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Invariance in Linear Systems

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INTRODUCTION

The study of invariance of linear systems derives its motivation from the need to develop an insight into ways in which a control system can be made insensitive to variations in its parameters.

Along these lines, Cruz and Perkins [1], have obtained necessary and sufficient conditions for the output y of the system characterized by the state equations

$$\dot{x} = Ax + Nv, \qquad x(0) = x_0, \tag{1}$$

$$y = Cx, \tag{2}$$

to be invariant to variations in v and A. These conditions are

$$CA^K D = 0, \qquad K = 0, 1, \dots, n - 1, \tag{3}$$

$$CA^K N = 0, \qquad K = 0, 1, \dots, n - 1 \tag{4}$$

and are related to the controllability and observability of the system in question.

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In this paper, we shall consider a more general problem [2], in which the system is characterized by equations of the form

$$\dot{x}(t) = A(t)x + u(t), \qquad x(0) = x_0 \tag{5}$$

and is subject to perturbations

$$A(t) \to A(t) + E(t), \tag{6}$$

$$u(t) \to u(t) + e(t). \tag{7}$$

Specifically, let the unperturbed system be described by

$$\dot{x} = A(t)x + u(t) \tag{8}$$

and the pertubed system by

$$\dot{\tilde{x}} = A(t)\,\tilde{x} + E(t)\,\tilde{x} + u(t) + e(t), \tag{9}$$

where $E(t)$ is the parameter error and $e(t)$ is the signal error over the time interval $[0, T]$. The performance measures of the two systems are defined as follows:

$$y(\tau, p, t) = x'(\tau, p, t)\,Q(\tau)\,x(\tau, p, t), \tag{10}$$

$$\tilde{y}(\tau, p, t) = \tilde{x}'(\tau, p, t)\,Q(\tau)\,\tilde{x}(\tau, p, t) \tag{11}$$

and

$$Y(T, p, t) = \int_t^T x'(\tau, p, t)\,Q(\tau)\,x(\tau, p, t)\,d\tau, \tag{12}$$

$$\tilde{Y}(T, p, t) = \int_t^T \tilde{x}'(\tau, p, t)\,Q(\tau)\,\tilde{x}(\tau, p, t)\,d\tau, \tag{13}$$

where $x(\tau, p, t)$ and $\tilde{x}(\tau, p, t)$ are solutions of (8) and (9), respectively, as functions of τ such that

$$x(t, p, t) = p \qquad \text{and} \qquad \tilde{x}(t, p, t) = p.$$

Now the property of invariance will be defined in terms of the performance measures y, $\tilde{y}$, Y, and $\tilde{Y}$.

Parameter Invariance

Definition 1. The system performance for the systems (8) and (9) is parameter-invariant over the time interval $[0, T]$, with respect to the transformation of the matrix $A(t)$ into $A(t) + E(t)$ if for each initial state $p \in R^n$ and $t \in [0, T]$

$$\hat{Y}(T, p, t) = Y(T, p, t)$$

with $e(t) \equiv 0$ in (9).

DEFINITION 2. The system performance for systems (8) and (9) is weakly parameter-invariant over the time interval $[0, T]$ with respect to the transformation of matrix $A(t)$ into $A(t) + E(t)$ if for each initial state $p \in R^n$,

$$\hat{Y}(T, p, 0) = Y(T, p, 0)$$

with $e(t) \equiv 0$ in (9).

DEFINITION 3. The system performance for system (8) and (9) is strongly parameter-invariant over the time interval $[0, T]$ with respect to the transformation $A(t)$ into $A(t) + E(t)$ if for each initial state $p \in R^n$ and $t \in [0, T]$.

$$y(\tau, p, t) = \tilde{y}(\tau, p, t), \qquad t \leqslant \tau \leqslant T.$$

with $e(t) \equiv 0$ in (9).

SIGNAL INVARIANCE

DEFINITION 4. The system performance for system (8) and (9) is signal invariant over the time interval with respect to changing the input $u(t)$ to $u(t) + e(t)$ if for each $p \in R^n$ and $t \in [0, T]$,

$$\hat{Y}(T, p, t) = Y(T, p, t)$$

with $E(t) \equiv 0$ in (9).

Weak signal invariance and strong signal invariance may be defined similarly.

SIGNAL AND PARAMETER INVARIANCE

DEFINITION 5. The system performance for the systems (8) and (9) is simultaneously signal and parameter-invariant over the time interval $[0, T]$ with respect to the transformation of $A(t)$ to $A(t) + E(t)$ and $u(t)$ to $u(t) + e(t)$ if for each initial state $p \in R^n$ and $t \in [0, T]$,

$$\hat{Y}(T, p, t) = Y(T, p, t).$$

Weak as well as strong simultaneous signal and parameter invariance may be defined similarly.

It may be observed that strong invariance implies invariance, and invariance implies weak invariance. Invariance, does not, in general imply strong invariance, and weak invariance does not, in general, imply invariance.

LEMMA 1. *Let $Y(T, p, t)$ be defined as above. Then for $p \in R^n$, and $t \in [0,T]$, Y satisfies the differential equation*

$$\frac{\partial Y(T,p,t)}{\partial t} + \left[\frac{\partial Y(T,p,t)}{\partial p}\right]' [A(t)\, p + u(t)] + p'Q(t)\, p = 0, \qquad t \in [0, T], \tag{14}$$

and $Y(T, p, T) = 0$.

Proof. Fix $p_0 \in R^n$. If $x(\tau, p_0, 0)$ is the solution of (8) with $x(0, p_0, 0) = p_0$, for $t \in [0, T]$, we define p by $p \equiv x(t, p_0, 0)$. Then $x(\tau, p_0, 0) = x(\tau, p, t)$ for $t \leqslant \tau \leqslant T$. Then for $t \in [0, T]$,

$$\begin{aligned}\frac{d}{dt}\{Y(T,p,t)\} &= \frac{d}{dt}\{Y(T, x(t, p_0, 0), t)\}\\ &= \frac{\partial Y(T,p,t)}{\partial t} + \left[\frac{\partial Y(T,p,t)}{\partial p}\right]' [A(t)\, x(t, p_0, 0) + u(t)]\\ &= \frac{\partial Y(T,p,t)}{\partial t} + \left[\frac{\partial Y(T,p,t)}{\partial p}\right]' [A(t)\, p + u(t)].\end{aligned}$$

Let

$$F(p) = \int_0^t x'(\tau, p_0, 0)\, Q(\tau)\, x(\tau, p_0, 0)\, d\tau + \int_t^T x'(\tau, p_0, 0)\, Q(\tau)\, x(\tau, p_0, 0)\, d\tau.$$

On observing that $x(\tau, p_0, 0) = x(\tau, p, t)$ for $t \leqslant \tau \leqslant T$, the expression for $F(p)$ reduces to

$$\int_0^t x'(\tau, p_0, 0)\, Q(\tau)\, x(\tau, p_0, 0)\, d\tau + Y(T, p, t).$$

On differentiating, one obtains

$$\frac{dY(T,p,t)}{dt} = -x'(t, p_0, 0)\, Q(t)\, x(t, p_0, 0) = -p'Q(t)\, q,$$

and on combining the two expressions for dY/dt,

$$\frac{\partial Y(T,p,t)}{\partial t} + \left[\frac{\partial Y(T,p,t)}{\partial p}\right]' [A(t)\, p + u(t)] + p'Q(t)\, p = 0.$$

Since given any $p \in R^n$ and $t \in [0, T]$, one can find $p_0 \ni x(t, p_0, 0) = p$, the above equation holds for all $p \in R^n$ and $t \in [0, T]$. A similar equation holds for $\hat{Y}(T, p, t)$.

THEOREM 1. *A necessary and sufficient condition for the system output to be simultaneously parameter invariant and signal invariant on* $[0, T]$ *is that* $E(t)$ *and* $e(t)$ *satisfy*

$$P(t)\,E(t) + E'(t)\,P(t) = 0, \tag{15}$$

$$2P(t)\,e(t) + E'(t)\,r(t) = 0, \tag{16}$$

$$r'(t)\,e(t) = 0 \tag{17}$$

on $[0, T]$, *where* $P(t)$ *is a symmetric matrix which satisfies*

$$\dot{P}(t) + PA + A'P + Q = 0 \qquad P(T) = 0$$

and $r(t)$ *is n-vector which satisfies*

$$\dot{r}(t) + A'r + 2Pu = 0, \qquad r(T) = 0.$$

Proof. Let $P(t)$, $r(t)$ and $s(t)$ satisfy the following set of equations on $[0, T]$:

$$\dot{P} + PA + A'P + Q = 0 \qquad P(T) = 0 \tag{18}$$

$$\dot{r} + A'r + 2Pu = 0 \qquad r(T) = 0 \tag{19}$$

$$\dot{s} + r'u = 0 \qquad s(T) = 0. \tag{20}$$

Then the solution $Y(T, p, t)$ of (14) is given by

$$Y(T, p, t) = p'P(t)\,p + p'r(t) + s(t).$$

Similarly, the solution $\tilde{Y}(T, p, t)$ of the equation for $\tilde{Y}$ similar to (14) is given by

$$\tilde{Y}(T, p, t) = p'\tilde{P}(t)\,p + p'r(t) + s(t)$$

where $\tilde{P}$, $\tilde{r}$, and $\tilde{s}$ are solutions to

$$\dot{\tilde{P}} + \tilde{P}(A + E) + (A' + E')\,\tilde{P} + Q = 0 \qquad \tilde{P}(T) = 0$$

$$\dot{\tilde{r}} + (A' + E')\,\tilde{r} + 2P(u + e) = 0 \qquad \tilde{r}(T) = 0$$

$$\dot{\tilde{s}} + \tilde{r}'(u + e) = 0 \qquad \tilde{s}(T) = 0.$$

In order that

$$Y(T, p, t) = \tilde{Y}(T, p, t)$$

for all $p \in R^n$ and $t \in [0, T]$, it is necessary and sufficient that

$$\tilde{P}(t) = P(t),$$
$$\tilde{r}(t) = r(t),$$
$$\tilde{s}(t) = s(t).$$

Hence the theorem.

COROLLARY 1. *Assume $Q(t)$ is positive definite on $[0, T]$, then the conditions* (15), (16), (17) *reduce to*

$$E(t) = S(t)\,P(t)$$
$$e(t) = \tfrac{1}{2}S(t)\,r(t)$$

where $P(t)$ and $r(t)$ satisfy (18) *and* (19) *respectively and $S(t)$ is a skew symmetric matrix for $t \in [0, T]$.*

Proof. The solution of (18) for $P(t)$ may be written as

$$P(t) = \int_t^T \varphi'(\tau, t)\,Q(\tau)\,\varphi(\tau, t)\,d\tau$$

where $\varphi(\tau, t)$ is the state transition matrix satisfying

$$\frac{d\varphi(\tau, t)}{d\tau} = A(\tau)\,\varphi(\tau, t), \varphi(t, t) = I$$

and I is the identity matrix. Now for $Z_0 \neq 0$, and $t \in [0, T]$,

$$Z(\tau)\,Q(\tau)\,Z(\tau) > 0, \qquad \tau \in [t, T]$$

(since $Q(\tau) > 0$), where

$$\frac{dZ(\tau)}{d\tau} = A(\tau)\,Z(\tau), \qquad \text{for} \quad \tau > t, \qquad Z(0) = Z_0\,.$$

Then, $\int_t^T Z(\tau) Q(\tau)\,Z(\tau)\,d\tau > 0$ for all $Z_0 \neq 0$, and $t \in [0, T]$, since $Z(\tau) = \varphi(\tau, t)Z_0$, where

$$Z_0 \left\{ \int_t^T \varphi'(\tau, t)\,Q(\tau)\,\varphi(\tau, t)\,d\tau \right\} Z_0 > 0$$

for all $Z_0 \neq 0$; hence $P(t) > 0$ on $[0, T]$. If $P(t) > 0$, the condition (15)

$$E'P + PE = 0$$

is equivalent to $E = SP$, where S is skew symmetric. Clearly $E = SP$ satisfies the condition (15). Conversely, if the condition is satisfied then PE is skew symmetric, or $EP^{-1} = P^{-1}PEP^{-1}$ is skew, so $E = SP$. The condition (16) is then equivalent to

$$2P(t)\,e(t) + P'(t)\,S'(t)\,r(t) = 0$$

or

$$2e(t) + S'(t)\,r(t) = 0$$

or

$$e(t) = \tfrac{1}{2}S(t)\, r(t).$$

Condition (17) then becomes $\frac{1}{2}r'(t)\, S(t)\, r(t)$, which is always satisfied since $S(t)$ is skew symmetric. This proves the corollary.

Since the performance measure matrix $Q(t)$ is usually positive definite, the Corollary 1 is a more useful result. Another reason behind Corollary 1 being more useful is that the error matrices are expressed explicitly.

COROLLARY 2. *If $Q(t) > 0$, then the necessary and sufficient condition for parameter invariance is*

$$E(t) = S(t)\, P(t)$$

where $S(t)$ is a skew-symmetric matrix satisfying $S(t)\, r(t) = 0$, on $[0, T]$ and $P(t), r(t)$ are solutions of (18), (19), *on* $[0, T]$.

If it turns out that $u(t) \equiv 0$, then (19) gives $r(t) = 0$, so that the condition $S(t)\, r(t) = 0$ is automatically satisfied.

COROLLARY 3. *If $Q(t) > 0$, then the necessary and sufficient condition for signal invariance is that*

$$e(t) = 0, \qquad \text{on} \quad [0, T].$$

Some observations. It may be observed that if the system performance is invariant on $[0, T]$, then it does not imply in general that it is invariant on $[0, T']$ where $T' < T$. This is easily seen by looking at $P(t)$, since in one case $P(T) = 0$ and in the other case $P(T') = 0$. This also shows that invariance does not imply strong invariance.

Under various hypotheses above, we have given several statements of necessary and sufficient conditions for invariance. Because of the relationship between strong invariance, invariance, and weak invariance, we have also derived results for weak invariance and strong invariance. In any of the above results, we may replace "necessary and sufficient" by sufficient if "invariance" is replaced by "weak invariance," and "necessary and sufficient" by necessary if "invariance" is replaced by "strong invariance."

INVARIANCE UNDER COMPARISON

We consider a slightly different class of systems for which the previous results apply. The system equations for the unperturbed and the perturbed systems are still given by (8) and (9); however, the quantity whose invariance is considered is

$$\int_t^T \epsilon'(\tau)\, Q(\tau)\, \epsilon(\tau)\, d\tau, \tag{21}$$

where $\epsilon'(\tau) = x(\tau, p, t) - q(t)$. Here $q(t)$ is a continuously differentiable vector function defined on $[0, T]$. The problem now is to determine the conditions that the output (21) is invariant for all $p \in R^n$ and $t \in [0, T]$. These conditions follow directly from previous results by defining a new set of state equations in the variable ϵ. These are

$$\dot{\epsilon}(t) = A\epsilon + u(t) + Aq - \dot{q},$$

$$\dot{\bar{\epsilon}}(t) = (A + E)\, \epsilon + u(t) + Aq - \dot{q} + e(t) + Eq.$$

Necessary and sufficient and corresponding conditions for invariance are

$$P(t)\, E(t) + E'(t)\, P(t) = 0, \tag{22}$$

$$2P(t)\{e(t) + E(t)\, q(t)\} + E'(t)\, r(t) = 0, \tag{23}$$

$$r'(t)(e(t) + E(t)\, q(t)) = 0, \tag{24}$$

where $P(t)$ and $r(t)$ satisfy

$$\dot{p} + PA + A'P + Q = 0, \tag{25}$$

$$\dot{r} + A'r + 2P(u + Aq - \dot{q}) = 0. \tag{26}$$

If $Q(t) > 0$, then the necessary and sufficient conditions are

$$E(t) = S(t)\, P(t), \tag{27}$$

$$e(t) = \tfrac{1}{2} S(t)\, r(t) - S(t)\, P(t)\, q(t), \tag{28}$$

where $P(t)$ and $r(t)$ satisfy (25) and (26).

Additional results are possible as explained earlier concerning weak invariance and strong invariance.

Invariance Conditions and State Transition Matrix

Let $\varphi(\tau, t)$ be the state transition matrix for the system

$$\dot{Z}(\tau) = A(\tau)\, Z(\tau),$$

i.e.,

$$\frac{d\varphi(\tau, t)}{d\tau} = A(\tau)\, \varphi(\tau, t), \qquad \varphi(t, t) = I,$$

and the state transition matrix for the adjoining system satisfies the following matrix equation

$$\frac{d\varphi'(\tau, t)}{dt} = -A'(t)\,\varphi'(\tau, t), \qquad \varphi(\tau, \tau) = I.$$

The solution to the equations (18) and (19) may now be written as

$$P(t) = \int_t^T \varphi'(\tau, t)\, Q(\tau)\, \varphi(\tau, t)\, d\tau$$

and

$$r(t) = 2 \int_t^T \varphi'(s, t)\, P(s)\, u(s)\, ds$$

or

$$= 2 \int_t^T \varphi'(s, t) \left[\int_t^T \varphi'(\tau, s)\, Q(\tau)\, \varphi(\tau, s)\, d\tau \right] u(s)\, ds.$$

So the conditions of Theorem 1 and various corollaries may be written in terms of $\varphi(\tau, t)$.

Conclusions

It may be observed by considering the above results that, in general, there will be many choices of $E(t)$ and $e(t)$ which satisfy various conditions for invariance. However, the results are sufficient and necessary in a restricted sense, i.e., with respect to all initial times t satisfying $t \leqslant T$. It is a disadvantage that one must solve for $P(t)$ backwards in time or $\varphi(t, t_2)$. Both processes must be done "off line." It may be desirable in certain applications to have conditions that may be computed "on-line."

References

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