIFOLDS

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We consider the class $\mathcal{C}$ of all compact topological manifolds, boundaries permitted. It is known that there are only a countable number of homotopy types in $\mathcal{C}$, [1] and [2]. The subclass $\mathcal{C}_{PL}$ of piecewise linear manifolds has only a countable number of topologically distinct elements, since each could be regarded as a finite simplicial complex and a simple argument shows there are only a countable number of those, up to isomorphism. Recently, however, Kirby and Siebenmann [3] have discovered some examples of topological manifolds admitting no PL structure, so that route for counting homeomorphism types in $\mathcal{C}$ is rather unpromising (whether manifolds can be triangulated without a PL structure is still open).

We take a direct approach and with the aid of a very useful result of Edwards and Kirby [4] examine overlapping coordinate neighborhoods to show that $\mathcal{C}$ has only countably many elements up to homeomorphism. Our argument is similar to one in a differential setting in [3].

Siebenmann has informed us that he and Kirby obtained the same result earlier for closed manifolds having dimension at least 6, and for bounded manifolds of dimension at least 7, as a result of their topological handlebody theory [6]. This approach will not give the general result, however, since they also show that there is a closed manifold of dimension 4 or 5 that has no handle decomposition. Our proof is much more elementary and it is purely geometric. An interesting by-product is a topological submersion theorem and a proof of a key lemma in [5]. See the remark after the proof.

THEOREM. There are precisely a countable number of compact topological manifolds (boundary permitted), up to homeomorphism.

Proof. We first consider those with empty boundaries. Suppose not, then there are an uncountable number of some dimension, say $n$. Let $\{M_x^n\}_{x \in A}$ be such a collection. For each $M_x^n$ find a collection of imbeddings $h_{xj}: B(2) \to M_x^n$, $j = 1, 2, \ldots, k_x$, such that $\{h_{xj}(B(1))\}_{j=1}^{k_x}$ covers $M_x^n$, where $B(r)$ is the closed ball of radius $r$ and center 0 in $R^n$. By possibly choosing an uncountable subcollection, we can assume without loss of generality that $k_x = k$ for all $x$. We can also assume, by reparametrizing, that $h_{xj} | B(1)$ can be extended to an imbedding of $B(k+1)$ into $M_x$, $j = 1, \ldots, k$. We shall also regard each $M_x^n$ as a subset of $R^1$, e.g. let $I = 2n + 1$. If $d$ is the metric in $R^1$, define $e_{xjn} = d(h_{xj}(B(m)), M_x - h_{xj}(B(m+1)))$.

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and let \( \varepsilon_n = \min \{ \varepsilon_{j,m} \} \). By choosing a subcollection again we can assume there exists an \( \varepsilon > 0 \) such that \( \varepsilon_n > \varepsilon \) for all \( x \) in \( A^* \). Each \( M_{x_*} \) determines an imbedding \( g_x : B(k + 1) \to R^k = R^1 \times R^1 \times \cdots \times R^1 \) by \( g_x(x) = (h_x(1)(x), \ldots, h_x(k)(x)) \). The set of all such imbeddings under the uniform metric: \( d_g(g_x, g_y) = \max_{x \in B(k + 1)} d(g_x(x), g_y(x)) \), is separable metric, hence some \( g_{x_0} \) is a limit point of a sequence of distinct imbeddings \( g_{x_1}, g_{x_2}, \ldots \). We will show that \( M_{x_0} \) is homeomorphic to \( M_{x_i} \), for \( i \) sufficiently large. Furthermore, this homeomorphism can be taken to be arbitrarily close to the identity as measured by the metric \( d \), which we use in all that follows.

Let \( V_j(m) = h_{x_0}(B(m)) \), \( j = 1, 2, \ldots, k \), \( m = 1, 2, \ldots, k + 1 \). To simplify notation we denote \( M_{x_i} \) by \( M' \), for fixed but arbitrarily large \( i \), and we denote \( h_{x_0}(B(m)) \subset M' \) by \( V_j(m) \), \( j = 1, 2, \ldots, k \), \( m = 1, 2, \ldots, k + 1 \). Now let

\[
U_j(m) = \bigcup_{p=1}^{j} V_p(m) \quad \text{and} \quad U_j'(m) = \bigcup_{p=1}^{j} V'_p(m).
\]

Note that \( U_1(1) = M \) and \( U_k'(1) = M' \). Define \( f_j : V_j(k + 1) \to V_j'(k + 1) \) as \( h_{x_0} \circ h_{x_0}^{-1} \), \( j = 1, \ldots, k \) and note that each \( f_j \) can be taken as close to the identity as we like for \( M' \) sufficiently far out in the sequence \( M_{x_1}, M_{x_2}, \ldots \). We now proceed to construct a small homeomorphism from \( M \) to \( M' \) inductively on the sets \( U_j(m) \) as follows.

Suppose we can construct an imbedding \( g_j : U_j(m) \to M' \) as close as we like to the identity by choosing \( M' \) sufficiently far out in the sequence. We will show that we can construct an imbedding \( g_{j+1} : U_{j+1}(m - 1) \to M' \) as close to the identity as we please. Hence, starting off by letting \( g_1 = f_1 \) and \( m = k \), in \( k - 1 \) steps we will have an imbedding \( g_k : U_k(1) \to M' \), the desired homeomorphism.

First we see that \( g_j(U_j(m) \cap V_{j+1}(m)) \subset V_{j+1}(m + 1) \) if \( M' \) is chosen sufficiently far out and \( g_j \) is close to the identity (relative to our previous \( \varepsilon \)). Then \( f_{j+1} \circ g_j \) is defined on \( U_j(m) \cap V_{j+1}(m) \) and close to the identity. Letting \( N \) be an open set in \( M \) with \( U_j(m - 1) \cap V_{j+1}(m - 1) \subset N \subset U_j(m) \cap V_{j+1}(m) \), we can extend \( f_{j+1} \circ g_j \) \( : N \to V_{j+1}(m) \) to an onto homeomorphism \( h : V_{j+1}(m) \to V_{j+1}(m) \) close to the identity, using the theorem of [4].

Now define \( g_{j+1} : U_{j+1}(m - 1) \to M' \) by

\[
g_{j+1}(x) = \begin{cases} g_j(x) & \text{for } x \in U_j(m - 1) \\ f_{j+1} \circ h(x) & \text{for } x \in V_{j+1}(m - 1). \end{cases}
\]

By the definition of \( h \), \( g_{j+1} \) is well-defined. Since \( g_{j+1} \) can be extended to \( N \), as well, using either half of the definition, it is easily seen that \( g_{j+1} \) is a local homeomorphism, since it is an imbedding on the two open sets \( U_j(m - 1) \cup (N \cap U_{j+1}(m - 1)) \) and \( V_{j+1}(m - 1) \cup (N \cap U_{j+1}(m - 1)) \). It would fail to be an imbedding only if \( g_{j+1}(x) = g_{j+1}(y) \) for some \( x \) and \( y \) in \( U_j(m - 1) \cap N \) and \( V_{j+1}(m - 1) \cap N \) respectively, two compact disjoint sets with

\[\text{We are indebted to Mr. A. Shilepsky for pointing out that this condition is necessary to obtain an inclusion we need later.}\]
a positive distance between them. Since these two sets are independent of the choice of $M_i$ and the $g_j$ we start off with, we choose $M'$ sufficiently far out in the sequence and $g_j$, and hence $g_{j+1}$, close enough to the identity so that $g_{j+1}$ is 1–1. This completes the induction and the proof of the non-bounded case.

To obtain the theorem in case the boundary is non-empty we vary the argument by assuming (without loss of generality, using the preceding case) that each $M_n$ is an $n$-dimensional manifold with boundary homeomorphic to a fixed $(n - 1)$-manifold $B^{n-1}$. Then each has a collar given by an imbedding $h_{2i}: B^{n-1} \times [0, k + 1] \to M_2^n$ with $h_{2i}(b, 0) = b$. We will also assume $h_{2j}: B(k + 1) \to M_2^n$, $j = 2, \ldots, k$ are imbeddings such that $\{h_{2i}(B^{n-1} \times [0, 1]), h_{22}(B(1)), \ldots, h_{2k}(B(1))\}$ covers $M_2^n$. As before, each $M_2^n$ is a subset of $R^i$, and we can define a map $g_2: B^{n-1} \times [0, k + 1] \times B(k + 1) \to R^i$ by $g_2(x, t, y) = (h_{2i}(x, t), h_{22}(y), \ldots, h_{2k}(y))$. Find $g_{20}$ a limit point of $g_{21}, g_{22}, \ldots$ Let $V_i(m) = h_{2i1}(B^{n-1} \times [0, m])$ and define $V_j(m)$, $j = 2, \ldots, k$, and $U_j(m), j = 1, \ldots, k$, $m = 1, 2, \ldots, k + 1$, as before. The proof now proceeds in exactly the same way to produce a small homeomorphism from $M_{20}$ to $M_2$, for $t$ sufficiently large.

**Remark.** Let $f: W \to Y$ be a proper monotone map satisfying the following condition. For every $x$ in $W$ there are closed neighborhoods $U$ of $f(x)$ in $Y$ and $V$ of $x$ in $W$ and a homeomorphism $h: B(2) \times U \to V$ such that $f \circ h$ is the projection map onto $U$. Letting $M_y = f^{-1}(y)$, it is easy to see $M_y$ is a compact topological manifold, and if we fix $y_0$ we can find a collection of imbeddings $\{h_{y_j}: B(2) \to M_{y_j}\}_{j=1}^{\infty}$, where $U$ is a closed neighborhood of $y_0$ in $Y$, and $\{h_{y_j}(B(1))\}_{j=1}^{\infty}$ covers $M_{y_j}$. Furthermore, for fixed $j$, the $h_{y_j}$ vary continuously in $y$ as imbeddings of $B(2)$ in $W$ (here we assume a metric on $W$ and use the uniform topology).

Our method of constructing a homeomorphism $g_y: M_{y_0} \to M_y$, for $y$ near $y_0$, say $y$ in a small neighborhood $U'$ of $y_0$, is canonical and continuous in $y$, since the result we used from [3] was also. Then $g: M_{y_0} \times U' \to W$, defined by $g(x, y) = g_y(x)$ is a local trivialization of $f$.

This shows that if $W$ is a compact connected topological $(n + 1)$-manifold, $Y = [0, 1]$, and $f$ is a topological Morse function without critical points, then $f$ is a trivial bundle map with fiber a compact $n$-manifold, a useful result in the topological Morse Theory in [5].

**REFERENCES**


_The University of Michigan_