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Flows on Nonorientable 2-Manifolds

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1. INTRODUCTION

This paper extends the study initiated in [1] where a class $\mathscr{M}$ of orientable 2-manifolds, including all compact ones, was defined and it was shown that the central sequence (defined below) of any flow on any manifold in $\mathscr{M}$ terminates in at most two steps. It was also shown that any nonorientable 2-manifold supports a flow whose central sequence does not terminate in two steps. In this paper we define a class $\mathscr{N}$ of nonorientable 2-manifolds, including all compact ones, and show that the central sequence of any flow on a member of $\mathscr{N}$ terminates in at most three steps.

Preliminary results, mainly topological, are presented in Section 2 and in Section 3 the main results are established. The rest of this section contains the basic definitions and the notation of the paper. For more details and a brief history of the problem the reader is referred to [1] and [2].

Let $\{h_t \mid t \in R\}$ be a flow on a space X. By X^1 we mean the set of all *nonwandering points*; thus, $x \in X^1$ if, and only if, for every neighborhood U of x and every $t \in R$, there exists $s \in R$, $s > t$ such that $h_s(U) \cap U \neq \varphi$. Clearly X^1 is closed in X and invariant under the flow. The *central sequence* (of the flow) is the possibly transfinite chain: $X^1 \supseteq X^2 \supseteq \cdots \supseteq X^\alpha \supseteq \cdots$, where $X^\alpha = (X^\beta)^1$ if $\alpha = \beta + 1$ and if α is a limit ordinal, $X^\alpha = \bigcap \{X^\beta \mid \beta < \alpha\}$.

The *center* (of the flow) is $\bigcap X^\alpha$ which may be described intrinsically in the following way. For each point x let $\Omega(x)$ ($A(x)$) be the set of points y such that $h_{t_n}(x) \to y$ for some sequence $t_n \to +\infty (t_n \to -\infty)$. A point x is called *Poisson stable in the positive* (*negative*) *direction* provided $x \in \Omega(x) (x \in A(x))$ in which case we shall write, "x is $P^+(P^-)$ stable." A point which is both P^+ and P^- stable is called *Poisson stable*. It is known, Theorem 5.08 on p. 358 of [2], that *the center is the closure of the set of Poisson stable points*.

The *depth* of the center is the least ordinal α such that $X^\alpha = X^{\alpha+1} = \cdots$; that is, α is the least ordinal such that X^α is the center. For the manifolds considered in this paper it will be shown the depth of the center of any flow is at most 3.

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2. Preliminary Results

Let S^2 denote the 2-sphere. The manifold obtained by removing n pairwise disjoint open disks from S^2 and sewing in n cross-caps will be denoted $S^2 \cup c_1 \cup \cdots \cup c_n$. A simple closed curve in a 2-manifold is called *one-sided* in case it has a closed neighborhood homeomorphic to a Möbius band. For $n \geqslant 1$, we shall say that a 2-manifold M is of *type n* provided it contains n pairwise disjoint one-sided simple closed curves but does not contain $n + 1$ such curves. All manifolds will be metrizable (hence, separable) connected 2-manifolds.

Theorem 2.1. *Let M be a manifold without boundary. Then M is of type n if, and only if, M is homeomorphic to $S^2 \cup c_1 \cup \cdots \cup c_n - F$, where F is a closed and totally disconnected subset of S^2.*

Proof. Suppose that M is of type $n \geqslant 1$. Let $S_1, ..., S_n$ be pairwise disjoint one-sided simple closed curves and let $N_1, ..., N_n$ be closed pairwise disjoint neighborhoods of the S_i with each N_i homeomorphic to a Möbius band. Then $M_1 = M - (N_1 \cup \cdots \cup N_n)$ is a manifold without boundary which is separated by every simple closed curve. To see this last fact, suppose S is a simple closed curve lying in M_1 and that $M_1 - S$ is connected. The boundary of N_1 is a simple closed curve in M which we denote B_1. Let a_1, b_1, a_2, b_2 be four points on B_1 such that $a_1 < b_1 < a_2 < b_2$ relative to some orientation of B_1. Using the fact that S does not separate M_1 we can choose disjoint arcs α_0 and β_0 such that $\alpha_0(\beta_0)$ has endpoints a_1 and a_2 (b_1 and b_2) and except for the endpoints α and β lie in M_1 (see Fig. 1).

Let α_1 and β_1 be arcs in N_1 with endpoints a_1, a_2 and b_1, b_2, respectively, which wind around N_1 once as pictured in Fig. 2.

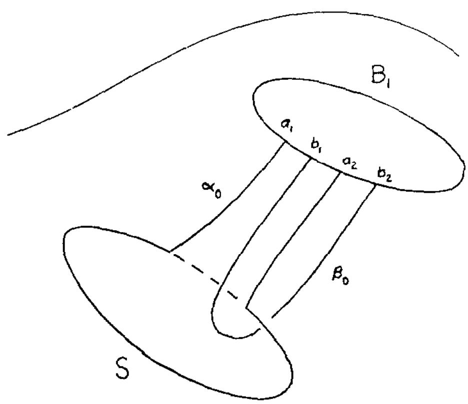

Figure 1

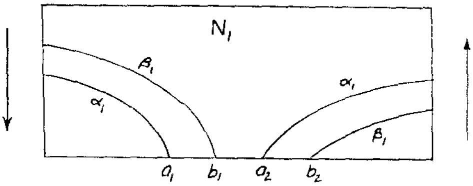

Figure 2

Then $\alpha_0 \cup \alpha_1$, $\beta_0 \cup \beta_1$, S_2,..., S_n are disjoint one-sided curves in M, a contradiction. Thus every simple closed curve in M_1 separates M_1 and by a theorem of Zippen [3], M_1 is homeomorphic to $S^2 - F_1$, where F_1 is closed and totally disconnected. It follows easily that M is homeomorphic to $S^2 \cup c_1 \cup \cdots \cup c_n - F$, where F is closed and totally disconnected.

The reverse implication is trivial.

The argument used above may also be used to prove the following:

THEOREM 2.2. *Let Γ be a continuum (compact, connected set) in $M = S^2 \cup c_1 \cup \cdots \cup c_n$. If some component of $M - \Gamma$ is of type n then there is an open cell in M which contains Γ.*

Proof. Suppose W is a component of $M - \Gamma$ which contains n pairwise disjoint one-sided simple closed curves S_1,..., S_n. Choose neighborhoods N_1,..., N_n of the S_i as in the proof of Theorem 2.1 with each $N_i \subset W$. As shown above, there is a homeomorphism h of $M - (N_1 \cup \cdots \cup N_n)$ onto $S^2 - F$, where F is closed and totally disconnected. Clearly F is a set of n points in S^2 and F lies in some component of $S^2 - h(\Gamma)$, namely, the component $h[W - (N_1 \cup \cdots \cup N_n)]$. It follows that there is an open cell U in $S^2 - F$ which contains $h(\Gamma)$ and the preimage of U is an open cell in M containing Γ.

We next state two results which will be used repeatedly in the next section. Let $\{h_t\}$ be a flow on a manifold M and let x be a point in the interior of M. If x is not fixed then there is an arc in M, say $\alpha = \alpha(p)$, $-1 \leqslant p \leqslant 1$, with $\alpha(0) = x$, and an $\epsilon > 0$ such that the mapping $(t, p) \to h_t[\alpha(p)]$ is a homeomorphism of $K = (-\epsilon, \epsilon) \times [-1, 1]$ into M. The image of K under this homeomorphism is called a *flow box neighborhood of* x and the arc α is called a *transversal through* x.

THEOREM 2.3 [3; p. 333]. *If $\{h_t\}$, M and x are as above, there is a flow box neighborhood of x.*

We shall also need

LEMMA 2.4. *Let y be a nonwandering point of a flow $\{h_t\}$ on a manifold M. Let U be an open cell in M, let α be a transversal lying in U and suppose that* *there exist s, t, in R with $s < t$ such that $h_s(y)$ and $h_t(y)$ are on α and $\{h_r(y) \mid s \leqslant r \leqslant t\}$ lies in U. Then y is periodic.*

The proof is a restatement of the argument given in Section 3 of [1] and is therefore omitted.

3. The Main Theorems

One principal result of this paper is:

THEOREM 3.1. *Let M be a connected open subset of $S^2 \cup c_1 \cup \cdots \cup c_n$. Then for any flow on M, the depth of the center is at most 3.*

Proof. Consider the propositions stated for $n \geqslant 1$ as follows:

$A(n)$: For any flow on $S^2 \cup c_1 \cup \cdots \cup c_k$ where $1 \leqslant k \leqslant n$ the depth of the center is at most 3.

$B(n)$: For any flow on a manifold without boundary of type k where $1 \leqslant k \leqslant n$, the depth of the center is at most 3.

Supposing for the moment that all these propositions have been verified, let M be as in the theorem. Then M is a manifold and is either nonorientable, in which case some $B(n)$ applies, or else M is orientable, in which the depth is at most 2 by the results of [1].

Thus, it suffices to prove the $A(n)$ and $B(n)$. We begin with $A(1)$. Let $\{h_t\}$ be a flow on $M = S^2 \cup c_1$ (the projective plane). We first observe that if there is a P^+ stable point which is not fixed or periodic then the depth of the center is at most 2. For let y be such a point and let $\Gamma = \Omega(y)$. If Γ lies in a cell U then so does some "tail," $\{h_t(y) \mid t \geqslant t_0\}$, of $0(x)$. But Lemma 2.4 would then apply to show that y is periodic. Thus Γ does not lie in a cell and from Theorem 2.2 it follows that every component of $M - \Gamma$ is orientable. By [1], $(M - \Gamma)^2$ is the center of $\{h_t \mid_{M-\Gamma}\}$ and since Γ surely lies in the center of h_t we conclude that M^2 is the center of $\{h_t\}$.

A similar argument applied to P^- stable points shows that in proving $A(1)$ we may assume that

2.5. *Every P^+ or P^- stable point is periodic.*

Now suppose there is $x \in M^3 - M^4$ and let α be a transversal through x. There exists a sequence $\{y_n\}$ with the following properties:

(1) $y_n \in M^2 - M^3$

(2) $y_n \in \alpha$ and $d(y_n, x) < 1/n$, where d is a metric for M.

(3) For some $t_n > 0$, $h_{t_n}(y_n) \in \alpha$ and $d(h_{t_n}(y_n), x) < 1/n$.

By choosing subsequences we many assume that either all y_n lie on a single orbit, $0(y)$, or that all the $0(y_n)$ are distinct. In the first case, x belongs to $\Omega(y)$ or $A(y)$, say $\Omega(y)$, and we let $\Gamma = \Omega(y)$. In the second case, we let $\Gamma = \limsup 0(y_n) = \bigcap_{k=1}^{\infty} cl[\bigcup_{k \geqslant n} (y_n)]$.

In either case Γ is a continuum in M which contains x. Moreover the sequence $\{y_n\}$ may be chosen so that

2.6. *No y_n belongs to Γ.*

If $\Gamma = \Omega(y)$ this is simply the statement that y cannot be P^+ stable which follows from condition 1 and Remark 2.5 above. If $\Gamma = \limsup 0(y_n)$ then it may be necessary to extract a proper subsequence with the desired properties. A detailed inductive process for doing this is given in Section 6 of [1].

Suppose Γ lies in an open cell U. If $\Gamma = \Omega(y)$ then some tail of $0(y)$ lies in U and meets a transversal through x twice. By Lemma 2.4, y is periodic contradicting condition 1. If $\Gamma = \limsup 0(y_n)$ we have the same contradiction since all but finitely many of the $0(y_n)$ lie in U.

Thus, Γ does not lie in a cell and, by Theorem 2.2, every component of $M - \Gamma$ is orientable. Since $0(y_1)$ lies in $M - \Gamma$ and in M^2 it follows from [1] that y_1 is in M^3 which again is a contradiction.

This final contradiction completes the argument for proposition $A(1)$.

We next observe that $A(n)$ implies $B(n)$ for all n. Let M be a manifold satisfying the hypotheses of $B(n)$ and let $\{h_t\}$ be a flow on M. By 2.1, there is a homeomorphism h of M onto $S^2 \cup c_1 \cup \cdots \cup c_k - F$, where F is closed and totally disconnected. It is easy to check that the family $\{g_t\}$ of mappings defined on $S^2 \cup c_1 \cup \cdots \cup c_k$ by

$$g_t(x) = \begin{cases} hh_th^{-1}(x) & x \notin F \\ x & x \in F \end{cases}$$

is a flow on $S^2 \cup c_1 \cup \cdots \cup c_k$. If $A(n)$ is true the length of the central sequence of $\{g_t\}$ is at most 3 and clearly the same is then true for $\{h_t\}$.

Finally, we need to show that $B(n)$ implies $A(n+1)$. The proof of $A(1)$ given above can be modified to handle this case. We need only observe that if a continuum Γ in $S^2 \cup c_1 \cup \cdots \cup c_{n+1}$ does not lie in a cell then each component of the complement of Γ is either orientable or satisfies the hypotheses of $B(n)$. Thus where we appealed to [1] in the proof of $A(1)$, we appeal either to [1] or to $B(n)$. With this change the induction step goes through easily and the proof of Theorem 3.1 is complete.

For manifolds with boundary we can extend the preceding result quite easily.

THEOREM 3.2. *Let M be a manifold of type n where $n \geqslant 1$. Then for any flow on M the length of the central sequence is at most* 3.

Proof. Let $\tilde{M}$ be the manifold obtained by attaching to each boundary component K of M a copy of $\{(x, y) \in R^2 \mid x^2 + y^2 < 1, 0 \leqslant y\}$ or

$$\{(x, y, z) \in R^3 \mid x^2 + y^2 = 1, \quad 0 \leqslant z < 1\}$$

according as K is an open arc or a simple closed curve. $\tilde{M}$ then satisfies the hypotheses of Theorem 3.1. Given a flow on M we may extend this to a flow on $\tilde{M}$ with the property that points of $\tilde{M} - M$ are wandering. By Theorem 3.1 it follows that the central sequence of the extended flow has length at most 3 and, by construction, the same will be true of the original flow on M.

Finally, we observe that the result of Theorem 3.2 is clearly valid for disconnected manifolds since each component is invariant under the flow.

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