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# A MATHEMATICAL MODEL TO DETERMINE VISCOELASTIC BEHAVIOR OF *IN VIVO* PRIMATE BRAIN*

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**Abstract**—Determination of mechanical properties of the constituents of the head is very essential for the construction of various theoretical and experimental head injury models. This paper represents a mathematical model for the evaluation of viscoelastic behavior of *in vivo* primate brain. From a theoretical mechanics point of view, the problem being considered is that of the steady state response characteristics of a solid sphere of linear viscoelastic material whose mating surface with the rigid container is free from shear stresses. The external load is taken to be a local radial harmonic excitation. First, the response of the elastic material is determined; later the elastic response solution is converted to viscoelastic response solution through the use of the correspondence principle applicable to steady state oscillations. The paper is concluded with a discussion of a method which enables the determination of the complex dynamic shear modulus of *in vivo* primate brain.

## 1. INTRODUCTION

VULNERABILITY of the human head and the resulting fatalities from various injuries to the head is a well-established fact. The gravity of the situation has attracted many investigators from both experimental and theoretical fields of physical sciences. Previous research to give a proper description of the head injury has been either on determination of mechanical properties of the constituents of the head or on analyses of various theoretical head injury models. While investigations on these two categories are numerous, only a few representative ones will be mentioned here.

Among the numerous theories proposed for brain damage, the one mainly advocated by Holbourn (1943) and supported by the mathematical analyses of Anzelius (1943) and Güttinger (1950) received the most attention. According to Holbourn the main cause of brain damage is the shearing effect produced by the severe deformation or fracture of the skull at the vicinity of the impact or by rotations of the brain within the skull. Anzelius and Güttinger considered the effect of a sudden impulsive load on a mass of inviscid fluid contained in a *rigid* closed spherical shell (or container). Their formulations are essentially identical and involve an axisymmetric solution of the wave equation in spherical coordinates. They concluded that an initial compression wave arises from the point of impact (coup), and due to the rigidity of the shell, a tension (rarefaction) wave is emitted instantaneously from the counterpole, both travelling towards the geometric center of the system. The collision (superposition) of the two waves at the center, which produces large pressure gradients, was considered to be the cause of brain damage. Hayashi (1968) treated a one-dimensional version of the Anzelius–Güttinger model. His model consists of a rigid vessel (skull) containing inviscid fluid (brain). The vessel is attached to a linear spring, which represents the composite elastic properties of the skull, scalp, etc. Approximate solutions were obtained for the limiting cases of very soft and very hard impacts. Although this simple model has the

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advantage of being easy to interpret, it has the similar shortcomings of the Anzelius–Güttinger model. Some of these shortcomings are: (a) due to rigidity and geometrical assumption, there is no way to determine the possible locations of skull fracture and (b) the effects of skull deformation on the intracranial pressure distribution can not be determined. Recently, Engin (1969) removed the major restrictions of previous models by obtaining analytical and numerical solutions for the dynamic response of a fluid-filled elastic spherical shell. His model consists of an elastic spherical shell filled with inviscid compressible fluid. The shell material and fluid are considered to be homogenous and isotropic. The loading pattern is taken to be local, radial, impulsive and axisymmetric. Since the load is applied locally the combined linear shell theory which includes membrane and bending effects of the shell has been used for the proper description of the wave propagation. The conclusions of his paper include the possible locations of brain damage and skull injury on the basis of the numerical computations.

Further extensions of Engin's model is possible if one knows the viscoelastic properties of brain; with this knowledge one can replace the inviscid fluid occupying the interior space of the shell with a viscoelastic material. In literature, there are only four papers on the mechanical properties of brain. Franke (1954) determined the coefficient of shear viscosity from impedance measurements of glass sphere vibrating within fresh pig brain. Creep experiments were performed by Dodgson (1962) and Koeneman (1966) who also studied dynamic cyclic properties from rabbits, rats, and pigs. Recently, Fallenstein, *et al.* (1969) developed an electromechanical device with a small driving point impedance probe which was placed in direct contact with the pia-arachnoid through a hole (diameter is approximately $\frac{1}{8}$ in.) in the skull. By means of this apparatus *in vivo* as well as *in vitro* tests on Rhesus monkeys were performed. In this paper, we will give the theoretical analysis of such a test conducted on the brain. The theoretical model for the mathematical analysis is shown in Fig. 1. From a mechanics point of view, the problem being considered is that of the steady state response characteristics of a solid sphere of linear viscoelastic material whose mating surface with the rigid container is free from the tangential shear stresses. In particular, we will be interested in the response of the viscoelastic material to a local radial harmonic excitation. First, the response of the elastic material will be determined; later elastic response solutions will be converted to viscoelastic response solutions through the use of the elastic–viscoelastic correspondence principle applicable to steady state oscillations. We will conclude this paper with a discussion of a method which enables the determination of the linear viscoelastic parameters of the brain.

## 2. THEORETICAL ANALYSES

As mentioned in the Introduction, the theoretical analyses of the model in consideration will be given in two parts, namely, (a) *Elastic response*, and (b) *Viscoelastic response*. We shall use the same model, Fig. 1, for both parts; the only difference will be in the type of material which occupies the rigid spherical shell.

### (a) *Elastic response*

The linear equations of motion of an elastic medium, in vector form, are given by Fung (1965)

$$(\lambda+2G)\nabla(\nabla\,.\,\bar{u})-2G\nabla\times\bar{\omega}=\rho\frac{\partial^2\bar{u}}{\partial t^2} \qquad (1)$$

where $\bar{u}$ and $\bar{\omega}$ represent the displacement and rotation vectors respectively, $\rho$ is the mass density of medium, $\lambda$ and $G$ are the elastic material constants. These equations can be expressed in spherical coordinates, $r, \theta, \phi$, and introduction of axisymmetry and precluding torsional displacements mean

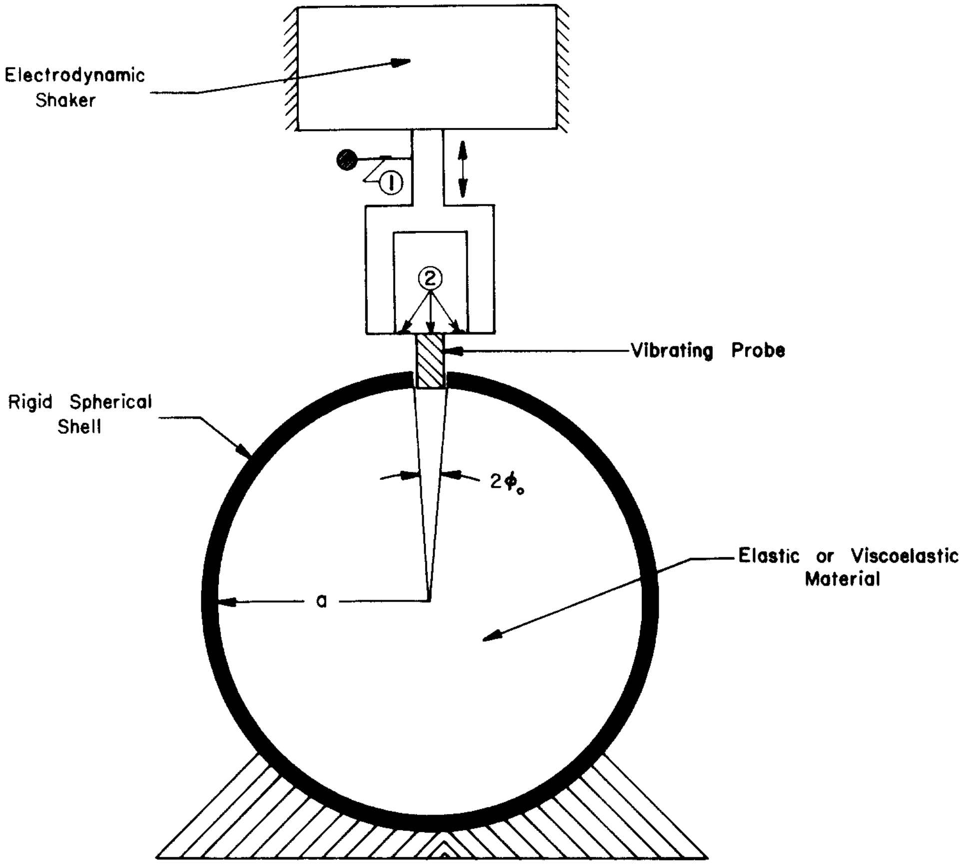


Fig. 1. Model for theoretical analysis.

$$\frac{\partial \Box}{\partial \theta} = 0, v = 0, \tag{2}$$

where $v$ is the displacement component in the $\theta$ direction. The remaining components of the displacement vector are along $\phi$ and along the radial coordinate, $r$, and they are $u$ and $w$ respectively. In view of conditions (2) the equation (1) yields the following two equations in spherical coordinates

$$(\lambda+2G)\frac{\partial \Delta}{\partial r} - \frac{2G}{r\sin\phi}\frac{\partial}{\partial \phi}(\omega_\theta \sin\phi) = \rho\frac{\partial^2 w}{\partial t^2}, \tag{3}$$

$$\frac{(\lambda+2G)}{r}\frac{\partial \Delta}{\partial \phi} + \frac{2G}{r}\frac{\partial}{\partial r}(r\omega_\theta) = \rho\frac{\partial^2 u}{\partial t^2} \tag{4}$$

where $\Delta$ is the cubical dilatation defined by

$$\Delta = \nabla \cdot \bar{u} = \frac{\partial w}{\partial r} + \frac{2w}{r} + \frac{1}{r}\frac{\partial u}{\partial \phi} + \frac{\cot\phi}{r}u, \tag{5}$$

and $\omega_\theta$ is a component of $\frac{1}{2}(\nabla \times \bar{u})$ and its value is

$$\omega_\theta = \frac{1}{2r}\left[\frac{\partial(ru)}{\partial r} - \frac{\partial w}{\partial \phi}\right]. \tag{6}$$

The equations of motion (3) and (4) can be uncoupled by assuming the displacement vector in the form of the gradient of a scalar function, $\Phi$, plus the curl of a vector potential, $\bar{\Psi}$, i.e.

$$\bar{u} = \nabla\Phi + \nabla \times \bar{\Psi}. \tag{7}$$

In spherical coordinates for axisymmetric motion the components of $\bar{u}$ from equation (7) are found to be

$$w = \left[\frac{\partial \Phi}{\partial r} + \frac{1}{r \sin\phi}\frac{\partial}{\partial \phi}(\psi \sin\phi)\right] \tag{8}$$

$$u = \left[\frac{1}{r}\frac{\partial \Phi}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(\psi r)\right] \tag{9}$$

where $\psi$ is the component of $\Psi$ along $\theta$-direction.

Since the excitation is harmonic and applied locally on the spherical surface, the following expansions are considered for $\Phi$ and $\psi$:

$$\Phi = \sum_{n=0}^{\infty} \Phi_n(r) P_n(\cos\phi) e^{i\omega t} \tag{10}$$

$$\Psi = \sum_{n=1}^{\infty} \Psi_n(r) P_n'(\cos\phi) e^{i\omega t} \tag{11}$$

where $P_n(\cos\phi)$ are Legendre polynomials of the first order, first kind. In view of the fact that the second solutions of the Legendre equations are singular at the poles they are not included in the expansions (10) and (11). In equations (10) and (11) $\omega$ and $t$ are the frequency of harmonic excitation and time respectively. Next, let us substitute equations (10) and (11) into equations (8) and (9) and defining $dP_n/d\phi \equiv \dot{P}_n$, etc. and with the relation $P_n'(\cos\phi) = \dot{P}_n(\cos\phi)$ we obtain the following

$$w = \left\{\frac{d\Phi_0}{dr} + \sum_{n=1}^{\infty}\left[\frac{d\Phi_n}{dr}P_n + \frac{\psi_n}{r}(\cot\phi \dot{P}_n + \ddot{P}_n)\right]\right\} e^{i\omega t} \tag{12}$$

$$u = \frac{1}{r}\left\{\sum_{n=1}^{\infty}\left(\Phi_n - \psi_n - r\frac{d\psi_n}{dr}\right)\dot{P}_n\right\} e^{i\omega t}. \tag{13}$$

For brevity the arguments of the Legendre polynomials in equations (12), (13) and in the subsequent equations are not shown. The equations of motion (3) and (4) contain terms like $\Delta$, cubical dilatation, and $\omega_\theta$. We evaluate these by substituting the expressions of displacement components, equations (12) and (13), into equations (5) and (6). The resulting equations are

$$\Delta = \left\{\frac{d^2\Phi_0}{dr^2} + \frac{2}{r}\frac{d\Phi_0}{dr} + \sum_{n=1}^{\infty}\left[\left(\frac{d^2\Phi_n}{dr^2} + \frac{2}{r}\frac{d\Phi_n}{dr}\right)P_n + \left(\cot\phi \dot{P}_n + \ddot{P}_n\right)\frac{\Phi_n}{r^2}\right]\right\} e^{i\omega t} \tag{14}$$

$$\omega_\theta = \frac{1}{2r^2}\left\{\sum_{n=1}^{\infty}\left[\left(\frac{\psi_n}{\sin^2\phi} - 2r\frac{d\psi_n}{dr} - r^2\frac{d^2\psi_n}{dr^2}\right)\dot{P}_n - \psi_n(\cot\phi \ddot{P}_n + \dddot{P}_n)\right]\right\} e^{i\omega t}. \tag{15}$$

Here we make a note that the Legendre polynomials satisfy the following differential equation

$$\ddot{P}_n + \dot{P}_n \cot\phi + \lambda_n P_n = 0 \tag{16}$$

where $\lambda_n = n(n+1)$.

Substitution of equations (12)–(15) into the first of the equations of motion, namely, equation (3) and repeated use of equation (16) in various places, after rather lengthy manipulation, yields the following expression

$$(\lambda + 2G)\frac{d}{dr}\left\{\sum_{n=0}^{\infty}\left[\frac{d^2\phi_n}{dr^2} + \frac{2}{r}\frac{d\Phi_n}{dr} - \frac{\lambda_n}{r^2}\Phi_n\right]P_n\right\} - \frac{G}{r}\left\{\sum_{n=1}^{\infty}\left[\frac{d^2\psi_n}{dr^2} + \frac{2}{r}\frac{d\psi_n}{dr} - \frac{\lambda_n}{r^2}\psi_n\right]\lambda_n P_n\right\} = -\rho\omega^2 \sum_{n=0}^{\infty}\left[\frac{d\Phi_n}{dr} - \frac{\psi_n}{r}\lambda_n\right]P_n. \tag{17}$$

Similarly the second equation of motion, equation (4), can be expressed as

$$(\lambda + 2G)\frac{1}{r}\sum_{n=0}^{\infty}\left[\frac{d^2\Phi_n}{dr^2} + \frac{2}{r}\frac{d\Phi_n}{dr} - \frac{\lambda_n}{r^2}\Phi_n\right]\dot{P}_n - \frac{G}{r}\left\{\sum_{n=1}^{\infty}\left[\frac{d^2\psi_n}{dr^2} + \frac{2}{r}\frac{d\psi_n}{dr} - \frac{\lambda_n}{r^2}\psi_n\right]\dot{P}_n + r\frac{d}{dr}\sum_{n=1}^{\infty}\left[\frac{d^2\psi_n}{dr^2} + \frac{2}{r^2}\frac{d\psi_n}{dr} - \frac{\lambda_n}{r}\psi_n\right]\dot{P}_n\right\} - \rho\omega^2\sum_{n=1}^{\infty}\left[\frac{\Phi_n}{r} - \frac{\psi_n}{r} - \frac{d\psi_n}{dr}\right]\dot{P}_n. \tag{18}$$

One can easily see that equation (17) is satisfied if the solutions of the following differential equations are found

$$(1)\quad \frac{d^2\Phi_n}{dr^2}+\frac{2}{r}\frac{d\Phi_n}{dr}-\frac{\lambda_n}{r^2}\Phi_n=-\frac{\rho\omega^2}{\lambda+2G}\Phi_n \qquad n=0,1,2,\ldots \quad (19)$$

$$(2)\quad \frac{d^2\psi_n}{dr^2}+\frac{2}{r}\frac{d\psi_n}{dr}-\frac{\lambda_n}{r^2}\psi_n=\frac{-\rho\omega^2}{G}\psi_n \qquad n=1,2,3,\ldots \quad (20)$$

The solutions of these two equations also satisfy equation (18). Thus, we can state that equations (19) and (20) are the two sets of differential equations that have to be solved.

Equations (19) and (20) can also be rewritten as

$$r^2\frac{d^2\Phi_n}{dr^2}+2r\frac{d\Phi_n}{dr}+(k_1{}^2r^2-\lambda_n)\Phi_n=0 \qquad n=0,1,2,\ldots \quad (21)$$

$$r^2\frac{d^2\psi_n}{dr^2}+2r\frac{d\psi_n}{dr}+(k_2{}^2r^2-\lambda_n)\psi_n=0 \qquad n=1,2,3,\ldots \quad (22)$$

where

$$k_1{}^2=\frac{\rho\omega^2}{\lambda+2G} \text{ and } k_2{}^2=\frac{\rho\omega^2}{G}.$$

The finite solutions of the above differential equations are the spherical Bessel function of the first kind

$$\Phi_n=a_nj_n(k_1r) \qquad n=0,1,2,\ldots$$
$$\psi_n=b_nj_n(k_2r) \qquad n=1,2,3,\ldots$$

where $j_n(k_1r)$ and $j_n(k_2r)$ are the spherical Bessel functions with arguments $k_1r$ and $k_2r$ respectively; $a_n$ and $b_n$ are the constants to be determined later.

Substitution of the solutions $\Phi_n$ and $\psi_n$ into equations (10)–(13) yields

$$\Phi=\sum_{n=0}^{\infty}a_nj_n(k_1r)P_ne^{i\omega t}$$
$$\psi=\sum_{n=1}^{\infty}b_nj_n(k_2r)P_n'e^{i\omega t}$$

$$w=\left\{a_0k_1j_0'(k_1r)+\sum_{n=1}^{\infty}\left[a_nk_1j_n'(k_1r)-\frac{\lambda_n}{r}b_nj_n(k_2r)\right]P_n\right\}e^{i\omega t}$$

$$u=\left\{\frac{1}{r}\sum_{n=1}^{\infty}\left[a_nj_n(k_1r)-b_nj_n(k_2r)-rb_nk_2j_n'(k_2r)\right]\dot{P}_n\right\}e^{i\omega t} \quad (23)$$

where (′) denotes differentiation with respect to argument.

The coefficients $a_n$ and $b_n$ are determined by utilizing the following appropriate boundary conditions

(1) Vanishing of the shear stress at the interface of elastic (or viscoelastic) material and the rigid boundary, i.e. $\tau_{r\phi}(a,\phi)=0$.

(2) Local application of the radial displacement, i.e.

$$w(a,\phi)=W(\phi)e^{i\omega t}, \text{ in particular,}$$

$$W(\phi)=\begin{cases}W_0 & 0\le\phi\le\phi_0\\ 0 & \phi_0<\phi\le\pi\end{cases}$$

where $W_0$ is the maximum amplitude of excitation.

From the first boundary condition we obtain

$$\left.\left(\frac{1}{r}\frac{\partial w}{\partial\phi}-\frac{u}{r}+\frac{\partial u}{\partial r}\right)\right|_{r=a}=0. \quad (24)$$

Substitution of displacement components from equation (23) into equation (24) yields

$$\sum_{n=1}^{\infty}\left\{2a_n\left[k_1j_n'(k_1a)-\frac{j_n(k_1a)}{a}\right]-b_n\left[\left(\lambda_n-2\right)\frac{j_n(k_2a)}{a}+k_2{}^2aj_n''(k_2a)\right]\right\}\dot{P}_n=0$$

for each $n\ge 1$

$$b_n=\frac{2\left[k_1j_n'(k_1a)-\dfrac{j_n(k_1a)}{a}\right]}{\dfrac{(\lambda_n-2)j_n(k_2a)}{a}+k_2{}^2aj_n''(k_2a)}a_n. \quad (25)$$

The second boundary condition in the view of equation (23) give the following relation

$$a_0 k_1 j_0'(k_1 a) + \sum_{n=1}^{\infty} \left[ a_n k_1 j_n'(k_1 a) - \frac{\lambda_n}{a} b_n j_n(k_2 a) \right] P_n = W(\phi). \quad (26)$$

Before proceeding further we expand the function $W(\phi)$ in a series of Legendre polynomials of the form

$$W(\phi) = \sum_{n=0}^{\infty} c_n P_n(\cos\phi) \quad (27)$$

where the coefficients $c_n$ are found, by the usual methods, to be

$$c_n = \tfrac{1}{2} W_0 [P_{n-1}(\cos\phi_0) - P_{n+1}(\cos\phi_0)] \qquad n = 0, 1, 2, \ldots \quad (28)$$

it being realized, of course, that $P_{-1}(\cos\phi_0) \equiv 1$. Substituting equations (25) and (27) into equation (26) yields

$$a_0 k_1 j_0'(k_1 a) + \sum_{n=1}^{\infty} a_n \left\{ k_1 j_n'(k_1 a) - \frac{2\lambda_n}{a} \frac{[k_1 a j_n'(k_1 a) - j_n(k_1 a)] j_n(k_2 a)}{(\lambda_n - 2)\, j_n(k_2 a) + k_2^2 a^2 j_n''(k_2 a)} \right\} \times P_n(\cos\phi) = \sum_{n=0}^{\infty} c_n P_n(\cos\phi).$$

Comparison of coefficients in the previous equation give the following
for $n = 0$

$$a_0 = \frac{c_0}{k_1 j_0'(k_1 a)} \quad (29)$$

and for $n \geqslant 1$

$$a_n = \frac{a[(\lambda_n - 2) j_n(k_2 a) + k_2^2 a^2 j_n''(k_2 a)] c_n}{k_1 a j_n'(k_1 a)[(\lambda_n - 2) j_n(k_2 a) + k_2^2 a^2 j_n''(k_2 a)] - 2\lambda_n [k_1 a j_n'(k_1 a) - j_n(k_1 a)] j_n(k_2 a)} \quad (30)$$

also from equations (25) and (30) for $n \geqslant 1$

$$b_n = \frac{2a[k_1 a j_n'(k_1 a) - j_n(k_1 a)] c_n}{k_1 a j_n'(k_1 a)[(\lambda_n - 2) j_n(k_2 a) + k_2^2 a^2 j_n''(k_2 a)] - 2\lambda_n [k_1 a j_n'(k_1 a) - j_n(k_1 a)] j_n(k_2 a)}. \quad (31)$$

Here we note that second derivative of spherical Bessel function appearing in equations (30) and (31) can be eliminated by utilization of the differential equation whose solutions are the spherical Bessel functions.

Having determined the coefficients $a_n$ and $b_n$ we can now obtain displacement components $w$ and $u$ from equation (23). For an isotropic elastic material, the stress, $\sigma_{ij}$, and strain, $\epsilon_{ij}$, tensors are related in the following manner

$$\sigma_{ij} = \lambda \Delta \delta_{ij} + 2G e_{ij}, \quad \delta_{ij} \begin{cases} = 1 & \text{for } i = j \\ = 0 & \text{for } i \neq j \end{cases} \quad (32)$$

where $\lambda = G(E-2G)/(3G-E)$, $E$ and $G$ are modulus of elasticity and shear modulus respectively. Substituting equation (23) into equation (5) yields the cubical dilatation, $\Delta$, for the axisymmetric motion of the material

$$\Delta = -k_1^2 \sum_{n=0}^{\infty} a_n j_n(k_1 r) P_n(\cos\phi)\, e^{i\omega t}.$$

Since we are interested in the normal stress in the radial direction $\epsilon_{rr}$ is obtained from

$$\epsilon_{rr} = \frac{\partial w}{\partial r} = \left\{ \sum_{n=0}^{\infty} a_n \left[ -\frac{2k_1}{r} j_n'(k_1 r) - k_1^2 \left(1 - \frac{\lambda_n}{k_1^2 r^2}\right) j_n(k_1 r) \right] P_n(\cos\phi) + \sum_{n=1}^{\infty} \lambda_n b_n \left[ \frac{j_n(k_2 r)}{r^2} - \frac{k_2 j_n'(k_2 r)}{r} \right] P_n(\cos\phi) \right\} e^{i\omega t}. \quad (34)$$

Thus, from equations (32) and (34) the final form of the normal stress, $\sigma_{rr}$, is

$$\sigma_{rr} = \sum_{n=0}^{\infty} a_n \Big\{ -\lambda k_1^2 j_n(k_1 r) - 2G \Big[ \frac{2k_1}{r} j_n'(k_1 r) - k_1^2 \Big(1 - \frac{\lambda_n}{k_1^2 r^2}\Big) j_n(k_1 r) \Big] \Big\} \times P_n(\cos\phi)\, e^{i\omega t} + 2G \sum_{n=1}^{\infty} \lambda_n b_n \Big[ \frac{j_n(k_2 r)}{r^2} - \frac{k_2 j_n'(k_2 r)}{r} \Big] P_n(\cos\phi)\, e^{i\omega t}. \quad (35)$$

This completes the elastic solution.

## (b) *Viscoelastic response*

The elastic solutions obtained in the part (a) can be converted to viscoelastic response solutions through the use of the elastic–viscoelastic correspondence principle applicable to steady state oscillations as discussed by Bland (1960). According to this principle the two independent elastic constants such as the elastic shear modulus, $G$, and the modulus of elasticity, $E$, are replaced by the complex shear modulus $G^* = G' + iG''$ and complex modulus of elasticity, $E^* = E' + iE''$ respectively. Both real and imaginary parts of $G^*$ and $E^*$ are, in general, functions of frequency.

Since $G$ and $E$ are replaced by $G^*$ and $E^*$, $k_1$, $k_2$ and $\lambda$ should be replaced by $k_1^*$, $k_2^*$ and $\lambda^*$. They are defined to be

$$\lambda^* = \frac{G^*(E^* - 2G^*)}{3G^* - E^*}$$

$$k_1^* = \left(\frac{\rho\omega^2}{\lambda^* + 2G^*}\right)^{1/2}$$

$$k_2^* = \left(\frac{\rho\omega^2}{G^*}\right)^{1/2}. \quad (36)$$

The coefficients $a_n$ and $b_n$ which were defined in the preceding sections now become complex functions, $a_n^*$ and $b_n^*$, of $k_1^*$, $k_2^*$ and spherical Bessel functions of complex arguments. In view of this the normal stress, $\sigma_{rr}$, will take the following form

$$\sigma_{rr} = \sum_{n=0}^{\infty} a_n^* \Big\{ -\lambda^* k_1^{*2} j_n(k_1^* r) - 2G^* \Big[ \frac{2k_1^*}{r} j_n'(k_1^* r) + k_1^{*2} \Big(1 - \frac{\lambda_n}{k_1^{*2} r^2}\Big) j_n(k_1^* r) \Big] \Big\} \times P_n(\cos\phi)\, e^{i\omega t} + 2G^* \sum_{n=0}^{\infty} \lambda_n b_n^* \Big[ \frac{j_n(k_2^* r)}{r^2} - \frac{k_2^* j_n'(k_2^* r)}{r} \Big] P_n(\cos\phi)\, e^{i\omega t}. \quad (37)$$

The procedure of separating the complex stress, $\sigma_{rr}$, into the real and imaginary parts are shown, in some detail, in the Appendix. Having performed this we obtain the following expression for the radial normal stress

$$\begin{aligned} \sigma_{rr}(r, \phi, t) &= \sum_{n=0}^{\infty} [Z_{n_1}(r) + iZ_{n_2}(r)] P_n(\cos\phi)\, e^{i\omega t} \\ &= [Z_1(r, \phi) + iZ_2(r, \phi)]\, e^{i\omega t} \\ &= |\sigma_{rr}|\, e^{i(\omega t + \delta)} \end{aligned} \quad (38)$$

where

$$Z_1(r, \phi) = \sum_{n=0}^{\infty} Z_{n_1}(r) P_n(\cos\phi)$$

$$Z_2(r, \phi) = \sum_{n=0}^{\infty} Z_{n_2}(r) P_n(\cos\phi)$$

$$|\sigma_{rr}| = [Z_1^2(r, \phi) + Z_2^2(r, \phi)]^{1/2}$$

$$\delta = \tan^{-1} \frac{Z_2(r, \phi)}{Z_1(r, \phi)}.$$

The definitions of $Z_{n_1}$ and $Z_{n_2}$ are given in the appendix. In equation (38) $\delta$ is the phase angle between the variation of stress and the variation of strain.

The radial normal force under the probe is given by

$$F_r = \int_0^{\phi_0} \sigma_{rr}(a, \phi, t)\, dA$$

where $dA = 2\pi a^2 \sin\phi \, d\phi$. In view of equation (38) above integral can be written as

$$F_r = 2\pi a^2 \int_0^{\phi_0} [Z_1(a,\phi) + iZ_2(a,\phi)] \sin\phi \, d\phi \, e^{i\omega t}. \quad (39)$$

### 3. DISCUSSION

As shown schematically in Fig. 1 the test apparatus has an acceleration transducer 1 and force transducer 2 which measures a composite signal consisting of the force caused by the acceleration of the probe mass and the force transferred to the test object. By a proper

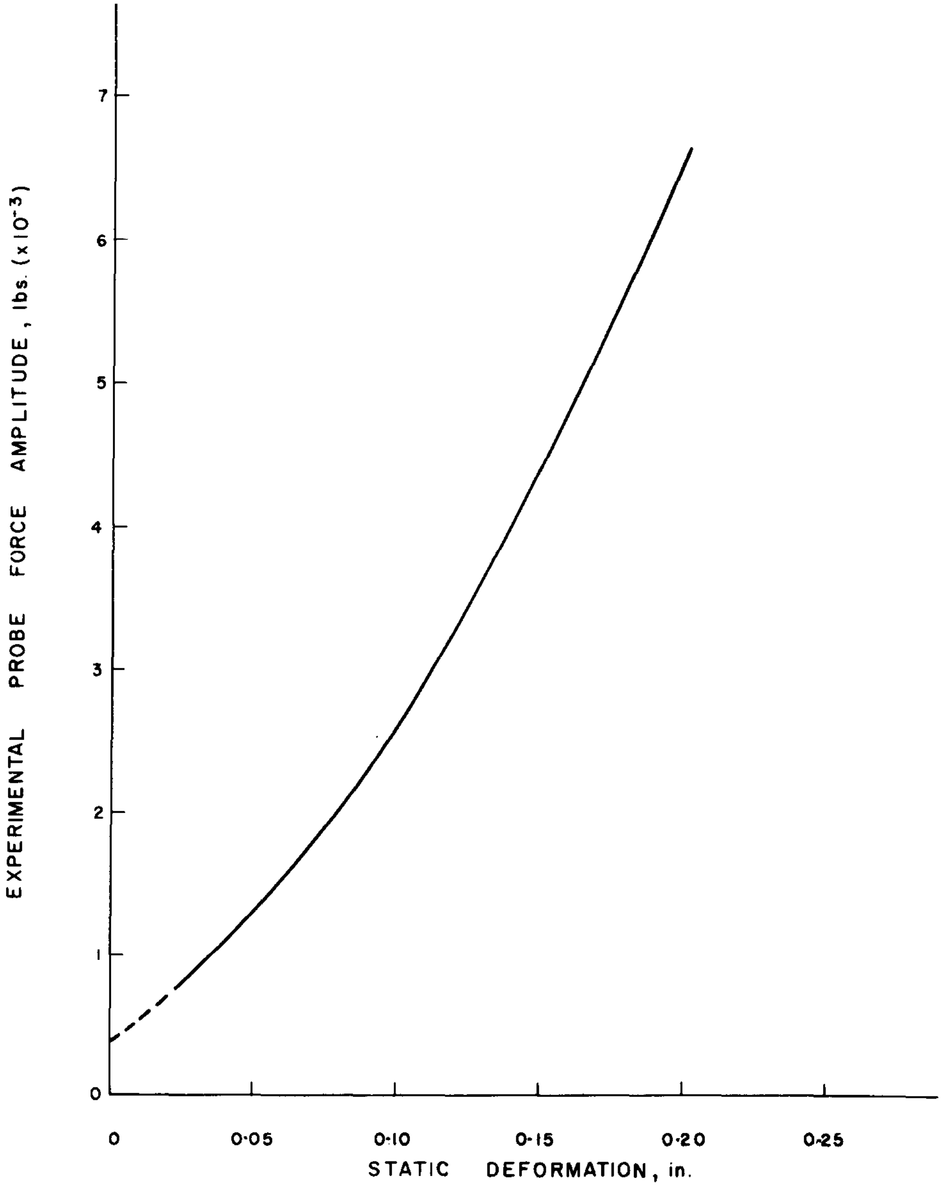


Fig. 2. Experimental probe force amplitude vs. static deformation (amplitude of dynamic probe displacement is kept constant).

calibration of these two transducers the shape and magnitudes of force and displacement quantities can be obtained.

Before the test. due to the irregular surface condition of the brain, application of a certain amount of static deformation on the pia-arachnoid is necessary. In Fig. 2 the least square fit of an experimental data is shown. On this figure, the static deformation corresponding to the starting point of solid curve is assumed to be the minimum static deformation for meaningful test results. Projection of this curve (dotted line) gives the probe force corresponding to zero static deformation, the knowledge of which is essential for the theoretical analysis. A typical test supplies two sets of information: namely, phase relations between the force and displacement and the magnitude of force. Since the brain is essentially incompressible, we can assume that the viscoelastic material contained in the rigid spherical shell is incompressible. For an incompressible viscoelastic material $3G^* = E^*$, thus the knowledge of the two material constants (or functions if one seeks frequency dependent relations) is sufficient. Let us choose $G'$ and $G''$ to be determined from a combined relationship of theoretical analysis and experimental data. For this task we carry on the following steps:

(a) From equation (39) obtain the numerical value of the complex force that the material exerts on the probe. For a viscoelastic material, mathematical analysis will give a complex force, the real part of which is in phase with displacement and the imaginary part 90° out of phase. Hence, the ratio of the imaginary part of the force to the real part will be the tangent of the phase angle between displacement and force.

(b) Plot the theoretically obtained phase angles vs. $G''$ for various values of $G'$. Here

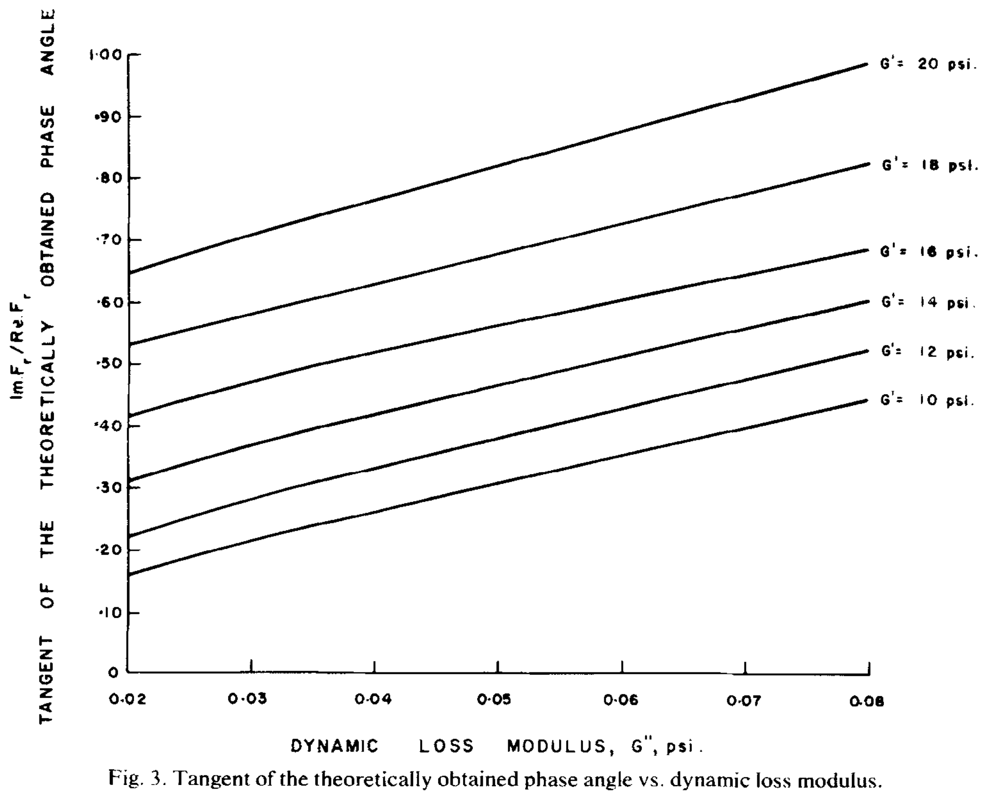


Fig. 3. Tangent of the theoretically obtained phase angle vs. dynamic loss modulus.

the values of $G'$ and $G''$ can be initially chosen arbitrarily. This plot is in the form of a family of curves as shown in Fig. 3. On this plot a line drawn passing through the experimental value of the phase angle and parallel to $G''$ axis will intersect the family of curves at various points which define pairs of values for $G'$ and $G''$.

(c) Using these pairs of $G'$ and $G''$ the numerical values of theoretical force are obtained and these force values vs. $G'$ are plotted. This plot will be only a single curve as shown on Fig. 4. The value of the experimental force corresponding to zero static deformation determines a point marked with a small circle on this curve. $G'$ and $G''$

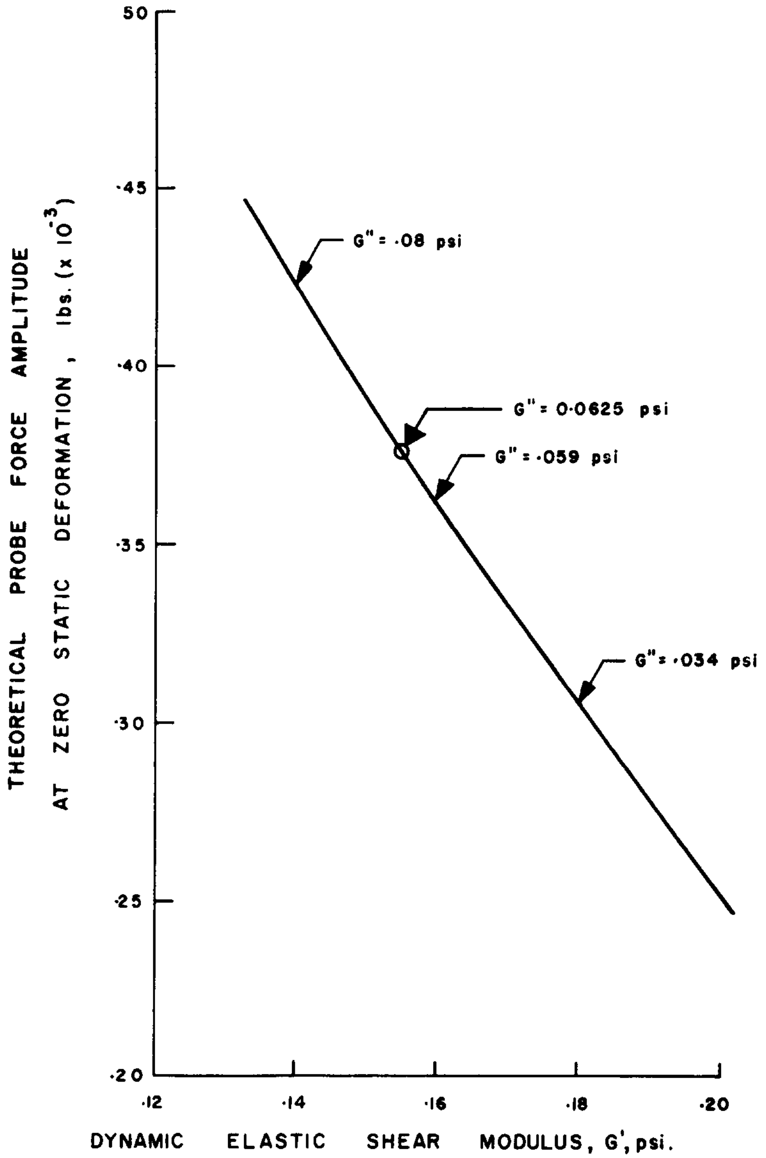


Fig. 4. Theoretical probe force amplitude (at zero static deformation) vs. dynamic elastic shear modulus.

defined by this point are the proper material constants for the corresponding frequency.

Utilizing the method outlined above one can obtain the real and imaginary parts of $G^*$ for various frequencies. The knowledge of $G^*$ as a function of frequency is very essential for the construction of transient response of the viscoelastic material.

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## REFERENCES

Anzelius. A. (1943) The effect of an impact on a spherical liquid mass. *Acta path microbiol. scand.* **48**.

Bland. D. R. (1960) *The Theory of Linear Viscoelasticity*, p. 67. Pergamon Press. Oxford.

Dodgson. M. C. H. (1962) Colloidal structure of brain. *Biorheology* **1**. 21–30.

Engin. A. E. (1969) The axisymmetric response of a fluid-filled spherical shell to a local radial impulse—a model for head injury. *J. Biomechanics* **2**. 325–341.

Fallenstein. G. T., Hulce. V. D. and Melvin. J. W. (1969) Dynamic mechanical properties of human brain tissue. *J. Biomechanics* **2**. 217–226.

Franke. E. K. (1954) The response of the human skull to mechanical vibrations. *Wright-Patterson Air Force Base, Ohio. WADC Tech. Rep.* pp. 24–54.

Fung. Y. C. (1965) *Foundations of Solid Mechanics*. 154–156. Prentice-Hall. Englewood Cliffs. N.J.

Güttinger. W. (1950) Der Stosseffekt auf eine Flüssigkeitskugel als Grundlage einer Physikalischen Theorie der Entstehung von Gehirnverletzungen. *Z. Naturf.* **A5**. 622–628.

Holbourn. A. H. S. (1943) *Lancet* **2**. 438–441.

Koeneman. J. B. (1966) Viscoelastic properties of brain tissue. M.S. Thesis. Case Institute of Technology.

## NOMENCLATURE

$E$ modulus of elasticity
$E^*$ complex modulus of elasticity, $E' + iE''$
$E', E''$ real and imaginary parts of $E^*$ respectively
$F_r$ radial force
$G^*$ complex shear modulus, $G' + iG''$
$G', G''$ real and imaginary parts of $G^*$ respectively
$P_n(\cos\phi)$ Legendre polynomials of the first kind
$P_n^1(\cos\phi)$ associated Legendre polynomials of the first kind and first order
$\Phi_n$ coefficients of the Legendre polynomial expansion of $\Phi$
$\Psi_n$ coefficients of the Legendre polynomial expansion of $\Psi$
$a$ radius of sphere
$a_n, b_n$ coefficients of $\Phi_n$ and $\Psi_n$ respectively
$c_n$ coefficients of the Legendre polynomial expansion of locally applied radial displacement
$j_n(z)$ spherical Bessel functions, $(\pi/2z)^{1/2}J_{n+1/2}(z)$
$k_1, k_2$ wave numbers for the dilatational and shear waves respectively
$k_1^*, k_2^*$ complex wave numbers for the dilatational and shear waves respectively
$r, \theta, \phi$ spherical coordinates
$t$ time
$\bar{u}$ displacement vector
$u, v, w$ components of the displacement vector in spherical coordinates
$\epsilon_{ij}, \sigma_{ij}$ strain and stress tensors respectively
$\lambda$ Lamé first constant, $G(E-2G)/(3G-e)$
$\lambda_n$ $= n(n+1)$, where $n$ are integers
$\rho$ mass density
$\omega$ circular frequency
$\Delta$ cubical dilatation, $\dfrac{\partial w}{\partial r} + \dfrac{2w}{r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \phi} + \dfrac{\cot\phi}{r}u$
$\nabla$ gradient operator
$\nabla x$ curl operator.

Note that every quantity which has superscript star is a complex number or function.

## APPENDIX

For the viscoelastic material the complex moduli are defined to be

$$G^* = G' + iG''$$
$$E^* = E' + iE''. \qquad (A1)$$

Also Lame's constant. $\lambda$, in viscoelastic case takes the form of

$$\lambda^* = \frac{G^*(E^* - 2G^*)}{3G^* - E^*}. \qquad (A2)$$

Substitution of (A1) into (A2) and rationalizing the resulting equation gives

$$\lambda^* = Re(\lambda^*) + iIm(\lambda^*) \qquad (A3)$$

where

$$Re(\lambda^*) = \frac{[G'(E'-2G') - G''(E''-2G'')](3G'-E') + [G''(E'-2G') + G'(E''-2G'')](3G''-E'')}{(3G'-E')^2 + (3G''-E'')^2}$$

$$Im(\lambda^*) = \frac{[G''(E'-2G') + G'(E''-2G'')](3G'-E') - [G'(E'-2G') - G''(E''-2G'')](3G''-E'')}{(3G'-E')^2 + (3G''-E'')^2}.$$

Note that $Re(\ )$ and $Im(\ )$ denote the real and the imaginary parts of the complex function inside of the paranthesis respectively.

The arguments of the spherical Bessel functions contain

$k_1$ and $k_2$ which involve $\lambda^*$ and $G^*$; hence $k_1$ and $k_2$ become complex and they are

$$k_1^* = \left(\frac{\rho\omega^2}{\lambda^* + 2G^*}\right)^{1/2} = Re(k_1^*) + iIm(k_1^*)$$

and

$$k_2^* = \left(\frac{\rho\omega^2}{G^*}\right)^{1/2} = Re(k_2^*) + iIm(k_2^*) \qquad (A4)$$

where

$$Re(k_1^*) = \frac{\omega\rho^{1/2}\{[(Re(\lambda^*)+2G')^2 + (Im(\lambda^*)+2G'')^2]^{1/2} + Re(\lambda^*)+2G'\}^{1/2}}{[2(Re(\lambda^*)+2G')^2 + 2(Im(\lambda^*)+2G'')^2]^{1/2}}$$

$$Im(k_1^*) = \frac{\omega\rho^{1/2}\{[(Re(\lambda^*)+2G')^2 + (Im(\lambda^*)+2G'')^2]^{1/2} - Re(\lambda^*)-2G'\}^{1/2}}{[2(Re(\lambda^*)+2G')^2 + 2(Im(\lambda^*)+2G'')^2]^{1/2}}$$

$$Re(k_2^*) = \frac{\omega\rho^{1/2}[(G'^2+G''^2)^{1/2}+G']^{1/2}}{[2G'^2+2G''^2]^{1/2}}$$

$$Im(k_2^*) = \frac{\omega\rho^{1/2}[(G'^2+G''^2)^{1/2}-G']^{1/2}}{[2G'^2+2G''^2]^{1/2}}. \qquad (A5)$$

To obtain (A5) the following relation has been used

$$(x \pm iy)^{1/2} = \left(\frac{r+x}{2}\right)^{1/2} \pm \left(\frac{r-x}{2}\right)^{1/2}$$

where

$$r = (x^2+y^2)^{1/2}.$$

For the viscoelastic material the relationship between $a_n$ and $b_n$ becomes complex in the following manner

$$b_n^* = \frac{2k_1^* j_n(k_1^* a) - \dfrac{2j_n(k_1^* a)}{a}}{\dfrac{(\lambda_n - 2)}{a} j_n(k_2^* a) + k_2^{*2} a j_n''(k_2^* a)} a_n^* \qquad (A6)$$

let

$$X_n = 2\left[k_1^* j_n'(k_1^* a) - \frac{j_n(k_1^* a)}{a}\right]$$

$$= Re(X_n) + iIm(X_n)$$

$$Y_n = \frac{\lambda_n - 2}{a} j_n(k_2^* a) + k_2^{*2} a j_n''(k_2^* a)$$

$$= Re(Y_n) + iIm(Y_n)$$

where

$$Re(X_n) = 2Re(k_1^*)Re[j_n'(k_1^* a)] - 2Im(k_1^*)Im[j_n'(k_1^* a)] - 2Re[j_n(k_1^* a)]/a$$

$$Im(X_n) = 2Im(k_1^*)Re[j_n'(k_1^* a)] + 2Re(k_1^*)Im[j_n'(k_1^* a)] - 2Im[j_n(k_1^* a)]/a$$

$$Re(Y_n) = \left[\frac{2(\lambda_n - 1)}{a} - aRe(k_2^{*2})\right]Re[j_n(k_2^* a)] + aIm(k_2^{*2})Im[j_n(k_2^* a)] - 2Re(k_2^*)Re[j_n'(k_2^* a)] - 2Im(k_2^*)Im[j_n'(k_2^* a)]$$

$$Im(Y_n) = \left[\frac{2(\lambda_n - 1)}{a} - aRe(k_2^{*2})\right]Im[j_n(k_2^* a)] - aIm(k_2^{*2})Re[j_n(k_2^* a)] - 2Im(k_2^*)Re[j_n'(k_2^* a)] + 2Re(k_2^*)Im[j_n'(k_2^* a)].$$

Now (A6) can be written as

$$b_n^* = \frac{Re(X_n) + iIm(X_n)}{Re(Y_n) + iIm(Y_n)} a_n^*$$

$$= \left[Re\left(\frac{X_n}{Y_n}\right) + iIm\left(\frac{X_n}{Y_n}\right)\right] a_n^* \qquad (A7)$$

where

$$Re\left(\frac{X_n}{Y_n}\right) = \frac{Re(X_n)Re(Y_n) + Im(X_n)Im(Y_n)}{[Re(Y_n)]^2 + [Im(Y_n)]^2}$$

$$Im\left(\frac{X_n}{Y_n}\right) = \frac{Im(X_n)Re(Y_n) - Re(X_n)Im(Y_n)}{[Re(Y_n)]^2 + [Im(Y_n)]^2}.$$

From equation (26)

$$a_0^* k_1^* j_0'(k_1^* a) + \sum_{n=1}^{\infty}\left[a_n^* k_1^* j_n'(k_1^* a) - \frac{\lambda_n}{a} b_n^* j_n(k_2^* a)\right]P_n = \sum_{n=0}^{\infty} c_n P_n$$

$$a_0^* = \frac{c_0}{k_1^* j_1'(k_1^* a)}$$

and

$$a_n^* = \frac{c_n}{\left[k_1^* j_n'(k_1^* a) - \dfrac{\lambda_n}{a}\dfrac{X_n}{Y_n} j_n(k_2^* a)\right]} \qquad n = 1, 2, 3, \ldots$$

or

$$a_n^* = \frac{c_n}{\left[A_r - Re\left(\dfrac{X_n}{Y_n} j_n\right)\right] + i\left[A_i - Im\left(\dfrac{X_n}{Y_n} j_n\right)\right]} \qquad n = 1, 2, 3, \ldots \qquad (A8)$$

where

$$A_r = Re(k_1^*)Re[j_n'(k_1^* a)] - Im(k_1^*)Im[j_n'(k_1^* a)]$$

$$A_i = Im(k_1^*)Re[j_n'(k_1^* a)] + Re(k_1^*)Im[j_n'(k_1^* a)]$$

$$Re\left(\frac{X_n}{Y_n} j_n\right) = \frac{\lambda_n}{a}\left\{Re\left(\frac{X_n}{Y_n}\right)Re[j_n(k_2^* a)] - Im\left(\frac{X_n}{Y_n}\right)Im[j_n(k_2^* a)]\right\}$$

$$Im\left(\frac{X_n}{Y_n} j_n\right) = \frac{\lambda_n}{a}\left\{Im\left(\frac{X_n}{Y_n}\right)Re[j_n(k_2^* a)] + Re\left(\frac{X_n}{Y_n}\right)Im[j_n(k_2^* a)]\right\}$$

equation (A8) now becomes

$$a_n^* = Re(a_n^*) + iIm(a_n^*) \qquad n = 1, 2, 3, \ldots \qquad (A9)$$

where

$$Re(a_n^*) = \frac{c_n\left[A_r - Re\left(\frac{X_n}{Y_n}j_n\right)\right]}{\left[A_r - Re\left(\frac{X_n}{Y_n}j_n\right)\right]^2 + \left[A_i - Im\left(\frac{X_n}{Y_n}j_n\right)\right]^2}$$

$$Im(a_n^*) = \frac{-c_n\left[A_i - Im\left(\frac{X_n}{Y_n}j_n\right)\right]}{\left[A_r - Re\left(\frac{X_n}{Y_n}j_n\right)\right]^2 + \left[A_i - Im\left(\frac{X_n}{Y_n}j_n\right)\right]^2}.$$

Substituting equation (A9) into equation (A7) we get

$$b_n^* = Re(b_n^*) + iIm(b_n^*)$$

where

$$Re(b_n^*) = Re\left(\frac{X_n}{Y_n}\right)Re(a_n^*) - Im\left(\frac{X_n}{Y_n}\right)Im(a_n^*)$$

$$Im(b_n^*) = Re\left(\frac{X_n}{Y_n}\right)Im(a_n^*) + Im\left(\frac{X_n}{Y_n}\right)Re(a_n^*).$$

Putting all the above equations (A1)–(A9) into the equation (37) we get the following expressions for the complex normal stress

$$\begin{aligned}
\sigma_{rr} = {} & \sum_{n=0}^{\infty} a_n^* \Big\langle -[Re(\lambda^*) + iIm(\lambda^*)][Re(k_1^{*2}) \\
& + iIm(k_1^{*2})]\{Re[j_n(k_1^* r)] + iIm[j_n(k_1^* r)]\} \\
& - \frac{4}{r}(G' + iG'')[Re(k_1^*) + iIm(k_1^*)] \\
& \times \{Re[j_n'(k_1^* r)] + iIm(j_n'(k_1^* r))\} \\
& - 2(G' + iG'')[Re(k_1^{*2}) + iIm(k_1^{*2})] \\
& \times \{Re[j_n(k_1^* r)] + iIm(j_n(k_1^* r))\} \\
& \cdot \left\{1 - \frac{\lambda_n}{[Re(k_1^{*2}) - iIm(k_1^{*2})]r^2}\right\}\Big\rangle P_n(\cos\phi)\, e^{i\omega t} \\
& + \sum_{n=0}^{\infty} 2\lambda_n (G' + iG'') \cdot [Re(b_n^*) + iIm(b_n^*)] \\
& \times \Big\langle \frac{Re[j_n(k_2^* r)] + iIm[j_n(k_2^* r)]}{r^2} \\
& - \frac{[Re(k_2^*) + iIm(k_2^*)]}{r} \\
& \times \{Re[j_n'(k_2^* r)] + iIm[j_n'(k_2^* r)]\}\Big\rangle P_n(\cos\phi)\, e^{i\omega t}.
\end{aligned} \quad \text{(A10)}$$

Next, let us define the following expressions:

$$\begin{aligned}
q_1 &= Re(\lambda^*)Re(k_1^{*2}) - Im(\lambda^*)Im(k_1^{*2}) \\
q_2 &= Re(\lambda^*)Im(k_1^{*2}) + Im(\lambda^*)Re(k_1^{*2}) \\
q_3 &= G'Re(k_1^*) - G''Im(k_1^*) \\
q_4 &= G'Im(k_1^*) + G''Re(k_1^*) \\
q_5 &= G'Re(k_1^{*2}) - G''Im(k_1^{*2}) \\
q_6 &= G'Im(k_1^{*2}) + G''Re(k_1^{*2}) \\
q_7 &= \{[Re(k_1^{*2})]^2 + [Im(k_1^{*2})]^2\}r^2 \\
q_8 &= G'Re(b_n^*) - G''Im(b_n^*) \\
q_9 &= G'Im(b_n^*) + G''Re(b_n^*) \\
q_{10} &= Re(k_2^*)Re[j_n'(k_2^* r)] - Im(k_2^*)Im[j_n'(k_2^* r)] \\
q_{11} &= Re(k_2^*)Im[j_n'(k_2^* r)] + Im(k_2^*)Re[j_n'(k_2^* r)]
\end{aligned} \quad \text{(A11)}$$

Equation (A10) in view of expressions defined by equation (A11) can be written as

$$\begin{aligned}
\sigma_{rr} = {} & \sum_{n=0}^{\infty} [Re(a_n^*) + iIm(a_n^*)] \\
& \Big\langle -\{q_1 Re[j_n(k_1^* r)] - q_2 Im[j_n(k_1^* r)] \\
& + iq_1 Im[j_n(k_1^* r)] + iq_2 Re[j_n(k_1^* r)]\} \\
& - \frac{4}{r}\{q_3 Re[j_n'(k_1^* r)] - q_4 Im[j_n'(k_1^* r)] \\
& + iq_3 Im[j_n'(k_1^* r)] + iq_4 Re[j_n'(k_1^* r)]\} \\
& - 2(q_5 + iq_6)\Big\{\frac{[q_7 - \lambda_n Re(k_1^{*2})]Re[j_n(k_1^* r)]}{q_7} \\
& - \frac{\lambda_n Im(k_1^{*2})Im[j_n(k_1^* r)]}{q_7} \\
& + i\frac{[q_7 - \lambda_n Re(k_1^{*2})]Im[j_n(k_1^* r)]}{q_7} \\
& + i\frac{\lambda_n Im(k_1^{*2})Re[j_n(k_1^* r)]}{q_7}\Big\}\Big\rangle \\
& \times P_n(\cos\phi)\, e^{i\omega t} + \sum_{n=0}^{\infty} 2\lambda_n \\
& \times \Big\langle\Big\{q_8 \frac{Re[j_n(k_2^* r)] - rq_{10}}{r^2} - q_9 \frac{Im[j_n(k_2^* r)] - rq_{11}}{r^2}\Big\} \\
& + i\Big\{q_8 \frac{Im[j_n(k_2^* r)] - rq_{11}}{r^2} + q_9 \frac{Re[j_n(k_2^* r)] - rq_{10}}{r^2}\Big\}\Big\rangle \\
& \times P_n(\cos\phi)\, e^{i\omega t}.
\end{aligned} \quad \text{(A12)}$$

After separating the real and imaginary parts of equation (A12) in $\langle\ \rangle$, it can be written as

$$\begin{aligned}
\sigma_{rr} = {} & \sum_{n=0}^{\infty} [Re(a_n^*) + iIm(a_n^*)](Q_1 + iQ_2)P_n(\cos\phi)\, e^{i\omega t} \\
& + \sum_{n=0}^{\infty} \frac{2\lambda_n}{r^2}(Q_3 + iQ_4)P_n(\cos\phi)\, e^{i\omega t}
\end{aligned} \quad \text{(A13)}$$

where

$$\begin{aligned}
Q_1 = {} & -q_1 Re[j_n(k_1^* r)] + q_2 Im[j_n(k_1^* r)] \\
& - 2(q_5 \cdot q_{12} - q_6 \cdot q_{13}) - \frac{4}{r}\{q_3 Re[j_n'(k_1^* r)] \\
& - q_4 Im[j_n'(k_1^* r)]\}
\end{aligned}$$

$$\begin{aligned}
Q_2 = {} & q_1 Im[j_n(k_1^* r)] - q_2 Re[j_n(k_1^* r)] - 2(q_6 q_{12} + q_5 q_{13}) \\
& - \frac{4}{r}\{q_3 Im[j_n'(k_1^* r)] + q_4 Re[j_n'(k_1^* r)]\}
\end{aligned}$$

$$\begin{aligned}
q_{12} = {} & \{[q_7 - \lambda_n Re(k_1^{*2})]Re[j_n(k_1^* r)] \\
& - \lambda_n Im[j_n(k_1^* r)]Im(k_1^{*2})\}/q_7
\end{aligned}$$

$$\begin{aligned}
q_{13} = {} & \{[q_7 - \lambda_n Re(k_1^{*2})]Im[j_n(k_1^* r)] \\
& + \lambda_n Re[j_n(k_1^* r)]Im(k_1^{*2})\}/q_7
\end{aligned}$$

$$Q_3 = q_8\{Re[j_n(k_2^*r)] - rq_{10}\} - q_9\{Im[j_n(k_2^*r)] - rq_{11}\}$$

$$Q_4 = q_8\{Im[j_n(k_2^*r)] - rq_{11}\} + q_9\{Re[j_n(k_2^*r)] - rq_{10}\}.$$

Finally, the normal stress can be expressed as

$$\sigma_{rr}(r, \phi, t) = \sum_{n=0}^{\infty} [Z_{n1}(r) + iZ_{n2}(r)] P_n(\cos\phi) e^{i\omega t} \quad \text{(A14)}$$

where

$$Z_{n1} = Re(a_n^*) \cdot Q_1 - Im(a_n^*) Q_2 + 2\lambda_n Q_3/r^2$$

$$Z_{n2} = Re(a_n^*) \cdot Q_2 + Im(a_n^*) Q_1 + 2\lambda_n Q_4/r^2 .$$

Equation (A14) is the desired expression to be shown