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GENERALIZED SENIORITY FOR FAVORED $J \neq 0$ PAIRS IN MIXED CONFIGURATIONS

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Abstract: By assigning a pseudo-spin and pseudo-orbital angular momentum (*b*-spin and *c*-spin) to the single-particle states in a mixed configuration of identical nucleons, it is possible to classify as $B = 0$ objects both the favored pair operators of the surface delta interaction (SDI) and the multipole moment operators, (the latter under suitable assumptions). The favored pair is defined for each J as the specific superposition of two-particle states acted upon by this separable interaction. The SDI is diagonal in the total B- and C-spins; the eigenvalues of a (B, C) multiplet are independent of the total angular momentum J, ($\boldsymbol{J} = \boldsymbol{B}+\boldsymbol{C}$); and all states with $B = \frac{1}{2}v$ are degenerate (v = total seniority). For the case of a degenerate doublet of levels $(l_j, (l+2)_{j+1})$, e.g. $(d_{\frac{3}{2}}g_{\frac{7}{2}})$ or $(f_{\frac{7}{2}}h_{\frac{9}{2}})$, the specification, $B = 0$, defines the favored pairs uniquely, and $2B$ counts the number of nucleons not members of favored pairs. Exact calculations for the $(d_{\frac{3}{2}}g_{\frac{7}{2}})$ system show that states with $B < \frac{1}{2}v$ cluster closely about their centers of gravity; therefore, to a good approximation the SDI can be replaced by a generalized pairing interaction depending only on B and v. Possible generalizations are discussed for the case of many degenerate single-particle levels, where this generalized pairing interaction is no longer a good approximation.

1. Introduction

The surface delta interaction (SDI) introduced by Green and Moszkowski [1,2]) has served as a remarkably good effective interaction for shell-model calculations in many regions of the periodic table [3-6]). The two-body matrix elements of this interaction and its modifications [7]) are also in good agreement with those derived from realistic interactions [8]) or directly from an analysis of experimental data [9]). One of the characteristic features of the SDI, when acting in mixed configurations of identical particles, is that it favors one specific superposition of two-particle states for each value of J. In the two-particle spectrum only a single one of the several possible states for each J-value is depressed in energy; the others have eigenvalues of zero. This property follows solely from the separability of the SDI and is therefore common to all separable two-body interactions [10]). The favored pair of the surface delta interaction with $J \neq 0$ has the additional property that it exhausts entirely the sum rule for a 2^J-pole transition connecting it to the $J = 0$ ground state [2]). Moreover, the favored $J = 0$ pair of the SDI is precisely the pair which is the basis of pairing theory for systems with valence particles filling several degenerate or nearly degenerate subshells. This $J = 0$ pair, the energetically most favored of all pairs, corresponds to the

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coherent superposition of $J = 0$ pairs which is the basis of the generalized seniority scheme for mixed configurations. The surface delta interaction is a scalar in quasispin space [2]), and its favored $J = 0$ pair is the pair with the largest possible value of the total quasispin quantum number which is related to the total or overall seniority number. Since the total seniority quantum number effectively counts the number of nucleons not members of favored $J = 0$ pairs, it is of interest to examine the question whether there is an additional generalization of the seniority concept with a quantum number which effectively counts the number of nucleons not members of the favored $J \neq 0$ pairs of the SDI or some modified form of this interaction.

The simplest pair of nuclear subshells consists of the two members of a spin-orbit doublet, that is the single-particle states with orbital angular momentum l and $j = l \pm \frac{1}{2}$. In a configuration based entirely on such a pair of single-particle levels and made up of identical nucleons, (protons *or* neutrons only), a delta interaction can act only on $S = 0$ pairs since a delta interaction vanishes in spatially antisymmetric states. In this simple case, therefore, the favored pairs are the pairs coupled to $S = 0$ and $L = J = 0, 2, 4, \ldots 2l$. (Whenever the single-particle states of a mixed configuration all have the same parity, the favored pairs have even values of J only.) If the single-particle levels were degenerate, (no spin-orbit coupling), the total spin S would be a good quantum number. Moreover, it would be a measure of the number of nucleons not members of favored pairs. Since strong spin-orbit coupling is one of the cornerstones of the nuclear shell model, this simple case never arises in practice. In real nuclei the single-particle states of the same parity which are nearly degenerate are pairs of states such as $\{s_{\frac{1}{2}}d_{\frac{3}{2}}\}$, $\{d_{\frac{5}{2}}g_{\frac{7}{2}}\}$, $\{p_{\frac{3}{2}}f_{\frac{5}{2}}\}$, $\{f_{\frac{7}{2}}h_{\frac{9}{2}}\}, \ldots$, that is pairs of states of the type $\{l_j, (l+2)_{j+1}\}$. It is the purpose of this investigation to show that it is possible to associate with such a pair of single-particle states a pseudo-spin and pseudo-orbital angular momentum (subsequently to be denoted by b-spin and c-spin, respectively; with single particle values $b = \frac{1}{2}$ and $c = l+1$ for the above doublets). The favored pairs are the pairs coupled to $B = 0$. The surface delta interaction as well as the multipole moment operators are b-space scalars. For states of total seniority v, the total B-spin has the possible values $\frac{1}{2}v, \frac{1}{2}v-1, \frac{1}{2}v-2, \ldots, 0$; and the number $2B$ plays the role of a generalized seniority quantum number since it counts the number of nucleons not members of favored pairs in the same sense in which the number v counts the number of nucleons not members of favored $J = 0$ pairs. Finally, for a degenerate single-particle doublet of the type $\{l_j, (l+2)_{j+1}\}$ it will be shown that the eigenvalues of the surface delta interaction are approximated quite well by a generalized pairing Hamiltonian whose eigenvalues depend only on the quantum numbers B and v, with a spectrum such that states with $2B \leqq v$ are repeated with the same spacing for each seniority greater than v.

Although single-particle doublets of the type $\{l_j, (l+2)_{j+1}\}$ are somewhat special, they can be taken as the basic building blocks for the nuclear shell model. The major nuclear shells are made up of such doublets, all of the same parity, up to a highest j-value of j_{max} to which is added a single state of opposite parity with $j = j_{max}+2$. In the limit in which these single-particle states are degenerate, the total

B-spin is again a good quantum number for the SDI. However, the B-spin is no longer sufficient to specify the favored pairs for each J completely. Additional quantum numbers are needed, or a generalization must be made of the symmetry associated with the b-space. The case of a full major nuclear shell is discussed briefly in sect. 4 in which some possible generalizations of the quantum number b are discussed. The present investigation concentrates on the case of the single-particle doublet of the type $\{l_j(l+2)_{j+1}\}$. As our prime example we take the $\{d_{\frac{5}{2}}g_{\frac{7}{2}}\}$ doublet, with $b = \frac{1}{2}$, $c = 3$. Recent experimental studies [11, 12]) on nuclei with 82 neutrons and $A = 135$–143 show that the valence protons with $Z > 50$ fill mainly the $1g_{\frac{7}{2}}$ and $2d_{\frac{5}{2}}$ levels. Detailed shell-model calculations for the configuration $(1g_{\frac{7}{2}}2d_{\frac{5}{2}})^n$ have recently been performed by Wildenthal [6]) and make it possible to compare the simple predictions of an extreme zeroth order generalized pairing model based on the favored $J \neq 0$ pairs of the SDI with the results of a full shell-model calculation.

2. Symmetry of the SDI

When acting in configurations of identical particles (neutrons *or* protons only) the SDI becomes a separable interaction which can be expressed in terms of the pair creation operators for the favored pairs, (and their hermitian conjugate pair annihilation operators). These are specific combinations of the pair creation operators coupled to total angular momentum J, defined by

$$A^{+}_{JM}(jj') = \sum_{m} \langle jmj'm'|JM\rangle a^{+}_{jm} a^{+}_{j'm'} . \tag{1}$$

The favored pairs of the SDI are [2])

$$\mathscr{A}^{+}_{JM} = \tfrac{1}{2} \sum_{jj'} (-1)^{l} h_J(jj') A^{+}_{JM}(jj'), \tag{2}$$

with

$$h_J(jj') = \left[\frac{(2j+1)(2j'+1)}{2J+1}\right]^{\frac{1}{2}} \langle j\tfrac{1}{2}j'-\tfrac{1}{2}|J0\rangle(-1)^{j'+\frac{1}{2}}. \tag{3}$$

For $J = 0$ the favored pair is that of ordinary pairing theory. In particular, the operator $\mathscr{A}^{+}_{00}$ is the total quasispin operator $\mathscr{S}_{+}$. In terms of the operators $\mathscr{A}^{+}_{JM}$ the surface delta interaction takes the form [2])

$$H_{\mathrm{SDI}} = -G \sum_{JMJ} \mathscr{A}^{+}_{JM} \mathscr{A}_{JM} . \tag{4}$$

To discuss the symmetry of the interaction it is convenient to introduce the unit tensor operators

$$u_{kq}(jj') = \sum_{mm'} \langle jmj'-m'|kq\rangle a^{+}_{jm} a_{j'm'} (-1)^{j'-m'} . \tag{5}$$

For a mixed configuration based on single-particle states of total degeneracy number $2\Omega = \sum_j (2j+1)$, the $(2\Omega)^2$ operators of type (5) generate the unitary group in 2Ω

dimensions, U(2Ω). Since the SDI is a scalar in quasispin space [2], the symplectic subgroup Sp(2Ω) is the physically relevant subgroup of the unitary group. It is generated by the $\Omega(2\Omega+1)$ infinitesimal operators

$$(u_{kq}(jj')+(-1)^{l+l'+j+j'+k}u_{kq}(j'j)). \tag{6}$$

The irreducible representations of Sp(2Ω) specify the total seniority. Finally, the total angular momentum operator, $\boldsymbol{J}$, is a specific linear combination of the operators (6) with $k = 1$, $(j' = j)$, which generates the three-dimensional rotation group R(3). The eigenstates of the SDI are classified by the group chain $U(2\Omega) \supset Sp(2\Omega) \supset R(3)$, with corresponding quantum numbers n, v and J. To examine the possibility that the SDI contains additional symmetries, a search must be made for subgroups of Sp(2Ω) which contain R(3). For this purpose it is convenient to associate with the nucleons a pseudo-spin and pseudo-orbital angular momentum, (b-spin and c-spin), by the relation

$$a^+_{bm_bcm_c} = \sum_j \langle bm_b cm_c|jm\rangle a^+_{jm}, \tag{7}$$

where c and b are integers and $\frac{1}{2}$-integers appropriately chosen to yield the single-particle j-values of the subshells of actual interest. (A single pair of b, c values may suffice, or a set of several may be required.)

The unit tensor operators which are scalars in c-space are of particular interest. They are defined by

$$B^{k_b}_{q_b} = \sum_{m_c}\sum_{m_bm'_b}\langle bm_b\, b-m'_b|k_b q_b\rangle a^+_{bm_bcm_c} a_{bm'_bcm_c}(-1)^{b-m'_b} \tag{8a}$$

and can be expressed in terms of the unit tensor operators of eq. (5) by

$$B^{k_b}_{q_b} = \sum_{jj'}(-1)^{j'+b+c+k_b}[(2j+1)(2j'+1)]^{\frac{1}{2}}\begin{Bmatrix} j & b & c\\ b & j' & k_b\end{Bmatrix} u_{k_bq_b}(jj'). \tag{8b}$$

(If more than one value of c is required, eqs. (8) may include sums over c with appropriate weighting coefficients.) The unit tensor operators which are scalars in b-space can be defined in analogous fashion

$$C^{k_c}_{q_c} = \sum_{jj'}(-1)^{j+b+c+k_c}[(2j+1)(2j'+1)]^{\frac{1}{2}}\begin{Bmatrix} j & c & b\\ c & j' & k_c\end{Bmatrix} u_{k_cq_c}(jj'). \tag{9}$$

Using the symmetry properties of the 6-j symbol, it can be seen that the operators $B^{k_b}_{q_b}$ with k_b odd, and similarly the operators $C^{k_c}_{q_c}$ with k_c odd belong to the class of eq. (6), provided single-particle states of the same parity are assigned values c of the same parity. These operators thus generate subgroups of the symplectic group Sp(2Ω). The operators with $k_b = 1$ and $k_c = 1$ are of greatest interest. With appropriate normalization factors (or reduced matrix elements) the operators

$$\boldsymbol{B}^{k=1}_q = \sum_{jj'}[\tfrac{1}{3}(2b+1)b(b+1)(2j+1)(2j'+1)]^{\frac{1}{2}}(-1)^{j'+b+c+1}\begin{Bmatrix} j & b & c\\ b & j' & 1\end{Bmatrix} u_{1q}(jj') \tag{10a}$$

and

$$C_q^{k=1} = [\tfrac{1}{3}(2c+1)c(c+1)(2j+1)(2j'+1)]^{\frac{1}{2}}(-1)^{j+b+c+1} \begin{Bmatrix} j & c & b \\ c & j' & 1 \end{Bmatrix} u_{1q}(jj') \tag{10b}$$

satisfy the relations

$$\begin{aligned} [B_+, B_-] &= 2B_0, \qquad [B_0, B_\pm] = \pm B_\pm; \\ [C_+ C_-] &= 2C_0, \qquad [C_0, C_\pm] = \pm C_\pm, \end{aligned} \tag{11a}$$

and

$$[\boldsymbol{B}, \boldsymbol{C}] = 0, \tag{11b}$$

$$\boldsymbol{J} = \boldsymbol{B} + \boldsymbol{C}. \tag{11c}$$

The operators $\boldsymbol{B}$ and $\boldsymbol{C}$ are commuting angular-momentum operators whose sum is the conventional total angular-momentum operator. The operators generate a direct product of two three-dimensional rotation groups $[\mathrm{O}(3)\times\mathrm{O}(3)]$, a subgroup of $\mathrm{Sp}(2\Omega)$, which itself contains the conventional three-dimensional rotation group. The irreducible representations are labeled by quantum numbers (B, C) related to the eigenvalues $B(B+1)$ and $C(C+1)$ of the operators $\boldsymbol{B}^2$ and $\boldsymbol{C}^2$. The angular momenta J contained in a representation (B, C) are given by the usual coupling rules: $J = B+C, \ldots, |B-C|$. To find the possible representations (B, C) contained in a given representation of $\mathrm{Sp}(2\Omega)$, (total seniority v), and to study the physical significance of the new subgroup and its relation to the symmetries of the SDI, it is best to consider some examples so that the values of b and c are specified.

3. The two-level case

The single-particle states of the same parity which are nearly degenerate in real nuclei are pairs of states such as $\{s_{\frac{1}{2}}d_{\frac{3}{2}}\}$, $\{d_{\frac{5}{2}}g_{\frac{7}{2}}\}$, $\{p_{\frac{3}{2}}f_{\frac{5}{2}}\}, \ldots$, that is, doublets of the type $\{l_j, (l+2)_{j+1}\}$. For such a doublet the single particle b- and c-spins can be assigned as $b = \frac{1}{2}$, $c = l+1$; e.g., $(b, c) = (\frac{1}{2}, 3)$ for a $(d_{\frac{5}{2}}g_{\frac{7}{2}})$ doublet. Antisymmetry requirements restrict the two-particle states (two identical particles) to those with $B = 0$, $C =$ even, $(J = C)$; and $B = 1$, C odd, $(J = C+1, C, C-1)$; and it can be shown that the favored pairs of the SDI are the pairs with $B = 0$. The pair creation operators coupled to $B = 0$, $C = J$ even, can be written as

$$\begin{aligned} A^+_{B=0\,M_B=0,\,C=JM} &= \sum_{m_b m_c} \langle bm_b\, b-m_b|00\rangle\langle cm_c\, cm'_c|JM\rangle a^+_{bm_b cm_c} a^+_{b-m_b cm'_c} \\ &= \sum_{jj'} (-1)^{b+c+j+J} \left[\frac{(2j+1)(2j'+1)}{2b+1}\right]^{\frac{1}{2}} \begin{Bmatrix} j & c & b \\ c & j' & J \end{Bmatrix} A^+_{JM}(jj'). \end{aligned} \tag{12}$$

With $b = \frac{1}{2}$, and the relation

$$(2c+1)\langle c0c0|J0\rangle(-1)^{j+j'} \begin{Bmatrix} j & c & \frac{1}{2} \\ c & j' & J \end{Bmatrix} = \langle j\tfrac{1}{2}j'-\tfrac{1}{2}|J0\rangle, \tag{13}$$

the creation operator for the favored pair of the SDI, eq. (2), becomes

$$\mathscr{A}^{+}_{JM} = [\tfrac{1}{2}(2c+1)]^{\frac{1}{2}}(-1)^{c}\langle c0J0|c0\rangle A^{+}_{B=0\ M_B=0,\ JM} \tag{14}$$

and, except for a proportionality constant, is the $B = 0$ pair creation operator. Since the favored pair creation operators are b-space scalars, they commute with the components of $\boldsymbol{B}$

$$[\mathscr{A}^{+}_{JM}, \boldsymbol{B}] = [\mathscr{A}_{JM}, \boldsymbol{B}] = 0. \tag{15}$$

Just as the favored pair annihilation operator with $J = 0$, (the total quasispin operator $\mathscr{S}_{-}$), can be used to determine the seniority of a state, the favored pair-annihilation operators with $J \neq 0$ can be used to measure a generalized seniority, (the B-spin). A state with nucleon number $n = v$ is entirely free of favored $J = 0$ pairs and satisfies

$$\mathscr{A}_{00}|n = v\rangle = 0, \tag{16}$$

If the action of $\mathscr{A}_{00}(\mathscr{S}_{-})$ on a state of n nucleons yields zero only after the successive application of $(x+1)$ such operators, the state has a seniority $v = n-2x$. From the commutation relations (15) it can be seen that the favored pair annihilation operator with $J \neq 0$ cannot change the B-value of a state. The operator $\mathscr{A}_{JM}$, however, lowers the nucleon number by two units, and consequently must give zero when acting on a state of B-spin high enough that such a B-spin is not found among the $(n-2)$ nucleon states. In particular,

$$\mathscr{A}_{JM}|B = \tfrac{1}{2}v\rangle = 0 \qquad \text{for all } J \neq 0, \tag{17}$$

that is, a state with $B = \frac{1}{2}v$ is entirely free of favored $J \neq 0$ pairs. Similarly, successive application of two favored pair annihilation operators with $J \neq 0$ must yield zero when acting on a state with $B = \frac{1}{2}v-1, \ldots$. The B quantum number, therefore, counts the number of favored $J \neq 0$ pairs; and states with $B = 0, 1, 2, \ldots$ can be said to have the values $0, 2, 4, \ldots$ for the new generalized seniority quantum number.

The possible (B, C) values for states with $v \geqq 3$ are given by the well-known techniques of spectroscopy. For the $(d_{\frac{5}{2}}g_{\frac{7}{2}})$ doublet, for example, with $(b, c) = (\frac{1}{2}, 3)$, they are identical with those for atomic f-shell spectroscopy, as given by Racah [13]. Some of these are shown in table 1 to illustrate the richness of the $v = 3$ and $v = 4$ spectrum, (the three- and four-quasiparticle states of pairing theory).

From the commutation relation (15) several simple properties of the SDI follow:

(i) The SDI is a scalar in B-space. Since it is a scalar in J-space, it is also a scalar in C-space. It is therefore diagonal in both B and C. If the single-particle levels are degenerate, both B and C are good quantum numbers. For the $(d_{\frac{5}{2}}g_{\frac{7}{2}})$ doublet it can be seen from table 1 that the matrix for H_{SDI} splits into submatrices which for $v \leqq 4$ are never larger than 3×3.

(ii) The eigenvalues of H_{SDI} are independent of J, and are functions only of v, B, C, and the additional quantum numbers needed to distinguish states with the same

TABLE 1

Allowed (BC) values for the $(d_{\frac{5}{2}}g_{\frac{7}{2}})$ system

v	(BC)								
1					$(\frac{1}{2}3)$				
2	(02)	(04)	(06)		(11)	(13)	(15)		
3	$(\frac{3}{2}6)$	$(\frac{3}{2}4)$	$(\frac{3}{2}3)$	$(\frac{3}{2}2)$	$(\frac{3}{2}0)$				
	$(\frac{1}{2}8)$	$(\frac{1}{2}7)$	$(\frac{1}{2}6)$	$(\frac{1}{2}5)^2$	$(\frac{1}{2}4)^2$	$(\frac{1}{2}3)$	$(\frac{1}{2}2)^2$	$(\frac{1}{2}1)$	
4	(26)	(24)	(23)	(22)	(20)				
	(19)	(18)	$(17)^2$	$(16)^2$	$(15)^3$	$(14)^3$	$(13)^3$	$(12)^2$	$(11)^2$
	(0, 10)	$(08)^2$	(07)	$(06)^2$	$(05)^2$	$(04)^3$	(03)	$(02)^3$	(00)

(B, C) for $v > 2$. Each (B, C) state is therefore a degenerate multiplet. Its components with $J = B+C, \ldots, |B-C|$ all have the same energy.

(iii) The eigenvalues of the SDI for all states with $B = \frac{1}{2}v = \frac{1}{2}n$ are zero for all values of C. All states $(B = \frac{1}{2}v, C)$ with different values of C are therefore degenerate.

The properties (i)–(iii) hold not only for the SDI but for a modified interaction in which the single strength coefficient G of eq. (4) is replaced by J-dependent coefficients; that is, properties (i)–(iii) hold for any interaction built from $B = 0$ pair operators. With these properties it is also easy to calculate the eigenvalues of $H_{\rm SDI}$ for $v > 2$ by means of a cfp expansion using cfp's known from atomic spectroscopy. The two-particle matrix elements are different from zero only in states with $B = 0$. For the two-level case they have the value

$$-G(2c+1)\langle c0J0|c0\rangle^2, \tag{18}$$

so that the n-particle matrix elements are given by the cfp expansion

$$\begin{aligned}\langle nv\alpha'(BC)|H_{\rm SDI}|nv\alpha(BC)\rangle &= -\tfrac{1}{2}Gn(n-1)(2c+1)\sum_J \langle c0J0|c0\rangle^2 \\ &\times \sum_{v_{n-2}\alpha_{n-2}B_{n-2}C_{n-2}} \langle n-2v_{n-2}\,\alpha_{n-2}(B_{n-2}\,C_{n-2});(0J)|\}nv\alpha(BC)\rangle \\ &\times \langle n-2v_{n-2}\,\alpha_{n-2}(B_{n-2}\,C_{n-2});(0J)|\}nv\alpha'(BC)\rangle. \end{aligned}\tag{19}$$

The quantum numbers α are needed whenever a (B, C) value occurs more than once for a given v. For $c = 3$, (the analogue of atomic f-shell spectroscopy), the quantum numbers α are given by the irreducible representation labels of the special group G_2, (Racah [13]). However, the group G_2 seems to have no particular physical significance for the SDI; ($H_{\rm SDI}$ is not diagonal in G_2). The cfp needed for eq. (19) for the $(d_{\frac{5}{2}}g_{\frac{7}{2}})$ doublet follow from the tabulations of Racah [13]). The eigenvalues of $H_{\rm SDI}$ for states with $v \leqq 4$ are shown in fig. 1 and table 2. As previously noted, states with $B = \frac{1}{2}v$, are all degenerate. The states with $B < \frac{1}{2}v$, though not degenerate, cluster quite closely about their centers of gravity. For example, the states with $v = 2$, $B = 0$, $J = 2, 4, 6$ lie $\frac{4}{15}$, $\frac{2}{11}$, and $\frac{100}{429}$ units below the $v = 2$, $B = 1$ states, (on a scale on which the $v = 0$ state lies one full unit below the $v = 2$, $B = 1$ states), and cluster quite closely about

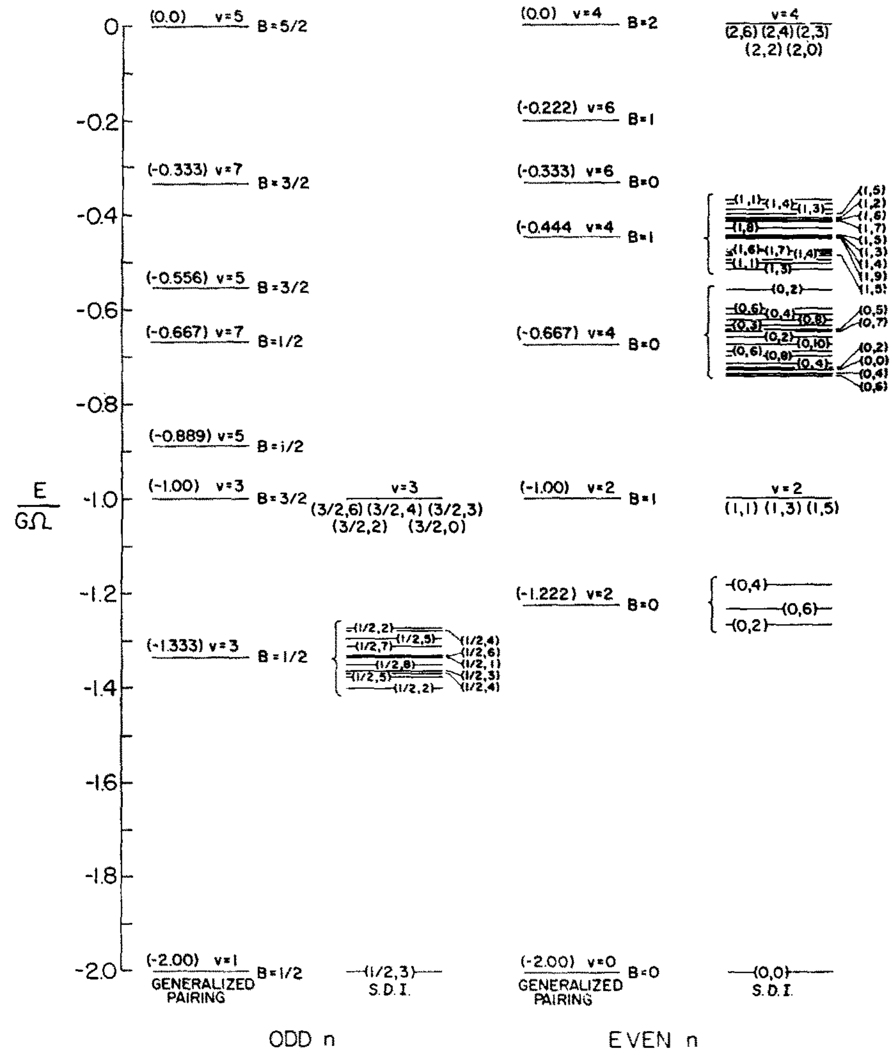

Fig. 1. Energy levels for a degenerate ($d_{\frac{5}{2}}g_{\frac{7}{2}}$) system. Exact eigenvalues for SDI are shown only for $v \leqq 4$. For the generalized pairing interaction only the low-lying higher v-states are shown.

their center of gravity at $\frac{2}{9}$ or $2/(2c+3)$ units. This suggests that the SDI, (in the two-level case at least), can be replaced to good approximation by a simplified interaction for which the effective interaction strength for favored $J \neq 0$ pairs is independent of J (but differs from the pairing strength for $J = 0$). This simplified form of the SDI will be called the generalized pairing interaction. It is given, in terms of the $B = 0$

TABLE 2

Energies of SDI for the $(d_{\frac{3}{2}}g_{\frac{7}{2}})$ system

$v = 1$	$(\frac{1}{2}, 3)$	-2.000	$v = 0$	$(0, 0)$	-2.000	$v = 2$	$(0, 2)$	-1.267
							$(0, 4)$	-1.182
$v = 3$	$(\frac{1}{2}, 1)$	-1.333					$(0, 6)$	-1.233
	$(\frac{1}{2}, 2)$	-1.400						
		-1.273	$v = 4$	$(0, 0)$	-0.727		$(1, 1)$	-0.504
	$(\frac{1}{2}, 3)$	-1.364		$(0, 2)$	-0.723			-0.369
	$(\frac{1}{2}, 4)$	-1.368			-0.660		$(1, 2)$	-0.494
		-1.279			-0.559			-0.409
	$(\frac{1}{2}, 5)$	-1.375		$(0, 3)$	-0.636		$(1, 3)$	-0.518
		-1.295		$(0, 4)$	-0.738			-0.448
	$(\frac{1}{2}, 6)$	-1.387			-0.717			-0.389
	$(\frac{1}{2}, 7)$	-1.329			-0.611		$(1, 4)$	-0.485
	$(\frac{1}{2}, 8)$	-1.350		$(0, 5)$	-0.742			-0.450
					-0.644			-0.381
				$(0, 6)$	-0.690		$(1, 5)$	-0.486
					-0.602			-0.445
				$(0, 7)$	-0.647			-0.400
				$(0, 8)$	-0.700		$(1, 6)$	-0.478
					-0.627			-0.412
				$(0, 10)$	-0.676		$(1, 7)$	-0.479
								-0.415
							$(1, 8)$	-0.431
								-0.452

pair operators $A^{+}_{B=0M_B=0,CM_C}$ of eq. (12), by

$$H_{\text{gen. pairing}} = -\tfrac{1}{2}G(2c+1)\left[A^{+}_{00,00}A_{00,00}+\frac{2}{2c+3}\sum_{J>0}\sum_{M}A^{+}_{00,JM}A_{00,JM}\right]. \quad (20)$$

It can be expressed in terms of the usual ($J = 0$) pairing Hamiltonian and a two-body pseudo spin-spin interaction modified by a monopole term of appropriate strength

$$H_{\text{gen. pairing}} = -G\frac{2c+1}{2c+3}\mathscr{A}^{+}_{00}\mathscr{A}_{00}+G\frac{2c+1}{2c+3}[2\sum_{i<j}^{n}\boldsymbol{b}_i\cdot\boldsymbol{b}_j-\tfrac{1}{2}\sum_{i<j}^{n}1]. \quad (21)$$

The generalized pairing term for $J \neq 0$ pairs thus leads to the two-body $\boldsymbol{B}^2$ operator, much as the ordinary pairing term leads to the quasispin $\mathscr{S}^2$ operator. (Racah and Talmi [14]) in discussing the pairing properties for the ordinary spin-orbit doublet of the l^n configuration cite the operator $\boldsymbol{s}_i\cdot\boldsymbol{s}_j$ as the simplest seniority preserving operator.) The eigenvalue of the generalized pairing Hamiltonian (21) is given by

$$E_{\text{gen. pairing}} = -G\frac{2c+1}{2c+3}[\tfrac{1}{4}(n-v)(4c+4-n-v)-B(B+1)+\tfrac{3}{4}n+\tfrac{1}{4}n(n-1)]. \quad (22)$$

Despite the apparent n-dependence, the relative spacings of these energies is n-independent. The interaction is a scalar in quasispin space, as is the SDI. The eigenvalues are plotted in fig. 1 alongside the exact eigenvalues of the SDI for the $(d_{\frac{5}{2}}g_{\frac{7}{2}})$ doublet, $(c = 3)$. It can be seen that the generalized pairing interaction gives a very good estimate of the position of any one of the very large number of states with $v \geqq 3$. States with $v = 4$ may be important in a calculation only insofar as they perturb the $v = 2$ states (through the single-particle energy splitting, for example). For this purpose their location by means of the generalized pairing interaction may serve as an excellent approximation. For $v \geqq 4$, the quantum number B becomes more important than v in ordering the energy levels. Thus states with $v = 4$, $B = 0$ lie close to the $v = 2$ states, while states with $v = 5$, $B = \frac{1}{2}$ lie close to the $v = 3$ states. Finally, it should be noted that the spectrum of states with $2B \leqq v$ is repeated with the same spacing for each seniority greater than v.

The energies shown in fig. 1 are for a degenerate single-particle doublet. If the single-particle energies ε_{j+1}, ε_j are unequal, the single-particle Hamiltonian must be considered. It can be put in the form

$$H_{\text{s.p.}} = n\left[\tfrac{1}{2}(\varepsilon_{j+1}+\varepsilon_j)+\frac{1}{2(2c+1)}(\varepsilon_{j+1}-\varepsilon_j)\right]+\frac{2}{2c+1}(\varepsilon_{j+1}-\varepsilon_j)\sum_{i=1}^{n}\boldsymbol{b}_i\cdot\boldsymbol{c}_i, \tag{23}$$

that is, apart from a constant term proportional to n, the single-particle Hamiltonian has the form of a one-body pseudo spin-orbit coupling term. Its matrix elements are therefore subject to the selection rules $|\Delta B| \leqq 1$, $|\Delta C| \leqq 1$; and in first order perturbation theory they can make no contribution to the $B = 0$ states. Calculations by Jones and Borgman [15]) for the $(1g_{\frac{7}{2}}2d_{\frac{5}{2}})$ levels of the 82-neutron nuclei show that the single-particle splitting can be treated in perturbation theory to good approximation. In these calculations the positions of states with $v > 2$ have been approximated by the generalized pairing value, eq. (22). The detailed shell-model calculations of Wildenthal [6]), however, do indicate that higher order terms may play some role since the final separation of states with $v = 4$, $B = 0$ and $v = 2$, $B = 1$, (or rather with predominant components of this type) have only about one half the separation energy predicted in zeroth order.

Finally, for a single (b, c) doublet, the favored pair operators generate a generalized quasispin group. This group can be identified from the work of Helmers [16]) as a symplectic group in $(4c+2)$ dimensions. Its infinitesimal operators are the $(2c+1)$ $(c+1)$ favored pair creation operators $\mathscr{A}^{+}_{JM}$ and their conjugate annihilation operators, together with the $(2c+1)^2$ operators $C^{k_c}_{q_c}$ of eq. (9). To make the correspondence with the infinitesimal operators of Sp$(4c+2)$ precise, it is convenient to uncouple these operators in c-space and write them in the form

$$A^{+}_{m_i m_k} = \sum_{m_b}(-1)^{b-m_b}a^{+}_{bm_b cm_i}a^{+}_{b-m_b cm_k} = A^{+}_{m_k m_i}, \tag{24a}$$

$$\mathscr{C}_{m_i m_k} = \sum_{m_b}a^{+}_{bm_b cm_i}a_{bm_b cm_k}-\tfrac{1}{2}(2b+1)\delta_{m_i m_k}. \tag{24b}$$

The correspondence between the $B = 0$ operators of eq. (24) and the generalized quasispin group operators is shown in table 3. With $b = \frac{1}{2}$, the eigenvalues of H_i can be at most $+1$. In this case, therefore, the irreducible representations of the generalized quasispin group Sp($4c+2$) are all of the form ($111 \ldots 00$). The number of 1's in the irreducible representation labels can be shown to be equal to $(2c+1-2B)$. The generalized quasispin group Sp($4c+2$) therefore does not lead to new quantum numbers in the two level case. The generalized pairing Hamiltonian of eq. (20) is, except for trivial factors, a combination of the Casimir invariants of the generalized quasispin group Sp($4c+2$) and its subgroup, the three-dimensional quasispin group based on the favored $J = 0$ pairs, whose eigenvalues are specified by the quantum numbers B and v, respectively. Finally, the orthogonal subgroup O($2c+1$) generated by the operators $C^{k_c}_{q_c}$ of eq. (9), with k_c odd, may introduce new quantum numbers in the case of relatively large c and v. For the smaller values of c and v of usual interest in the nuclear shell model, however, the irreducible representations of O($2c+1$) are almost always completely specified by the seniority number v alone [13]).

TABLE 3

Standard form for the infinitesimal operators of the generalized quasispin group, Sp($4c+2$)

Operators [a])	Infinitesimal generators [b]) (standard form)	Number of operators
$A^{\dagger}_{m_i m_i}$	E_α with: $\alpha = +2e_i$	$(2c+1)$
$A_{m_i m_i}$	$\alpha = -2e_i$	$(2c+1)$
$A^{\dagger}_{m_i m_k}$ $(m_k \neq m_i)$	$\alpha = e_i+e_k$	$c(2c+1)$
$A_{m_i m_k}$ $(m_k \neq m_i)$	$\alpha = -e_i-e_k$	$c(2c+1)$
$\mathscr{C}_{m_i m_k}$ $(m_k \neq m_i)$	$\alpha = e_i-e_k$	$c(2c+1)$
$\mathscr{C}_{m_k m_i}$ $(m_k \neq m_i)$	$\alpha = -e_i+e_k$	$c(2c+1)$
$\mathscr{C}_{m_i m_i}$	H_i	$(2c+1)$

[a]) See eq. (24). $m_i = c, c-1, \ldots, -c$ for $i = 1, \ldots, (2c+1)$.
[b]) The notation is that of ref. [17]).

4. The many-level case. Major nuclear shells

In the many-level case there may be several ways of assigning the one-particle b- and c-spins. The case of major nuclear shells will be illustrated by the ($s_{\frac{1}{2}}d_{\frac{3}{2}}$), ($d_{\frac{5}{2}}g_{\frac{7}{2}}$), ($h_{\frac{11}{2}}$) and the ($p_{\frac{1}{2}}$), ($p_{\frac{3}{2}}f_{\frac{5}{2}}$), ($f_{\frac{7}{2}}h_{\frac{9}{2}}$), ($i_{\frac{13}{2}}$) shells. If the favored pair is to be precisely the pair of eq. (2) favored by the SDI, it appears that all states of the same parity must be assigned a b-spin of $\frac{1}{2}$; so that the (bc) values for the two examples cited would be ($\frac{1}{2}1$), ($\frac{1}{2}3$), ($0\frac{11}{2}$), and ($\frac{1}{2}0$), ($\frac{1}{2}2$), ($\frac{1}{2}4$), ($0\frac{13}{2}$), respectively; or in general the (bc) spins for the major nuclear shells would be chosen as ($\frac{1}{2}$, $c_{\max}$), ($\frac{1}{2}$, $c_{\max}-2$), $\ldots$, ($0, j_0$), where the state with angular momentum j_0 (c-spin $= j_0$) has a parity opposite to that of the remaining states. Many other choices are of course possible. A ($p_{\frac{1}{2}}$, $p_{\frac{3}{2}}$, $f_{\frac{5}{2}}$) triplet, for example, could be assigned (bc) = ($1\frac{3}{2}$). However, if there is to be a precise correspondence between $B = 0$ pairs and the favored pairs of the SDI the choice based on b-spins of $\frac{1}{2}$ seems to be required. With this choice there are several ways of making $B = 0$ pairs of a given $J = C$.

$$A^{+}_{B=0\,M_B=0,\,JM} = \sum_{cc'} \alpha_{cc'}(J) \sum_{jj'} \left[\frac{(2j+1)(2j'+1)}{2b+1}\right]^{\frac{1}{2}} (-1)^{b+c+j+J} \begin{Bmatrix} j & c & b \\ c & j' & J \end{Bmatrix} A^{+}_{JM}(jj'), \tag{25}$$

where the coefficients $\alpha_{cc'}(J)$ are still arbitrary. In order to make the precise correspondence: $A^{+}_{00,JM} = \mathscr{A}^{+}_{JM}$, the coefficients $\alpha_{cc'}(J)$ must be chosen as

$$\alpha_{cc'}(J) = \left[\frac{(2c+1)(2c'+1)}{2(2J+1)}\right]^{\frac{1}{2}} \langle c0c'0|J0\rangle(-1)^{J} \qquad \text{for integral } c(b = \tfrac{1}{2}), \tag{26a}$$

$$\alpha_{j_0 j_0}(J) = (-1)^{j_0+\frac{1}{2}} \frac{2j_0+1}{2[(2J+1)]^{\frac{1}{2}}} \langle j_0 \tfrac{1}{2} j_0 -\tfrac{1}{2}|J0\rangle. \tag{26b}$$

The vector $\boldsymbol{B}$ is given by eq. (10a), with $b = \frac{1}{2}$; but a sum over the c-values of the $b = \frac{1}{2}$ levels is now implied. (The values j, j' determine c; note that $j, j' \neq j_0$.) The vector $\boldsymbol{C}$ is given by terms of the form given by eq. (10b) summed over all c, again with $b = \frac{1}{2}$, to which a term with $j = j' = j_0$ must be added where this term gives the j_0 contribution to the vector $\boldsymbol{J}$. With these definitions the vectors $\boldsymbol{B}$ and $\boldsymbol{C}$ again satisfy the properties of eqs. (11a, b, c) and (15). The c-space tensors C^k_q with $k > 1$ can now be chosen in several ways. Tensors with even k

$$C^k_q = \sum_{cc'} \gamma_{cc'}(k) \sum_{jj'} (-1)^{c+\frac{1}{2}+j+k} [(2j+1)(2j'+1)]^{\frac{1}{2}} \begin{Bmatrix} j & c & \frac{1}{2} \\ c & j' & k \end{Bmatrix} u_{kq}(jj') + \gamma_{j_0}(k) u_{kq}(j_0 j_0), \tag{27a}$$

with

$$\gamma_{cc'}(k) = (-1)^{c} \left[\frac{(2c+1)(2c'+1)}{2k+1}\right]^{\frac{1}{2}} \langle c0c'0|k0\rangle, \tag{27b}$$

and

$$\gamma_{j_0}(k) = (-1)^{j_0-\frac{1}{2}} \frac{2j_0+1}{[2k+1]^{\frac{1}{2}}} \langle j_0 \tfrac{1}{2} j_0 -\tfrac{1}{2}|k0\rangle,$$

are, except for a constant multiplier, the multipole moment operators [2])

$$Q(k, q) = \text{const.} \sum_{jj'} h_k(jj') u_{kq}(jj'). \tag{28}$$

In the approximation in which the variation of the radial integrals of the 2^k-pole operators with the different single-particle states are neglected, the electric multipole operators, and (in the same approximation) more general 2^k-pole operators are seen to be b-space scalars. In this approximation, therefore, their matrix elements are subject to the selection rule $\Delta B = 0$. A special case of this selection rule was noted by Arvieu and Moszkowski [2]) who pointed out that the 2^J-pole operator can connect the ground state of even nuclei ($v = B = C = 0$) only to the favored $J \neq 0$, $v = 2$

state, (again a state with $B = 0$) so that this transition probability takes up the full strength of the sum rule for a 2^J-pole transition. It is interesting to note that inelastic scattering experiments on odd nuclei of the 82 neutron family [18]), such as ^{141}Pr, with ground states which are predominantly $v = 1$, $B = \frac{1}{2}$ excite mainly a group of positive parity states centered about 1.5 MeV where the excited states have predominant components with $v = 3$, $B = \frac{1}{2}$, whereas no appreciable excitation to positive parity states is observed at higher energies in the region of the predicted $v = 3$, $B = \frac{3}{2}$ states.

One of the difficulties of the above assignments of (bc) values is that the quantum number B does not by itself give a unique definition of the favored pairs. For a degenerate set of single-particle levels with $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, ($p_{\frac{1}{2}}$, $p_{\frac{3}{2}}$, $f_{\frac{5}{2}}$, *or* $s_{\frac{1}{2}}$, $d_{\frac{3}{2}}$, $d_{\frac{5}{2}}$), there are two different pairs with $B = 0$ for both $J = 0$ and $J = 2$. The favored $J = 2$ pair is depressed below the $v = 2$, $B = 1$ states by 0.571 units, the favored $J = 4$ pair by 0.238 units on a scale on which the $v = 0$ state lies 1 unit below the $v = 2$, $B = 1$ states. The second $B = 0$, $J = 0$ and $J = 2$ states also remain at the unperturbed position of the $v = 2$, $B = 1$ states. The favored $J = 2$ pair is now depressed much more than the favored $J = 4$ pair. A generalized pairing approximation of the type of eqs. (20)–(22) is therefore no longer a good approximation. A much better approximation might be based on a model in which there is a single favored $J \neq 0$ pair, with $J = 2$, while the single $B = 0$, $J = 4$ pair is demoted and lumped in with the unperturbed states.

Similarly, for degenerate single-particle levels with $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$, the favored pairs of the SDI are depressed below the $v = 2$, $B = 1$ states as follows: $J = 2$ by 0.667 units, $J = 4$ by 0.394 units, and $J = 6$ by 0.163 units, again on a scale on which the favored $J = 0$ ($v = 0$) state is depressed by 1 full unit. The generalized pairing approximation which worked very well for the two-level case becomes poorer as the number of single-particle levels increases. Again, a much better approximation might be based on a model in which the favored $J \neq 0$ pairs are restricted to be those with $J = 2$ and $J = 4$, while the $J = 6$ state can to a good approximation be grouped with the two-particle states with zero eigenvalue which, (according to the above b, c assignments), include besides all of the $B = 1$ states, $B = 0$ states with $J = 0, 1, 2, 3,$ and 4.

Such models have the advantage that they can in principle be based on a single set of (bc) values. For the $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ family, for example, (bc) can be chosen to have the values $(1\frac{1}{2})$ or $(\frac{1}{2}1)$; while for the $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$, family (bc) values of $(\frac{3}{2}2)$ or $(2\frac{3}{2})$ are possible. In these models the vectors $\boldsymbol{B}$ and $\boldsymbol{C}$ are those of eq. (10), while the favored pairs are the $B = 0$ pairs of eq. (12) and are now uniquely defined by the specification: $B = 0$. A generalized quasispin group can be defined as in table 3, and the vectors $\boldsymbol{B}$, $\boldsymbol{C}$, can be imbedded in a group $[O(2b+1) \times O(2c+1)]$. While these models are simplest from the mathematical point of view, the question remains whether the $B = 0$ pairs defined in terms of a single (bc) multiplet are in relatively good agreement with the $J \neq 0$ pairs actually favored by nature. Unfortunately the correspondence does not seem to be very close. The $B = 0$ pairs based on a single (bc) multiplet seem to have the wrong j, j' components. This is illustrated in table 4 for the favored $J = 2$ pair of a $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, configuration. The table shows the j, j' components of the favored $J = 2$ two-particle states. It can be seen that the predictions of the SDI are in very good agreement with those based on the Kuo matrix elements [8]). (These numbers are the j, j' components of the lowest $J = 2$ two-particle state obtained by diagonalizing the Kuo two-particle matrix [8]) for a completely degenerate $(s_{\frac{1}{2}}d_{\frac{3}{2}}d_{\frac{5}{2}})$ single-particle multiplet. The phases in the various other columns are adjusted to correspond to these.) Unfortunately the j, j' components of the

TABLE 4

The j, j' components of the favored $J = 2$ state for the $j = \frac{1}{2}\ \frac{3}{2}\ \frac{5}{2}$ configuration, based on several models

j	j'	SDI	Kuo	$b = 1\ c = \frac{3}{2}$	$b = \frac{1}{2}\ c = 1$	SU_3 (40)
$\frac{5}{2}$	$\frac{5}{2}$	0.447	0.485	0.772	0.567	0.326
$\frac{3}{2}$	$\frac{3}{2}$	0.342	0.331	0.168	−0.495	0.250
$\frac{5}{2}$	$\frac{3}{2}$	0.316	0.288	−0.446	−0.491	0.231
$\frac{5}{2}$	$\frac{1}{2}$	0.592	0.592	0.188	0.415	0.683
$\frac{1}{2}$	$\frac{3}{2}$	0.483	0.472	0.377	−0.138	0.558

$B = 0$, $J = 2$ pairs for the $b = 1$, $c = \frac{3}{2}$ and $b = \frac{1}{2}$, $c = 1$ models differ appreciably from these. An assignment of (bc) values: $(\frac{1}{2}, c_{\max})$, $(\frac{1}{2}, c_{\max}-2)$, $(\frac{1}{2}, c_{\max}-4)$, ... therefore seems to be the best for the states of the same parity of a major nuclear shell. It is interesting to see how closely the favored pairs with $J = 0, 2, \ldots, 2c_{\max}$ agree with those of two particles coupled to the totally symmetric SU_3 representation $(2\lambda, 0)$, with $\lambda = c_{\max}$. Wigner coefficients for the coupling of totally symmetric representations of SU_3 in the chain $SU_3 \supset R_3$ have recently been given by Sharp *et al.* [19]). The reduced Wigner coefficient for the product $(\lambda 0)\times(\lambda 0) \to (2\lambda, 0)$ has the form

$$\langle(\lambda 0)c; (\lambda 0)c' \| (2\lambda, 0)J\rangle = [(2c+1)(2c'+1)]^{\frac{1}{2}}\langle c0c'0|J0\rangle \times \frac{f(\lambda, c)f(\lambda, c')}{f(2\lambda, J)}\left[\frac{2(2\lambda)!}{(2\lambda+2)!}\right]^{\frac{1}{2}}, \tag{29}$$

with

$$f(\lambda, c) = \left[\frac{(\frac{1}{2}(\lambda+c))!\lambda!2^c}{(\frac{1}{2}(\lambda-c))!(\lambda+c+1)!}\right]^{\frac{1}{2}}.$$

Except for the factors $f(\lambda, c)$ these coefficients would be the ones required by the SDI, (see eq. (26a)). Unfortunately, the factors $f(\lambda, c)$ again differ appreciably from unity. The results for a $J = 2$ two-particle state coupled to SU_3 representation (40) are shown in the last column of table 4.

No *complete* group theoretical description has been found for the favored $J \neq 0$ pairs, except in the case of the $\{l_j(l+2)_{j+1}\}$ single-particle doublet. The calculations for the case of many nearly degenerate single-particle states would therefore be similar to calculations for mixed configurations in atomic spectra where the (bc) values play the same role as ordinary spin and orbital angular momenta, with $b = \frac{1}{2}$ and c integral. An extension to configurations of both neutrons and protons will be the subject of a subsequent investigation.

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Note added in proof: The usefulness of pseudo-*LS* coupling schemes has recently also been pointed out by A. Arima, M. Harvey, and M. Shimizu (private communication) and by C. Quesne and R. Arvieu (private communication).

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