LOW FREQUENCY SCATTERING FROM AN OGIVE

by

Ergun Ar

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TABLE OF CONTENTS

I
INTRODUCTION

By means of a method developed by R. E. Kleinman (1965) of the Radiation Laboratory, it is now possible to solve iteratively the Dirichlet problem for the scalar Helmholtz equation in the regions exterior to a non-separable body imbedded in the Euclidean 3-space provided:

(a) k, the complex wave number, is sufficiently small,

(b) the solution of the potential Dirichlet problem for the body in question is known.

In the present work we consider the low frequency scattering of a plane wave at nose-on incidence from an ogive.

In Section II we find the integral representation for the static Dirichlet Green's function for the ogive. In Section III we give the series representation for this function. In Section IV, by means of the above method, we express the iterates for the scattered field explicitly in the form of integrals. In the Appendix, we indicate the orthogonality properites of the eigenfunctions arising in the problem.

II
THE STATIC GREEN'S FUNCTION OF THE FIRST KIND FOR THE OGIVE

2.1 The Separability of the Laplace Equation $\nabla^2\psi = 0$ in Bispherical Coordinates

Limiting ourselves to Euclidean 3-space and orthogonal curvilinear coordinate systems (u^1, u^2, u^3), we give two definitions.

Definition: If the assumption

$$\psi = \prod_{i=1}^{3} V^i(u^i) \tag{2.1}$$

permites the separation of the equation $\nabla^2\psi = 0$ into three ordinary differential equations, the Laplace equation is said to be simply separable.

Definition: If the assumption

$$\psi = \frac{1}{R(u^1, u^2, u^3)} \prod_{i=1}^{3} V^i(u^i) \tag{2.2}$$

permits the separation of the equation $\nabla^2\psi = 0$ into three ordinary differential equations, and if $R \neq$ constant, the equation is said to be R-separable.

Bispherical Coordinates

These orthogonal curvilinear coordinates (α, β, ϕ) are defined by

$$\begin{aligned} x &= \frac{c\sinh\beta}{\cosh\beta - \cos\alpha} \\ y &= \frac{c\sin\alpha\sin\phi}{\cosh\beta - \cos\alpha} \\ z &= \frac{c\sin\alpha\psi\cos\phi}{\cosh\beta - \cos\alpha} \end{aligned} \tag{2.3}$$

$$h_\beta = h_\alpha = \frac{c}{\cosh\beta - \cos\alpha} \qquad h_\phi = \frac{c\sin\alpha}{\cos\beta - \cos\alpha}$$

where

$$h_\alpha^2 = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right) + \left(\frac{\partial z}{\partial \alpha}\right)^2 ,$$

and similarly for h_β and h_ϕ.

In these coordinates

$$\nabla^2 \psi = \frac{1}{h_\beta^3}\left[\frac{\partial}{\partial \beta}\left(h_\beta \frac{\partial \psi}{\partial \beta}\right) + \frac{1}{\sin\alpha}\frac{\partial}{\partial \alpha}\left(h_\beta \sin\alpha \frac{\partial}{\partial \alpha}\right) + \frac{h_\beta^2}{\sin^2\alpha}\frac{\partial^2 \psi}{\psi \phi^2}\right]. \quad (2.4)$$

The ranges of the variables are $-\infty < \beta < \infty$, $0 \leqslant \alpha \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$.

The necessary and sufficient conditions for the separability of the Laplace (and Helmholtz) equations in various coordinate systems are given by Moon and Spencer (1952). Using their criteria for separation, we see that the Laplace equation is not simply separable in bispherical coordinates but is R-separable (whereas the Helmholtz equation is non-separable in either sense).

2.2 Definition of Ogive

In the bispherical coordinates the surface $\alpha = \alpha_1$(const.) is a surface formed by rotating about the x-axis that part of the circle, in the x-z plane, of radius $c \operatorname{cosec}\alpha_1$ with center $x = 0$, $z = c \cot\alpha_1$. The surface of revolution is called an ogive (Fig. 1). All the surfaces of constant α go through two points $x = \mp c$ ($y = z = 0$): and at these points $\beta = \mp\infty$ respectively. The surface $\alpha = 0$ is the x-axis for $x > c$ plus the sphere at infinity; the surface $\alpha = \pi/2$ is the sphere of radius c with center at the origin: and the surface $\alpha = \pi$ is the x-axis for $x < c$. The exterior region we are concerned with is $\alpha_1 \geqslant \alpha \geqslant 0$, $\infty > \beta > -\infty$, $2\pi \geqslant \phi \geqslant 0$. (there is another way of arriving at the bispherical coordinates, starting with the cylinderical coordinates, which will be considered later.)

2.3 The Green's Function

The Jacobian of the transformation (2.3) is

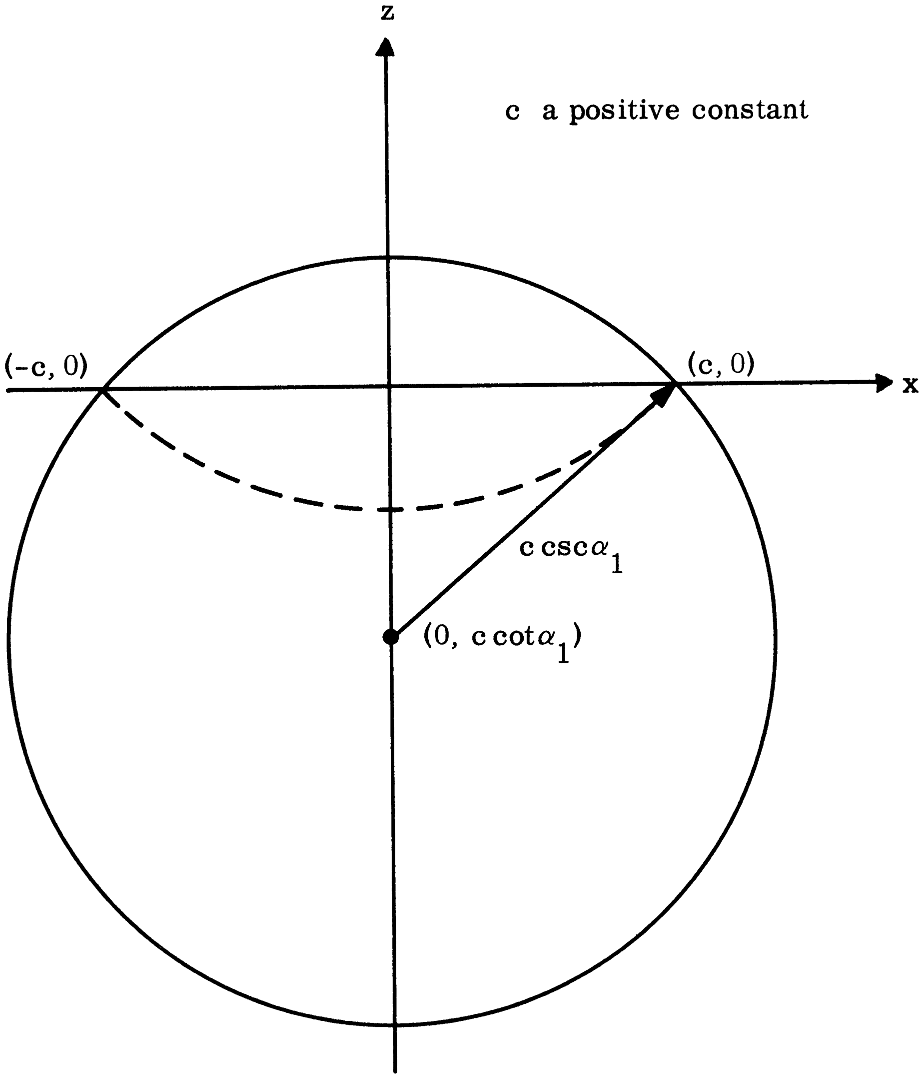

FIGURE 1

$$\frac{\partial(x, y, z)}{\partial(\alpha, \beta, \phi)} = \frac{c^3 \sin\alpha}{(\cosh\beta - \cos\alpha)^3} \,. \tag{2.5}$$

Also

$$\delta(x - x_o)\delta(y - y_o)\delta(z - z_o) = \left|\frac{\partial(x, y, z)}{\partial(\alpha, \beta, \phi)}\right|^{-1} \delta(\alpha - \alpha_o)\delta(\beta - \beta_o)\delta(\phi - \phi_o) \tag{2.6}$$ *

Therefore, the equation to be solved is

$$\nabla^2 G(\alpha, \beta, \phi; \alpha_o, \beta_o, \phi_o) = -4\pi \frac{(\cosh\beta - \cos\alpha)^3}{c^3 \sin\alpha} \cdot \delta(\alpha - \alpha_o) - (\beta - \beta_o)\delta(\phi - \phi_o) \tag{2.7}$$

where ∇^2 in bispherical coordinates is given by (2.4).

We find the necessary substitution for R-separability to be

$$G = \sqrt{\cosh\beta - \cos\alpha}\; g \,. \tag{2.8}$$

With this substitution (2.7) becomes

$$\frac{\partial^2 g}{\partial\beta^2} + \frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha \frac{\partial g}{\partial\alpha}\right) + \frac{1}{\sin^2\alpha}\frac{\partial^2 g}{\partial\phi^2} - \frac{1}{4} g = -4\pi \frac{\sqrt{\cosh\beta - \cos\alpha}}{c \sin\alpha} \cdot$$
$$\cdot\, \delta(\alpha - \alpha_o)\delta(\beta - \beta_o)\delta(\phi - \phi_o) \,. \tag{2.9}$$

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* Equality is understood in the sense of distributions. It is sufficient for our purposes to interpret the equality (2.5) (and any equality involving δ-functions as follows:

If the equation is multiplied by an arbitrary function $\phi \in C(-\infty, \infty)$ and integrated from $-\infty$ to ∞ with

$$\int_{-\infty}^{\infty} \delta(x)\, \phi(x)\, dx = \phi(0)$$

used to evaluate integrals involving δ-functions, then the resulting equality is correct in the ordinary sense. (Similarly, for the cases in more than one variable.) Also we shall assume the other δ-function formalism such as substitution, integration by parts, etc., and for proofs refer to, e.g., Gelfond and Schilow (1962).

We now require the periodicity of g in $0 \leqslant \phi \leqslant 2\pi$ and expand it in a (uniformly convergent) Fourier series

$$g(\alpha, \beta, \phi; \alpha_o, \beta_o, \phi_o) = \sum_{-\infty}^{\infty} A_m g_m(\alpha, \beta, \alpha_o, \beta_o) e^{im\phi} \quad \text{in } 0 \leqslant \phi \leqslant 2\pi \ . \tag{2.10}$$

Substituting (2.10) into (2.9), multiplying both sides by $e^{-in\phi}$ and integrating over ϕ from zero to 2π (and observing the orthogonality of the set $\{e^{in\phi}\}$), we see that

$$A_m = \frac{1}{2\pi} e^{-im\phi_o} \ ,$$

and (2.10) becomes

$$g(\alpha, \beta, \phi; \alpha_o, \beta_o, \phi_o) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} g_m(\alpha, \beta; \alpha_o, \beta_o) e^{im(\phi - \phi_o)} \ . \tag{2.11}$$

The equation satisfied by g_m is

$$\frac{\partial^2 g_m}{\partial \beta^2} + \frac{\partial^2 g_m}{\partial \alpha^2} + \frac{\cos\alpha}{\sin\alpha} \frac{\partial g_m}{\partial \alpha} - \left(\frac{1}{4} + \frac{m^2}{\sin^2\alpha}\right) g_m = -4\pi \frac{\sqrt{\cosh\beta - \cos\alpha}}{c \sin\alpha} \delta(\alpha - \alpha_o)\delta(\beta - \beta_o). \tag{2.12}$$

We now consider the Fourier transform of g_m*

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*If R is the distance between the field point $p = (\alpha, \beta, \phi)$ and the source point $p_o = (\alpha_o, \beta_o, \phi_o)$, then

$$\int_{-\infty}^{\infty} d\beta / R^2 < \infty$$

provided $p \neq p_o$, i.e., $1/R \in L^2(-\infty, \infty)$. Since the Green's function is regular at infinity (which will be shown after the completion of the construction of G), $G \in L^2(-\infty, \infty)$. Therefore the Fourier integral theorem is valid for G in the variable β. Or we simply note that G is a distribution and that the Fourier integral theorem is valid for distributions (Lighthill, 1960).

$$E\left[g_m\right] = \tilde{g}_m(\alpha, \nu; \alpha_o \beta_o) = \int_{-\infty}^{\infty} g_m(\alpha, \beta; \alpha_o, \beta_o)\, e^{i\nu\beta}\, d\beta \qquad (-\infty < \nu < \infty)$$

and the operational property

$$E\left[\frac{\partial^2 g_m}{\partial \beta^2}\right] = -\nu^2 \tilde{g}_m(\alpha, \nu; \alpha_o, \beta_o) \ .$$

Thus taking the Fourier transform of both sides of (2.12) with respect to β, we obtain

$$\frac{d^2 \tilde{g}_m}{d\alpha^2} + \frac{\cos\alpha}{\sin\alpha}\frac{d\tilde{g}_m}{d\alpha} - \left(\nu^2 + \frac{1}{4} + \frac{m^2}{\sin^2\alpha}\right)\tilde{g}_m = -4\pi \frac{e^{i\nu\beta_o}\sqrt{\cosh\beta_o - \cos\alpha}}{c\sin\alpha}\, \delta(\alpha - \alpha_o) \ . \tag{2.13}$$

Let

$$\cos\alpha = \xi \ , \qquad \frac{d\alpha}{d\xi} = -\frac{1}{\sin\alpha} \ , \tag{2.14}$$

then (2.13) becomes

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dg_m}{d\xi}\right] + \left[(i\nu - \frac{1}{2})(i\nu + \frac{1}{2}) - \frac{m^2}{1-\xi^2}\right]\tilde{g}_m = -\frac{4\pi}{c}\, e^{i\nu\beta_o}\sqrt{\cosh\beta_o - \xi}\ \delta(\xi - \xi_o). \tag{2.15}$$

We now proceed to solve the equation (2.15) $L\left[g_m\right] = v$ with

$$L \equiv \frac{d}{d\xi}\left[(1-\xi^2)\frac{d}{d\xi}\right] + \left[(i\nu - \frac{1}{2})(1\nu + \frac{1}{2}) - \frac{m^2}{1-\xi^2}\right]$$

(2.16)

$$v = -4\pi \frac{e^{i\nu\beta_o}}{c}(\cosh\beta_o - \xi)^{1/2}\,\delta(\xi - \xi_o) \ ,$$

with proper regularity conditions at infinity.

The two independent solutions of $L\left[\tilde{g}_m\right] = 0$ are $\tilde{g}_{m_2} = P^m_{i\nu - 1/2}(\xi)$ and $\tilde{g}_{m_1} = P^m_{i\nu - 1/2}(-\xi)$. Taking $m = 0$ for a moment we see that

$$P_{i\nu - 1/2}(\xi) = P_{i\nu - 1/2}(\cos\alpha) = {}_2F_1\left(\frac{1}{2} - i\nu,\ \frac{1}{2} + i\nu;\ \frac{\sin^2\alpha}{2}\right)$$

$$= 1 + \frac{4\nu^2 + 1}{2^2}\sin^2\alpha/2 + \frac{(4\nu^2 + 1^2)(4\nu^2 + 3^2)}{2^2 \cdot 4^2}\sin^4\alpha/2 + \ldots \tag{2.17}$$

is equal to 1 for $\alpha = 0$ ($\xi = 1$, $\sqrt{x^2 + y^2 + z^2} = \infty$).

By changing α to $\pi - \alpha$, we have

$$P_{i\nu - 1/2}(-\xi) = P_{i\nu - 1/2}(-\cos\alpha) = {}_2F_1\left(\frac{1}{2} - i\nu,\ \frac{1}{2} + i\nu; 1; \cos^2\alpha/2\right), \tag{2.18}$$

Hence $P_{i\nu - 1/2}(\xi)$ is not bounded when $\alpha = \pi (\xi = -1)$.

<u>The Wronskian</u>: $\tilde{g}_{m_1} \cdot \tilde{g}'_{m_2} - \tilde{g}_{m_2} \cdot \tilde{g}'_{m_1}$

We have (Magnus and Oberhettinger, 1949, p. 63)

$$P^\mu_\nu(\xi)\, Q^\mu_\nu(\psi) - P^{\mu\,\prime}_\nu(\xi)\, Q^\mu_\nu(\xi) = \frac{2^{2\mu}}{1 - \xi^2} \cdot \frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\nu+\mu+2}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu+2}{2}\right)} \tag{2.19}$$

$$P^\mu_\nu(-\xi) = \cos(\zeta + \mu)\pi\, P^\mu_\nu(\xi) - \frac{2}{\pi}\sin(\nu + \mu)\pi\, Q^\mu_\nu(\xi) \tag{2.20}$$

$$Q^\mu_\nu(\xi) = \frac{\pi}{2}\, \frac{1}{\sin(\nu+\mu)\pi}\left\{\cos(\nu+\mu)\pi\, P^\mu_\nu(\xi) - P^\mu_\nu(-\xi)\right\}. \tag{2.21}$$

From (2.19) - (2.21) we obtain

$$P^\mu_\nu(-\xi)P^{\mu\,\prime}_\nu(\xi) - P^\mu_\nu(\xi)P^{\mu\,\prime}_\nu(-\xi) = \frac{2}{\pi}\left[\sin(\nu+\mu)\pi\right]\, \frac{2^{2\mu}}{1 - \xi^2}\, \frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\nu+\mu+2}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu+2}{2}\right)}. \tag{2.22}$$

Using Legendre's duplication formula

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}) ,$$

replacing ν by $i\nu-\frac{1}{2}$ and μ by m (an integer), we finally obtain the Wronskian

$$W\left[\tilde{g}_{m_1}, \tilde{g}_{m_2}\right] = P^m_{i\nu-1/2}(-\xi)\frac{d}{d\xi}P^m_{i\nu-1/2}(\xi) - P^m_{i\nu-1/2}(\xi)\frac{d}{d\xi}P^m_{i\nu-1/2}(-\xi)$$

$$= \frac{1}{1-\xi^2}\, J\left[\tilde{g}_{m_1}, \tilde{g}_{m_2}\right] , \tag{2.23}$$

where J, the conjunct of $\tilde{g}_{m_1}$ and $\tilde{g}_{m_2}$, is given by

$$J\left[\tilde{g}_{m_1}, \tilde{g}_{m_2}\right] = \frac{2}{\pi}\cdot\left[\sin(i\nu+m-\tfrac{1}{2})\pi\right]\cdot\frac{\Gamma(i\nu+\frac{1}{2}+m)}{\Gamma(i\nu+\frac{1}{2}-m)} \tag{2.24}$$

<u>The Solution of (2.15)</u>: $L\left[g_m\right] = v$.

$$\tilde{g}_m = \tilde{g}_{m_2}\int^{\xi}\frac{v\tilde{g}_{m_1}\,du}{J\left[\tilde{g}_{m_1},\tilde{g}_{m_2}\right]} + g_{m_1}\int_{\xi}\frac{v\tilde{g}_{m_2}\,du}{J\left[\tilde{g}_{m_1},\tilde{g}_{m_2}\right]} \tag{2.25}$$

or

$$\tilde{g}_m(\xi,\xi_o) = \tilde{g}_{m_2}(\xi)\int_{-1}^{\xi}\frac{\tilde{g}_{m_1}(u)}{J\left[\tilde{g}_{m_1},\tilde{g}_{m_2}\right]}\,\frac{e^{i\nu\beta_o}(\cos\beta_o-u)^{1/2}}{c}\,\delta(u-\xi_o)\,du$$

$$+\,\tilde{g}_{m_1}(\xi)\int_{\xi}^{1}\frac{\tilde{g}_{m_2}(u)}{J\left[\tilde{g}_{m_1},\tilde{g}_{m_2}\right]}\,\frac{e^{i\nu\beta_o}(\cos\beta_o-u)^{1/2}}{c}\,\delta(u-\xi_o)\,du. \tag{2.26}$$

The first integral vanishes for $\xi \leqslant \xi_o$ while the second integral vanishes for $\xi \geqslant \xi_o$. Hence

$$\tilde{g}_m(\xi,\xi_o) = \frac{-4\pi\, e^{i\nu\beta_o}(\cosh\beta_o-\xi_o)^{1/2}}{c\, J\left[\tilde{g}_{m_1},\tilde{g}_{m_2}\right]_{\xi=\xi_o}} \begin{cases} \tilde{g}_{m_1}(\xi)\tilde{g}_{m_2}(\xi_o)\,, & \xi\leqslant\xi_o \\ \tilde{g}_{m_2}(\xi)\tilde{g}_{m_1}(\xi_o)\,, & \xi\geqslant\xi_o \end{cases}$$

$$= \frac{-2\pi^2}{c}\, \frac{e^{i\nu\beta_o}(\cosh\beta_o-\xi_o)^{1/2}}{\sin(i\nu+m-\frac{1}{2})\pi}\, \frac{\Gamma(i\nu+\frac{1}{2}-m)}{\Gamma(i\nu+\frac{1}{2}+m)} \begin{cases} P^m_{i\nu-1/2}(-\xi)P^m_{i\nu-1/2}(\xi_o)\,, & \xi\leqslant\xi_o \\ P^m_{i\nu-1/2}(\xi)P^m_{i\nu-1/2}(-\xi_o)\,, & \xi\geqslant\xi_o \end{cases} \tag{2.27}$$

A Representation for the Free Space Green's Function

Taking the inverse Fourier transform of $\tilde{g}_m$, from (2.27) it follows that

$$g_m(\xi,\beta;\xi_o,\beta_o) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\nu\beta}\, g_m(\xi,\nu;\xi_o,\beta_o)\, d\nu$$

$$= -\frac{\pi}{c}(\cosh\beta_o-\xi_o)^{1/2}\int_{-\infty}^{\infty} d\nu\, \frac{e^{i\nu|\beta_o-\beta|}}{\sin(i\nu+m-\frac{1}{2})\pi}\, \frac{\Gamma(i\nu+\frac{1}{2}-m)}{\Gamma(i\nu+\frac{1}{2}+m)}\,\cdot$$

$$\cdot\begin{cases} P^m_{i\nu-1/2}(-\xi)P^m_{i\nu-1/2}(\xi_o)\,, & \xi\leqslant\xi_o \\ P^m_{i\nu-1/2}(\xi)P^m_{i\nu-1/2}(-\xi_o)\,, & \xi\geqslant\xi_o \end{cases} \tag{2.28*}$$

Substituting (2.28) into (2.11) and the resulting expression into (2.8) we finally obtain the following integral representation for the inverse of the distance.

* The integral is understood in the cense of Cauchy principle value.

$$\frac{1}{R(\xi,\beta,\phi;\xi_o,\beta_o,\phi_o)} = -\frac{1}{2c}\sum_{-\infty}^{\infty} e^{im(\phi-\phi_o)}(\cosh\beta-\xi)^{1/2}(\cosh\beta_o-\xi_o)^{1/2} .$$

$$\cdot\int_{-\infty}^{\infty} d\nu \frac{e^{i\nu|\beta_o-\beta|}}{\sin(i\nu+m-\frac{1}{2})\pi}\,\frac{\Gamma(i\nu+\frac{1}{2}-m)}{\Gamma(i\nu+\frac{1}{2}+m)}\cdot\begin{cases} P^m_{i\nu-1/2}(-\xi)P^m_{i\nu-1/2}(\xi_o) , & \xi\leqslant\xi_o \\ P^m_{i\nu-1/2}(\xi)P^m_{i\nu-1/2}(-\xi_o) , & \xi\geqslant\xi_o \end{cases} \tag{2.29}$$

<u>The Green's Function with Boundary Condition</u>

To obtain the Green's function for the Dirichlet problem, we go back to the equation (2.15) and require that the condition $g_m(\alpha_1,\nu;\alpha_o,\beta_o)=0$ be satisfied on the surface of the ogive $\alpha=\alpha_1$ (const.).

Using the same notation as before, we define two functions

$$\phi_1 = \tilde{g}_{m_1}(\xi)\tilde{g}_{m_2}(\xi_1)-\tilde{g}_{m_2}(\xi)\tilde{g}_{m_1}(\xi_1)$$

$$\phi_2 = \tilde{g}_{m_2}(\xi)$$

with

$$\tilde{g}_{m_1}(\xi) = P^m_{i\nu-1/2}(-\xi), \quad \tilde{g}_{m_2}(\xi) = P^m_{i\nu-1/2}(\xi) . \tag{2.30}$$

ϕ_1, ϕ_2 are two linearly independent solutions of $L\left[\tilde{g}_m\right]=0$, since $\tilde{g}_{m_1}, \tilde{g}_{m_2}$ are, and L is a linear operator. Also, $\phi_1(\xi_1)=0$.

First, we observe that

$$\phi_1(\xi)\phi_2'(\xi) - \phi_1'(\xi)\phi_2(\xi) = g_{m_2}(\xi_1)\left[\tilde{g}_{m_1}(\xi)\tilde{g}'_{m_2}(\xi)-\tilde{g}'_{m_2}(\xi)-\tilde{g}_{m_1}(\xi)\tilde{g}_{m_2}(\xi)\right]$$

$$= \tilde{g}_{m_2}(\xi_1)W\left[\tilde{g}_{m_1},\tilde{g}_{m_2}\right] = \frac{\tilde{g}_{m_2}(\xi_1)\,J(\tilde{g}_{m_1},\tilde{g}_{m_2})}{1-\xi^2} \tag{2.31}$$

where W is the Wronskian and J is the conjunct of $\tilde{g}_{m_1}, \tilde{g}_{m_2}$.

Proceeding as before, we find that

$$g_m(\xi,\xi_o) = -4\pi \frac{e^{i\nu\beta_o}(\cosh\beta_o-\xi_o)^{1/2}}{c\tilde{g}_{m_2}(\xi_1)J(\tilde{g}_{m_1},\tilde{g}_{m_2})}\begin{cases}\phi_1(\xi)\phi_2(\xi_o)\,, & \xi\leqslant\xi_o\\ \phi_2(\xi)\phi_1(\xi_o)\,, & \xi\geqslant\xi_o\end{cases}$$

$$= -\frac{2\pi^2}{c}\cdot\frac{e^{i\nu\beta_o}(\cosh\beta_o-\xi_o)^{1/2}}{\sin(i\nu+m-\frac{1}{2})\pi}\cdot\frac{\Gamma(i\nu+\frac{1}{2}-m)}{\Gamma(i\nu+\frac{1}{2}+m)}\begin{cases}\phi_1(\xi)\phi_2(\xi_o)\,, & \xi\leqslant\xi_o\\ \phi_2(\xi)\phi_1(\xi_o)\,, & \xi\geqslant\xi_o\end{cases} \tag{2.32}$$

and that

$$G(\xi,\beta,\phi;\xi_o,\beta_o,\phi_o) = -\frac{1}{2c}\sum_{-\infty}^{\infty} e^{im(\phi-\phi_o)}(\cosh\beta-\xi)^{1/2}(\cosh\beta_o-\xi_o)^{1/2}\,\cdot$$

$$\int_{-\infty}^{\infty} d\nu\,\frac{1}{P^m_{i\nu-1/2}(\xi_1)}\,\frac{e^{i\nu|\beta_o-\beta|}}{\sin(i\nu+m-\frac{1}{2})\pi}\,\frac{\Gamma(i\nu+\frac{1}{2}-m)}{\Gamma(i\nu+\frac{1}{2}+m)}\,\cdot$$

$$\cdot\begin{cases}\left[P^m_{i\nu-1/2}(-\xi)P^m_{i\nu-1/2}(\xi_1)-P^m_{i\nu-1/2}(\xi)P^m_{i\nu-1/2}(-\xi_1)\right]P^m_{i\nu-1/2}(\xi_o)\,, & \xi\leqslant\xi_o\\ P^m_{i\nu-1/2}(\xi)\left[P^m_{i\nu-1/2}(-\xi_o)P^m_{i\nu-1/2}(\xi_1)-P^m_{i\nu-1/2}(\xi_o)P^m_{i\nu-1/2}(-\xi_1)\right]\,, & \xi\geqslant\xi_o\,.\end{cases} \tag{2.33}$$

With (2.29) and (2.33) we also have the solution to the exterior Dirichlet potential problem:

$$\nabla^2 u = 0$$

$$u_{boundary} = -\frac{1}{R}\,,$$

and u is regular at infinity (yet to be shown).

2.4 A Verification of the Result

In this section we shall use a different procedure for obtaining the bispherical coordinates, which is particularly suitable in treating the potential problems by considering a single (and/or double) charge layer on the boundary of the body. This will also serve as a check of the results of section 2.3.

Let (x, r, ϕ) be the cylindrical coordinates. Then the equation to be solved is

$$\nabla^2 G(x, r, \phi; x_o, r_o, \phi_o) = \frac{-4\pi}{r}\,\delta(x-x_o)\delta(r-r_o)\delta(\phi-\phi_o) \tag{2.34}$$

(with boundary conditions to be imposed later).

Assuming the periodicity of G in ϕ, with period 2π, we may expand it in a Fourier series

$$G(x, r, \phi; x_o, r_o, \phi_o) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} G_m(x, r, x_o, r_o)\, e^{im(\phi-\phi_o)} \tag{2.35}$$

Substituting into (2.34) we obtain

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - (m/r)^2\right] G_m(x, r; x_o, r_o) = \frac{-4\pi}{r}\,\delta(x-x_o)\delta(r-r_o)$$

Let

$$G_m = \frac{1}{\sqrt{r}}\, g_m\ . \tag{2.36}$$

Then g_m satisfies the differential equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{\frac{1}{4} - m^2}{r^2}\right] g_m(x, r; x_o, r_o) = \frac{-4\pi}{\sqrt{r}}\,\delta(x-x_o)\delta(r-r_o)\ . \tag{2.37}$$

Bipolar Coordinates:

In general, for the map

$$z = f(w) = x(\alpha, \beta) + ir(\alpha, \beta) \tag{2.38}$$

where f is analytic in its domain of definition, we have

$$\left|\frac{dz}{dw}\right|^2 = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial x}{\partial \beta}\right)^2 = \left(\frac{\partial r}{\partial \alpha}\right)^2 + \left(\frac{\partial r}{\partial \beta}\right)^2 \tag{2.39}$$

and

$$\frac{\partial x}{\partial \alpha} = \frac{\partial r}{\partial \beta}, \qquad \frac{\partial x}{\partial \beta} = \frac{\partial r}{\partial \alpha}. \tag{2.40}$$

Using (2.39) and (2.40) we obtain

$$\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}\right) g(\alpha, \beta) = \left|\frac{dz}{dw}\right|^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2}\right) g(x, r) .$$

In particular, if we consider the map

$$z = ic \cot \omega/2 = c \, \frac{1+e^{i\omega}}{1-e^{i\omega}} \tag{2.41}$$

with

$$z = x + ir , \qquad w = \alpha + i\beta ,$$

we obtain

$$x = \frac{c \sinh\beta}{\cosh\beta - \cos\alpha} , \qquad r = \frac{c \sin\alpha}{\cosh\beta - \cos\alpha} \tag{2.42}$$

and

$$\left|\frac{dz}{dw}\right|^2 = \frac{c^2}{(\cosh\beta - \cos\alpha)^2} . \tag{2.43}$$

The range and the domain of definition of this map is

$$\begin{aligned} -\infty < x < \infty , \qquad & 0 < r < \infty \\ 0 < \alpha < \pi , \qquad & -\infty < \beta < \infty \end{aligned} \tag{2.44}$$

The z-half plane is represented on the w-strip as shown in Fig. 2. Under this map the equation (2.37) transforms to

$$\left[\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} + \frac{\frac{1}{4} - m^2}{\sin^2\alpha}\right] g_m(\alpha, \beta; \alpha_o, \beta_o) = \left|\frac{dz}{dw}\right|^2 \left|\frac{\partial(x, r)}{\partial(\alpha, \beta)}\right|^{-1} \frac{-4\pi}{\sqrt{r}} \cdot \delta(\alpha - \alpha_o)\delta(\beta - \beta_o)$$

$$= \frac{-4\pi}{\sqrt{c}} \cdot \frac{(\cosh\beta - \cos\alpha)^{1/2}}{(\sin\alpha)^{1/2}} \cdot \delta(\alpha - \alpha_o)\delta(\beta - \beta_o) \tag{2.45}$$

with

$$x_o = \frac{c\sinh\beta_o}{\cosh\beta_o - \cos\alpha_o}, \quad r_o = \frac{c\sin\alpha_o}{\cosh\beta_o - \cos\alpha_o} .$$

We now define the bispherical coordinates (α, β, ϕ) by rotating the bipolar coordinates (α, β) around the x-axis. As in section 2.2 the ogive is the body of revolution obtained by rotating the coordinate surface $\alpha = \alpha_1$(const.) around the x-axis.

By proceeding as before, we solve the equation (2.45) and arrive at the same representations for $1/R$ and G of the previous section. Details are omitted.

2.5 The Regularity, the Existence and the Uniqueness

In bispherical coordinates

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\frac{\cosh\beta + \cos\alpha}{\cosh\beta - \cos\alpha}} . \tag{2.46}$$

Since α and β are real, $r \to \infty$ is equivalent to $\alpha \to 0$ and $\beta \to 0$. That $rG < \infty$ as $r \to \infty$ is clear by inspection of (2.33).

$$\frac{\partial G}{\partial r} = \frac{\partial G}{\partial\alpha} \cdot \frac{\partial\alpha}{\partial r} + \frac{\partial G}{\partial\beta} \cdot \frac{\partial\beta}{\partial r} \tag{2.47}$$

$$\frac{\partial\alpha}{\partial r} = - \frac{\sqrt{\cosh\beta + \cos\alpha}}{c\cosh\beta} \cdot \frac{(\cosh\beta - \cos\alpha)^{3/2}}{\sin\alpha} \tag{2.48a}$$

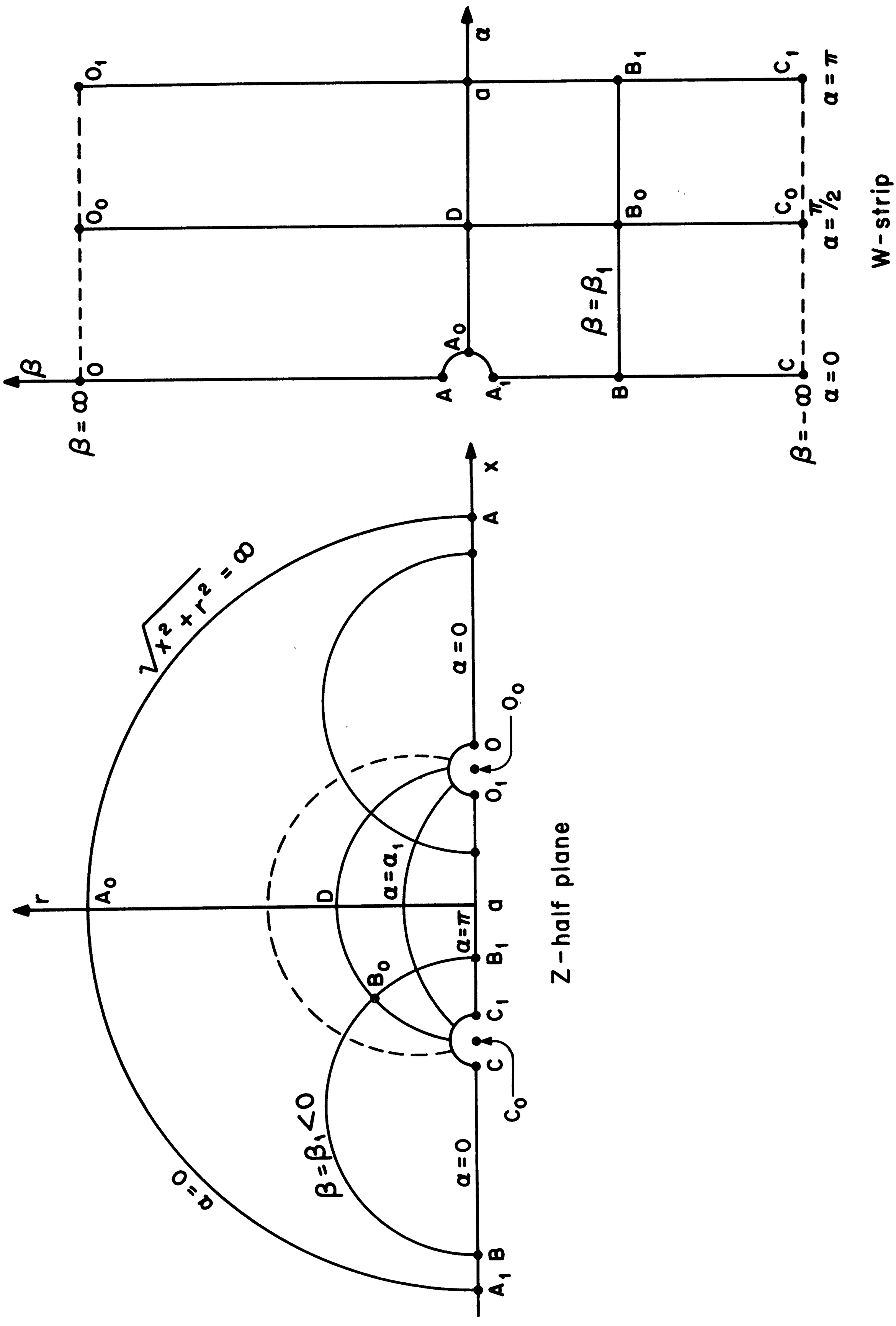

FIGURE 2

$$\frac{\partial\beta}{\partial r} = - \frac{\sqrt{\cosh\beta + \cos\alpha}}{c\cos\alpha} \cdot \frac{(\cosh\beta - \cos\alpha)^{3/2}}{\sinh\beta} \tag{2.48b}$$

$$\frac{\partial G}{\partial\alpha} = O\left(\frac{\sin\alpha}{\sqrt{\cosh\beta - \cos\alpha}}\right) \qquad \text{as } r \to \infty \tag{2.49a}$$

$$\frac{\partial G}{\partial\beta} = O\left(\frac{\sinh\beta}{\sqrt{\cosh\beta - \cos\alpha}}\right) \qquad \text{as } r \to \infty \tag{2.49b}$$

Substituting (2.48) and (2.49) into (2.47),

$$\frac{\partial G}{\partial r} = O(\cosh\beta - \cos\alpha) \qquad \text{as } r \to \infty$$

and hence

$$r^2 \frac{\partial G}{\partial r} = O(1) \qquad \text{as } r \to \infty .$$

Note that $r^{2+\epsilon} \cdot \frac{\partial G}{\partial r} \longrightarrow \infty$ as $r \longrightarrow \infty$ for an arbitrary $\epsilon > 0$. So the static Green's function (2.33) is regular at infinity (in the sense of Kellog).

The existence question does not arise in our particular problem, since we have actually constructed the Green's function, and have justified every step in the process (either by providing or by indicating the proofs or referring to the proper sources).

The uniqueness, of course, follows from the fact that the solution to the exterior potential problem

(1) $\nabla^2\psi = 0$

(2) ψ regular at infinity

is $\psi \equiv 0$.

2.6 Summary

In bispherical coordinates (α, β, ϕ) we have constructed the (unique) static Green's function of the first kind, i.e., a function such that

(a) $\nabla^2 G(\alpha, \beta, \phi; \alpha_o, \beta_o, \phi_o) = -4\pi \dfrac{(\cosh\beta - \cos\alpha)^3}{c^3 \sin\alpha} \cdot \delta(\alpha-\alpha_o)\delta(\beta-\beta_o)\delta(\phi-\phi_o)$

(b) $G(\alpha_1, \beta, \phi; \alpha_o, \beta_o, \phi_o) = 0$

(c) G is regular at infinity in the sense of Kellog.

Next we shall find the series representation for (2.33) because that form will be more suitable for generating the Green's function for the Helmholtz equation for the ogive.

III
THE SERIES FORMS

3.1 Series Representation for 1/R (Residue Series)

We have

$$\frac{1}{R} = -\frac{1}{2c}\sum_{-\infty}^{\infty} e^{im(\phi-\phi_o)}(\cosh\beta-\xi)^{1/2}(\cosh\beta_o-\xi_o)^{1/2}\,.$$

$$\int_{-\infty}^{\infty}\frac{e^{i\nu|\beta-\beta_o|}}{\sin(i\nu+m-\frac{1}{2})\pi}\,\frac{\Gamma(i\nu+\frac{1}{2}-m)}{\Gamma(i\nu+\frac{1}{2}+m)}\left[P^m_{i\nu-1/2}(-\xi)P^m_{i\nu-1/2}(\xi_o)\right], \qquad \xi\leqslant\xi_o \tag{3.1}$$

With the substitution $s = i\nu - 1/2$, this becomes

$$\frac{1}{R} = \frac{i}{2c}(\cosh\beta-\cos\alpha)^{1/2}(\cosh\beta_o-\cosh\alpha_o)^{1/2}\int_{-i\infty-1/2}^{i\infty-1/2} ds\,\frac{e^{(s+\frac{1}{2})|\beta-\beta_o|}}{\sin s\pi}\,.$$

$$\cdot\sum_{m=0}^{\infty}\epsilon_m\cos m(\phi-\phi_o)\cdot(-1)^m\frac{\Gamma(s+1-m)}{\Gamma(s+1+m)}\cdot P^m_s(-\cos\alpha)P^m_s(\cos\alpha_o)$$

$$\alpha\geqslant\alpha_o \tag{3.2}$$

where $\epsilon_o = 1$, $\epsilon_m = 2$ for $m = 1, 2, 3, \ldots$.

This, in turn, reduces by means of the addition theorem to

$$\frac{1}{R} = \frac{i}{2c}(\cosh\beta-\cos\alpha)^{1/2}(\cosh\beta_o-\cos\alpha_o)^{1/2}\int_{-i\infty-1/2}^{i\infty-1/2} ds\,\frac{e^{(s+\frac{1}{2})|\beta-\beta_o|}}{\sin s\pi}\,P_s(\cos\Theta) \tag{3.3}$$

where $\cos\Theta = -\cos\alpha\cos\alpha_o - \sin\alpha\sin\alpha_o\cos(\phi-\phi_o)$.

The only poles of the integrand are at the zeros of $\sin s\pi$; hence, they are simple and located at $s = 0, \mp 1, \mp 2, \ldots$. Furthermore, since

$$P_s(\cos\theta) \sim \sqrt{\frac{2}{s\pi \sin\theta}} \sin\left\{(s+\frac{1}{2})\theta + \frac{\pi}{4}\right\} \tag{3.4}$$

for large $|s|$ we see that the integrand vanishes if (and only if) $\mathrm{Re}\, s < -1/2$ as $|s| \to \infty$. This condition determines which way the contour is to be closed (Fig. 3). Since ∞ is the only limit point for the poles and the spacing between the poles remains uniform, no special analysis is required in "threading" the poles for large $|s|$ and we proceed in the usual way to obtain the residue series.

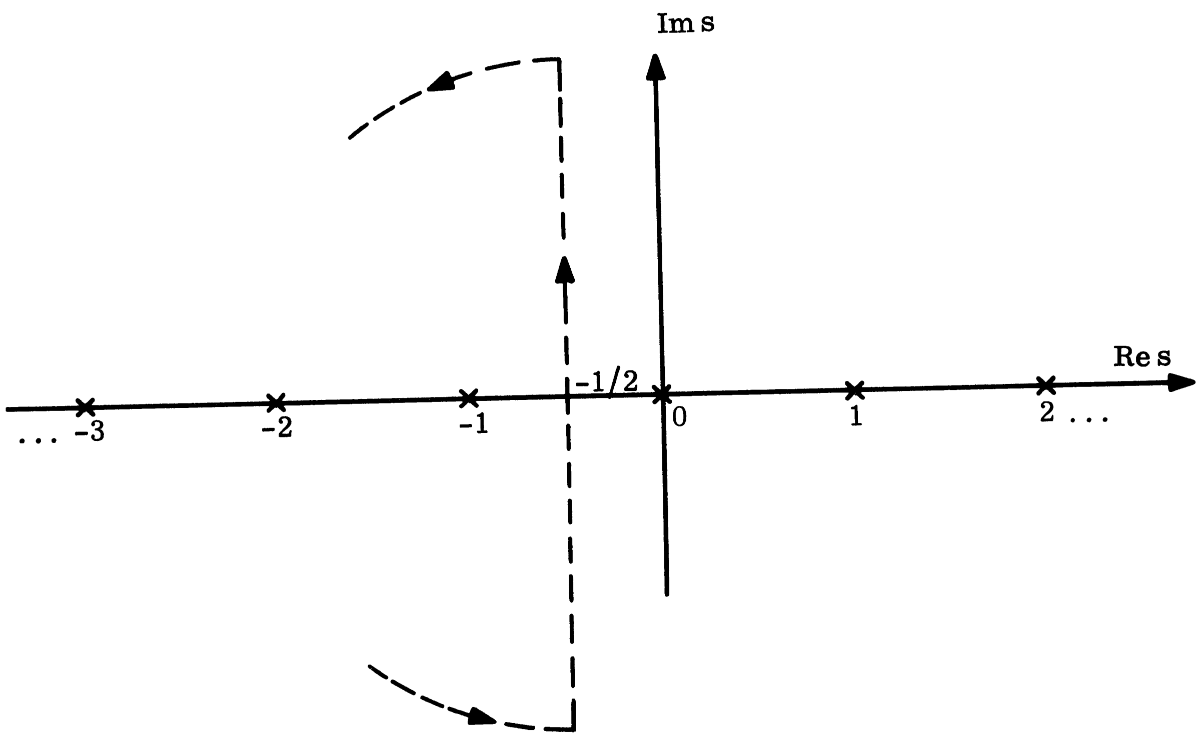

FIG. 3: THE CONTOUR AND THE POLES

$$\frac{1}{R} = -\frac{\pi}{c}(\cosh\beta - \cos\alpha)^{1/2}(\cosh\beta_o - \cos\alpha_o)^{1/2}\sum_{n=1}^{\infty} e^{(s_n+\frac{1}{2})|\beta-\beta_o|} P_{s_n}(\cos\Theta)\cdot$$

$$\cdot\ \frac{1}{\pi\cos s_n\pi} \tag{3.5}$$

where $s_n = -n$, $n = 1, 2, 3, \ldots$.

Replacing n by $n+1$ and observing that $P_{-n-1}(\cos\theta) = P_n(\cos\theta)$, (3.5) becomes

$$\frac{1}{R} = \frac{1}{c}(\cosh\beta - \cos\alpha)^{1/2}(\cosh\beta_o - \cos\alpha_o)^{1/2}\sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\ \beta-\beta_o} P_n(\cos\gamma), \tag{3.6}$$

where $\cos\gamma = -\cos\Theta = \cos\alpha\cos\alpha_o + \sin\alpha\sin\alpha_o\cos(\phi-\phi_o)$.

3.2 Series for 1/R (Directly)

We now use a well known procedure to find the series representations for $1/R$ directly, which will also serve as a check

In bispherical coordinates, as we noted earlier, with the substitution $G = \sqrt{\cosh\beta\cos\alpha}\ g$, $\nabla^2 G = 0$ yields the following equation for g,

$$\frac{\partial^2 g}{\partial\beta^2} + \frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\frac{\partial g}{\partial\alpha}\right) + \frac{1}{\sin^2\alpha}\frac{\partial g}{\partial\phi^2} - \frac{1}{4}g = 0, \tag{3.7}$$

which, by $g = A(\alpha)B(\beta)\Phi(\phi)$, separates into the three equations

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad 0 \leqslant \phi \leqslant 2\pi \tag{3.8}$$

$$\frac{d^2B}{d\beta^2} = (n+\frac{1}{2})B, \qquad -\infty < \beta < \infty \tag{3.9}$$

$$\frac{1}{\sin\alpha}\frac{d}{d\alpha}\left(\sin\alpha\frac{dA}{d\alpha}\right)-\frac{m^2A}{\sin^2\alpha}=-n(n+1)A, \qquad 0\leqslant\alpha\leqslant\pi \tag{3.10}$$

The assumptions of continuity in ϕ and the boundedness of A at $\alpha=0$, $\alpha=\pi$ restrict m to be zero or a positive integer and n to be an integer greater than or equal to m.

We want to solve the nonhomogeneous equation (3.7) with right side

$$-4\pi\frac{\sqrt{\cosh\beta-\cos\alpha}}{c\sin\alpha}\delta(\alpha-\alpha_o)\delta(\beta-\beta_o)\delta(\phi-\phi_o)\ . \tag{3.11}$$

We have the well known complete set of (normalized) eigenfunctions for our problem in α and ϕ, namely,

$$\psi_{nm}(\alpha,\phi)=\sqrt{\epsilon_m\cdot\frac{2n+1}{4\pi}\cdot\frac{(n-m)!}{(n+m)!}}\,P_n^m(\cos\alpha)\,{\cos \atop \sin}(m\phi) \tag{3.12}$$

with $\epsilon_o=1$, $\epsilon_m=2$ for $m=1,2,\ldots$. Substituting the expansion of g

$$g(\alpha,\beta,\phi;\alpha_o,\beta_o,\phi_o)=\sum_{n,m}B(\beta,\beta_o)A_{nm}(\alpha_o,\phi_o)\cdot\psi_{nm}(\alpha,\phi) \tag{3.13}$$

where $B(\beta,\beta_o)$, $A_{nm}(\alpha_o,\phi_o)$ are to be determined, into the differential equation for g and making use of the fact that the surface harmonics ψ_{nm} satisfy the differential equation

$$\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\sin\alpha\frac{\partial\psi_{nm}}{\partial\alpha}+\frac{1}{\sin^2\alpha}\frac{\partial^2\psi_{nm}}{\partial\phi^2}+n(n+1)\psi_{nm}=0 \tag{3.14}$$

we arrive at

$$\sum_{n,m}A_{nm}(\alpha_o,\phi_o)\psi_{nm}(\alpha,\phi)\left[\frac{d^2}{d\beta^2}-(n+\tfrac{1}{2})^2\right]B(\beta)= \tag{3.15}$$

$$-\frac{4\pi}{c}\frac{\sqrt{\cosh\beta-\cos\alpha}}{\sin\alpha}\delta(\alpha-\alpha_o)\delta(\beta-\beta_o)\delta(\phi-\phi_o)$$

Since

$$\int_\Omega d\Omega\, \psi_{mn} \psi_{n'm'} = \delta_{nmn'm'} ,$$

where

$$\int_\Omega d\Omega = \int_0^{2\Omega} d\phi \int_0^\pi \sin\alpha\, d\alpha ,$$

we multiply (3.15) by $\sin\alpha\psi_{nm}(\alpha,\phi)$ and integrate over α, ϕ to obtain

$$A_{nm}(\alpha_o,\phi_o)\left[\frac{d^2}{d\beta^2} - (n+\frac{1}{2})^2\right] B = \frac{4\pi}{c} \sqrt{\cosh\beta - \cos\alpha_o}\ \delta(\beta-\beta_o)\psi_{nm}(\alpha_o,\phi_o) . \tag{3.16}$$

Taking

$$A_{mn}(\alpha_o,\phi_o) = \psi_{nm}(\alpha_o,\phi_o) \tag{3.17}$$

we are left with the differential equation

$$L\left[B\right] = \left[\frac{d^2}{d\beta^2} - (n+\frac{1}{2})^2\right] B = \frac{4\pi}{c} \sqrt{\cosh\beta - \cos\alpha_o}\ \delta(\beta-\beta_o) . \tag{3.18}$$

The independent solutions of $L\left[B\right] = 0$ are

$$y_1 = e^{-(n+\frac{1}{2})\beta} , \qquad y_2 = e^{(n+\frac{1}{2})\beta}$$

with the Wronskian $W(y_1, y_2) = 2n+1$. Hence, the solution of (3.18) is

$$B(\beta, \beta_o) = \frac{4\pi}{c} \frac{\sqrt{\cosh\beta_o - \cos\alpha_o}}{2n+1} \begin{cases} e^{-(n+\frac{1}{2})(\beta-\beta_o)}, & \beta \geqslant \beta_o \\ e^{-(n+\frac{1}{2})(\beta_o-\beta)}, & \beta \leqslant \beta_o \end{cases} . \quad (3.19)$$

Substituting (3.17) and (3.19) into (3.13) and observing that $G = \sqrt{\cosh\beta - \cos\alpha}\ g$, we arrive at the following series representation for the free space static Green's function in bispherical coordinates

$$\frac{1}{R(\alpha, \beta, \phi, \alpha_o, \beta_o, \phi_o)} = \frac{1}{c}\sqrt{(\cosh\beta - \cos\alpha)(\cosh\beta_o - \cos\alpha_o)}\ \cdot$$

$$\cdot \sum_{n=0}^{\infty}\sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} \cos\left[m(\phi-\phi_o)\right] P_n^m(\cos\alpha)P_n^m(\cos\alpha_o)\ \cdot$$

$$\cdot \begin{cases} e^{-(n+\frac{1}{2})(\beta-\beta_o)}, & \beta \geqslant \beta_o \\ e^{-(n+\frac{1}{2})(\beta_o-\beta)}, & \beta \leqslant \beta_o \end{cases} , \qquad \begin{array}{l} \epsilon_o = 1 \\ \epsilon_m = 2 \text{ for } m = 1, 2, 3 \end{array} \quad (3.20)$$

If we let

$$\cos\gamma = \cos\alpha\cos\alpha_o + \sin\alpha\sin\alpha_o\cos(\phi - \phi_o) \quad (3.21)$$

then by the addition theorem

$$P_n(\cos\gamma) = P_n(\cos\alpha)P_n(\cos\alpha_o) + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!}\ \cdot$$

$$\cdot\ P_n^m(\cos\alpha)P_n^m(\cos\alpha_o)\cos m(\phi - \phi_o) \quad (3.22)$$

The above result for $1/R$ is expressed in the following more compact form

$$\frac{1}{R} = \frac{1}{c}\sqrt{(\cosh\beta - \cos\alpha)(\cosh\beta_o - \cos\alpha_o)}\sum_{n=0}^{\infty} P_n(\cos\gamma)\cdot e^{-(n+\frac{1}{2})|\beta-\beta_o|} . \tag{3.23}$$

which is equal to (3.6) as it should be.

3.3 The Series Representation for the Green's Function of the First Kind

With the substition $s = i\nu - 1/2$, our previously obtained integral representation for the Dirichlet Green's function is written as

$$G = \frac{i}{2c}(\cosh\beta - \cos\alpha)^{1/2}(\cosh\beta_o - \cos\alpha_o)^{1/2}\int_{-i\infty-1/2}^{i\infty-1/2} ds\, \frac{e^{(s+\frac{1}{2})|\beta-\beta_o|}}{\sin s\pi} \cdot$$

$$\cdot \sum_{m=0}^{\infty} \epsilon_m (-1)^m \cos m(\phi - \phi_o)\,\frac{\Gamma(s+1-m)}{\Gamma(s+1+m)} \cdot$$

$$\cdot \left\{ P_s^m(-\cos\alpha)P_s^m(\cos\alpha_o) + \frac{P_s^m(-\cos\alpha_1)}{P_s^m(\cos\alpha_1)} P_s^m(\cos\alpha)P_s^m(\cos\alpha_o)\right\},$$

$$\alpha_1 > \alpha \geqslant \alpha_o . \tag{3.24}$$

We now investigate the location and the nature of the poles of the integrand of (3.24).

<u>Observations</u>:

(1) First we note that since the behavior of $P_s^m(-\cos\alpha)P_s^m(\cos\alpha_o)$ is the same as

$$\frac{P_s^m(-\cos\alpha_1)}{P_s^m(\cos\alpha_1)} \cdot P_s^m(\cos\alpha)P_s^m(\cos\alpha_o)$$

for large $|s|$, the contour to be used for G is the same as that for $1/R$. Therefore, we only need to study the integrand in the half plane $\mathrm{Re}\, s < -1/2$.

(2) $\dfrac{1}{\sin s\pi} \dfrac{\Gamma(s+1-m)}{\Gamma(s+1+m)}$ has m poles of order two at $s=-1, -2, \ldots, -m$ and simple poles at $s=-m-n$, $n=1, 2, 3, \ldots$.

(3) In the s-plane $P_s^m(\cos\alpha)$ is an entire function. Zeros of P_s^m in the s plane are all real and distinct.*

(4) In the s-plane, the zeros of $P_s^m(\cos\alpha)$, $P_s^m(\cos\beta)$ are different for $\alpha \neq \beta$; the zeros of $P_s^m(-\cos\alpha)$, $P_s^m(\cos\alpha)$ are different.

(5) We recall that in constructing the Green's function in the variable α over the range $\alpha_1 > \alpha > \alpha_o$ we used the proper combination of two functions:

$$\phi_1 = P_s^m(-\xi)P_s^m(\xi_1) - P_s^m(\xi)P_s^m(-\xi_1)$$

$$\phi_2 = P_s^m(\xi) \ .$$

If s is an integer, then, since

$$P_s^m(\xi) = (-1)^m P_s^m(-\xi), \qquad P_s^m(\xi_1) = (-1)^m P_s^m(-\xi_1)$$

the total Green's function $G = 0$ for all α in $\alpha_1 > \alpha > 0$. This means that the factor

* H. M. Macdonald (1900) showed that for μ real and $\mu > 0$ $P_s^\mu(\cos\alpha)$ can have no complex zeros; all its zeros are real. He also showed that $P_s^\mu(\cos\alpha)$, $\mu > 0$, has an infinite number of distinct real zeros, and, in addition, at most 2k complex zeros, where k is the greatest integer contained in μ. In our case $m = 0, 1, 2 \ldots$ and we can exclude the possibility of complex zeros as can be seen from the relation

$$P_s^{-m}(\cos\alpha) = \frac{\Gamma(s-m+1)}{\Gamma(s+m+1)} (-1)^m P_s^m(\cos\alpha)$$

for integer m.

$\frac{1}{\sin s\pi}\,\frac{\Gamma(s+1-m)}{\Gamma(s+1+m)}$ has in fact no residue contribution to the integral and the only contribution is from the simple poles of $1/P_s^m(\cos\alpha_1)$ at the non-integer real values of s in the region $\mathrm{Re}\, s<-1/2$.

(6) Let $s=s_j$ denote the non-integer real solutions of the equation $P_s^m(\cos\alpha_1)=0$ in the region $\mathrm{Re}\, s<-1/2$ of the s-plane.

We can now by means of the observations (1) through (6) write down the final residue series for G

$$G=\frac{\pi}{c}(\cosh\beta-\cos\alpha)^{1/2}(\cosh\beta_o-\cos\alpha_o)^{1/2}\left\{\sum_{m=0}^{\infty}\epsilon_m(-1)^m\cos m(\phi-\phi_o)\,\cdot\right.$$

$$\left.\cdot\sum_j\frac{e^{(s_j+\frac{1}{2})|\beta-\beta_o|}}{\sin s_j\pi}\,\frac{\Gamma(s_j+1-m)}{\Gamma(s_j+1+m)}\,\frac{P_{s_j}^m(-\cos\alpha_1)}{\frac{d}{ds}\left\{P_{s_j}(\cos\alpha_1)\right\}}\,P_{s_j}^m(\cos\alpha)P_{s_j}^m(\cos\alpha_o)\right\}$$

$$\alpha_1>\alpha>\alpha_o \qquad (3.25)$$

IV
THE GREEN'S FUNCTION OF THE FIRST KIND FOR THE HELMHOLTZ EQUATION

4.1 The Objective and the Preliminaries

We are seeking the Green's function for the surface B of the ogive $\alpha = \alpha_1$ satisfying

(a) $(\nabla^2 + k^2) G_k(p, p_o) = -4\pi \cdot \delta\left[R(p, p_o)\right], \qquad p, p_o \in V$

(b) $G_k(p_B, p_o) = 0$, (4.1)

(c) the radiation condition, $\lim_{r \to \infty} \left| r \left(\frac{\partial G_k}{\partial r} - ikG_k \right) \right| = 0$, uniformly in all directions, where V denotes the volume exterior to the ogive surface B, $p(\alpha, \beta, \phi)$ the field point, $p_o(\alpha_o, \beta_o, \phi_o)$ the surface point, $p_B(\alpha_1, \beta, \phi)$ a point on the surface of the ogive $\alpha = \alpha_1$.

$R(p, p_o)$ is the distance between p and p_o, and $\delta\left[R(p, p_o)\right]$ has been given explicitly in bispherical coordinates in the static case. We recall $r = \sqrt{\frac{\cosh\beta + \cos\alpha}{\cosh\beta - \cos\alpha}}$ and $r \to \infty \Longleftrightarrow \alpha \to 0, \ \beta \to 0$. With our choice of δ-function and the radiation condition, the free space Green's function is e^{ikR}/R, and the decomposition of G_k into singular and regular parts is

$$G_k(p, p_o) = \frac{e^{ikR(p, p_o)}}{R(p, p_o)} + U_k(p, p_o) , \tag{4.2}$$

and

$$(\nabla^2 + k^2) U_k = 0 \quad . \tag{4.3}$$

U_k has no singularities in the closure $\bar{V}$, is twice differentiable, and satisfies the radiation condition. Also we note that if the Helmholtz equation is considered as reduced from the wave equation, our case corresponds to assuming the harmonic time dependence $e^{-i\omega t}$.

We now restate the representation theorem (Kleinman, 1965), which is another form of the Green's theorem, and the expansion theorem (Atkinson, 1949; Barrar and Kay; Wilcox, 1956) as applied to the ogive.

Theorem 1

If

(a) $w(p)$ is defined for all $p \in \bar{V}$

(b) $w \in C^2(V)$

(c) $|rw| < \infty$, $\left| r^2 \frac{\partial w}{\partial r} \right| < \infty$ as $r \to \infty$,

then $w(p)$ satisfied the integral equation

$$w(p) = \int_V G_o(p, p') \nabla^2 w(p') \, dv' + \int_B w(p_B) \frac{\partial}{\partial n} G_o(p, p_B) d\sigma_B \tag{4.4}$$

where the volume element is given by

$$dv = dx\,dy\,dz = \frac{\partial(x, y, z)}{\partial(\alpha, \beta, \phi)} d\alpha\, d\beta\, d\phi = \frac{C^3 \sin\alpha}{(\cosh\beta - \cos\alpha)^3} d\alpha\, d\beta\, d\phi \, ;$$

the surface element (for the surface $\alpha = \alpha_1$) by

$$d\sigma = \frac{C^2 \sin\alpha_1}{(\cosh\beta - \cos\alpha_1)^2} d\beta\, d\phi$$

the Laplacian ∇^2 in bispherical coordinates (α, β, ϕ) is as given in the static case; $\partial/\partial n$ is the normal derivative (in the direction out of V) and is given by

$$\frac{\partial}{\partial n} = -\frac{1}{h_\alpha} \frac{\partial}{\partial \alpha} = -\frac{1}{c} (\cosh\beta - \cos\alpha) \frac{\partial}{\partial \alpha} \, .$$

Theorem 2

The field scattered from the surface of the ogive may be written as

$$U_k(p, p_o) = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} f_n/r^n \qquad (4.5)$$

where the series converges absolutely and uniformly for $r \geqslant d+\epsilon$, $\epsilon > 0$. d is the radius of the sphere obtained by rotating $\alpha = \pi/2$ about the x-axis. The series may be differentiated term by term with respect to α, β or ϕ any number of times and the resulting series all converge absolutely and uniformly. The functions f_n depend on $\alpha_o, \beta_o, \phi_o$, and the parameter k.

Following Kleinman (1965), we want to represent the regular part U_k of the Green's function using Theorem 1. U_k is not regular at infinity as can be seen from Theorem 2. There is more than one way of making U_k regular.

Although the obvious form $e^{-ikr} U_k$, indicated by Theorem 2, is regular, in a particular problem such as ours the choice must be made more judicilusly to simplify the resulting equations and to enable us to carry out the integrations arising in connection with the iteration. Thus we define

$$\tilde{U} = e^{-ikf(\alpha, \beta)} U_k \qquad (4.6)$$

and call f the "eikonal" because of the apparent analogy to the corresponding entity in physical optics (e.g. Born and Wolf, 1959).

4.2 The Proper Choice of the Eikonal $f(\alpha, \beta)$

We have

$$\tilde{U} = e^{-ikf} U_k \qquad (4.6)$$

and

$$(\nabla^2 + k^2) e^{ikf} \tilde{U} = 0 . \qquad (4.8)$$

Therefore,

$$\nabla^2(e^{ikf}\tilde{U})+k^2 e^{ikf}\tilde{U} = \nabla\circ(ik e^{ikf}\tilde{U}\nabla f+e^{ikf}\nabla\tilde{U})+k^2 e^{ikf}\tilde{U}$$

$$= e^{ikf}\left[1-(\nabla f\circ\nabla f)\right]k^2\tilde{U}+e^{ikf}\nabla^2\tilde{U}$$

$$+ 2ik e^{ikf}\nabla\tilde{U}\circ\nabla f+ik e^{ikf}\tilde{U}\nabla^2 f = 0$$

or

$$\nabla^2\tilde{U}+2ik\nabla\tilde{U}\circ\nabla f+ik\tilde{U}\nabla^2 f+\left[1-(\nabla f\circ\nabla f)\right]k^2\tilde{U} = 0 \; . \tag{4.9}$$

We see from (4.9) that the first natural simplification is achieved by setting

$$\nabla f\circ\nabla f = 1 \; . \tag{4.10}$$

This is the "eikonal equation" for f.

Solution of $\nabla f \circ \nabla f = 1$

In bispherical coordinates

$$\nabla \equiv \frac{1}{c}(\cosh\beta-\cos\alpha)\left\{\hat{e}_\alpha\frac{\partial}{\partial\alpha}+\hat{e}_\beta\frac{\partial}{\partial\beta}+\frac{\hat{e}_\phi}{\sin\alpha}\frac{\partial}{\partial\phi}\right\} \; . \tag{4.11}$$

With (4.11), (4.10) becomes

$$\left(\frac{\partial f}{\partial\alpha}\right)^2+\left(\frac{\partial f}{\partial\beta}\right)^2 = \frac{c^2}{(\cosh\beta-\cos\alpha)^2} \; .$$

We note that

$$\cosh\beta-\cos\alpha = 2\sin\left(\frac{\alpha+i\beta}{2}\right)\cdot\sin\left(\frac{\alpha-i\beta}{2}\right) = 2\sinh\left(\frac{\beta+i\alpha}{2}\right)\sinh\left(\frac{\beta-i\alpha}{2}\right) \; .$$

Let

$$z = \beta+i\alpha, \qquad \bar{z} = \beta-i\alpha$$

then

$$\frac{\partial f}{\partial \alpha} = i\frac{\partial f}{\partial z} - i\frac{\partial f}{\partial \bar{z}}$$

$$\frac{\partial f}{\partial \beta} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}$$

$$\therefore \left(\frac{\partial f}{\partial \alpha}\right)^2 + \left(\frac{\partial f}{\partial \beta}\right)^2 = 4\frac{\partial f}{\partial z}\cdot\frac{\partial f}{\partial \bar{z}} \quad .$$

The equation to be solved is

$$4\,\frac{\partial f}{\partial z}\cdot\frac{\partial f}{\partial \bar{z}} = \frac{c^2}{4\sinh^2(z/2)\sinh^2(\bar{z}/2)}$$

or

$$2\,\frac{\partial f}{\partial z}\cdot 2\,\frac{\partial f}{\partial \bar{z}} = \frac{c}{2\sinh^2(z/2)}\cdot\frac{c}{2\sinh^2(\bar{z}/2)} \tag{4.12}$$

Assume $f = g(z) + g(\bar{z})$, then

$$2\,\frac{\partial g(z)}{\partial z} = \frac{c}{2\sinh^2 z/2} = -c\,\frac{\partial}{\partial z}\cdot\coth(z/2)$$

$$2\,\frac{\partial g(\bar{z})}{\partial \bar{z}} = \frac{c}{2\sinh^2 \bar{z}/2} = -c\,\frac{\partial}{\partial \bar{z}}\coth(\bar{z}/2)$$

$$\therefore\ g(z) = -\frac{c}{2}\coth(z/2)$$

$$g(\bar{z}) = -\frac{c}{2}\coth(\bar{z}/2)$$

$$f = \pm\frac{c}{2}\left\{\coth\left(\frac{\beta+i\alpha}{2}\right) + \coth\left(\frac{\beta-i\alpha}{2}\right)\right\}, \qquad c > 0 \tag{4.13}$$

(+ sign to be chosen in our case).

From $r = \sqrt{\frac{\cosh\beta + \cos\alpha}{\cosh\beta - \cos\alpha}}$ and (4.13) it is immediately seen that

$$f = O(r) \quad \text{as} \quad r \to \infty \ (\alpha \to 0, \ \beta \to 0). \tag{4.14}$$

From Theorem 2,

$$r\,U_k = e^{ikr} \sum_{n=0}^{\infty} f_n / r^n \ ;$$

and with $\tilde{U} = e^{-ikf} U_k$,

$$r\,\tilde{U} = e^{ik(r-f)} \sum_{n=0}^{\infty} f_n / r^n \ ,$$

so that

$$\left| r\tilde{U} \right| < \infty \quad \text{as} \quad r \to \infty \ . \tag{4.15}$$

Also

$$r^2 \frac{\partial \tilde{U}}{\partial r} = r^2 \frac{\partial}{\partial r} \left\{ e^{-ikf} U_k \right\} = r^2 \frac{\partial}{\partial r} \left\{ \frac{e^{ik(r-f)}}{r} \sum_{n=0}^{\infty} f_n / r^n \right\}$$

$$\sim r^2 \frac{\partial}{\partial r} (1/r) \quad \text{as} \quad r \to \infty;$$

therefore

$$\left| r^2 \frac{\partial \tilde{U}}{\partial r} \right| < \infty \quad \text{as} \quad r \to \infty \ . \tag{4.16}$$

Equations (4.15) and (4.16) show that

$$\tilde{U} = e^{-ikf} U_k$$

is regular at infinity.

We also note that $\nabla^2 f = 0$; therefore (4.9) yields

$$\nabla^2 \tilde{U} = -2ik \nabla\tilde{U} \circ \nabla f \tag{4.17}$$

But

$$\nabla\tilde{U} = \nabla(e^{-ikf} U_k) = -ike^{-ikf} U_k \nabla f + e^{-ikf} \nabla U_k ;$$

therefore

$$\nabla\tilde{U} \circ \nabla f = -ike^{-ikf} U_k (\nabla f \circ \nabla f) + e^{-ikf} \nabla U_k \circ \nabla f = -ike^{-ikf} U_k + e^{-ikf} \nabla U_k \circ \nabla f .$$

So we have

$$\nabla^2 \tilde{U} = -2(k^2 U_k + ik \nabla U_k \circ \nabla f) e^{-ikf} . \tag{4.18}$$

Now $\tilde{U}$ satisfies the hypothesis of Theorem 1. Taking w(p) = U in (4.4), we have

$$\tilde{U}(p, p_o) = \int_V dv' \, G_o(p, p') \nabla^2 \tilde{U}(p', p_o) + \int_B d\sigma \, \tilde{U}(p_B, p_o) \frac{\partial}{\partial n} G_o(p, p_B) , \tag{4.19}$$

where $\nabla^2 \tilde{U}$ is given by (4.17) or (4.18), or

$$\tilde{U}(p, p_o) = -2ik \int_V dv' \, G_o(p, p') \cdot \nabla f \circ \nabla\tilde{U}(p', p_o) + \int_B d\sigma \, \tilde{U}(p_B, p_o) \frac{\partial}{\partial n} G_o(p, p_B). \tag{4.20}$$

4.3 Scattering of a Plane Wave by the Ogive (Nose-on Incidence)

We write (4.20) in the operator form:

$$\tilde{U} = K \circ \tilde{U} + U^{(0)} \tag{4.21}$$

where

$$K \equiv -2ik \int_V dv\, G_o(p, p')\, \nabla f \circ \nabla \tag{4.22}$$

$$U^{(0)} = \int_B d\sigma\, \tilde{U}(p_B, p_o) \frac{\partial}{\partial n} G_o(p, p_B) \, . \tag{4.23}$$

The iterates are given by

$$U^{(N)} = \sum_{n=0}^{N} K^n U^{(0)} \tag{4.24}$$

or by

$$\tilde{U}^{(N)} = K \circ U^{(N-1)} + U^{(0)} \, , \quad N \geqslant 1 \, . \tag{4.25}$$

On the surface of the ogive $\alpha = \alpha_1$,

$$\tilde{U}(p_B, p_o) = e^{-ikf(\alpha_1, \beta)} U_k(p_B, p_o),$$

and the sum of the scattered field U_k and the incident field U^{inc} vanish on the surface; therefore (4.23) becomes

$$U^{(0)} = -\int_B e^{-ikf(\alpha_1, \beta)} U^{inc}(p_B) \frac{\partial}{\partial n} G_o(p, p_B) d\sigma \quad . \tag{4.26}$$

If we assume the incident field to be a plane wave propagating in the direction of the negative x-axis, that is, if

$$U^{inc} = e^{-ikx(\alpha, \beta)}$$

$$x(\alpha, \beta) = \frac{c \sinh\beta}{\cosh\beta - \cos\alpha} \, , \tag{4.27}$$

then we may assume m = 0 in the expression for the static Green's function

$$G_o = \frac{\pi}{c}(\cosh\beta - \cos\alpha)^{1/2}(\cosh\beta_o - \cos\alpha_o)^{1/2}\sum_j \frac{e^{(s_j+\frac{1}{2})|\beta-\beta_o|}}{\sin s_j\pi} \cdot$$

$$\cdot \frac{P_{s_j}(-\cos\alpha_1)}{\frac{d}{ds}\left\{P_{s_j}(\cos\alpha_1)\right\}} P_{s_j}(\cos\alpha)P_{s_j}(\cos\alpha_o) . \tag{4.28}$$

Observing that $\frac{\partial}{\partial n} = -\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha} = -\frac{1}{c}(\cosh\beta - \cos\alpha)\frac{\partial}{\partial\alpha}$ and that $P_{s_j}(\cos\alpha_1) = 0$, from (4.28) we obtain

$$\left.\frac{\partial G_o}{\partial n}\right|_{\alpha=\alpha_1} = \frac{\pi}{c^2}\sin\alpha_1(\cosh\beta - \cos\alpha_1)^{3/2}(\cosh\beta_o - \cos\alpha_o)^{1/2} \cdot$$

$$\cdot \sum_j \frac{e^{(s_j+\frac{1}{2})|\beta-\beta_o|}}{\sin s_j\pi}\,\frac{P_{s_j}(-\cos\alpha_1)}{\frac{d}{ds}\left\{P_{s_j}(\cos\alpha_1)\right\}}\left\{\frac{d}{d\alpha}\left[P_{s_j}(\cos\alpha_1)\right]\right\}P_{s_j}(\cos\alpha_o) \tag{4.29}$$

Substituting (4.27) and (4.29) into (4.26) and noting that

$$\int_B d\sigma = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} \frac{c^2\sin\alpha_1}{(\cosh\beta - \cos\alpha_1)^2}\, d\beta$$

we obtain

$$U^{(0)}(\alpha_o,\beta_o) = -2\pi^2\sin^2\alpha_1(\cosh\beta_o - \cos\alpha_o)^{1/2}\sum_j \frac{1}{\sin s_j\pi}\cdot\frac{P_{s_j}(-\cos\alpha_1)}{\frac{d}{ds}\left\{P_{s_j}(\cos\alpha_1)\right\}}\cdot$$

$$\frac{d}{d\alpha}\left\{P_{s_j}(\cos\alpha_1)\right\}P_{s_j}(\cos\alpha_o)\int_{-\infty}^{\infty} d\beta\,\frac{e^{-ik\left[f(\alpha_1,\beta)+x(\alpha_1,\beta)\right]}\,e^{(s_j+\frac{1}{2})|\beta-\beta_o|}}{(\cosh\beta - \cos\alpha_1)^{3/2}} \tag{4.30}$$

where

$$f(\alpha_1, \beta) = \frac{c}{2}\left\{\coth\left(\frac{\beta + i\alpha_1}{2}\right) + \coth\left(\frac{\beta - i\alpha_1}{2}\right)\right\},$$

$$x(\alpha_1, \beta) = \frac{c \sinh\beta}{\cosh\beta - \cos\alpha_1} \quad ,$$

and we recall that s_j are the real zeros of $P_{s_j}(\cos\alpha_1)$, $(s_j < -1/2)$.

The iterates $U^{(N)}$ for $N \geqslant 1$, given by (4.22) and (4.25) are

$$U^{(N)} = -2ik \int_V dv' \, G_o(p, p') \nabla f(\alpha', \beta') \circ \nabla U^{(N-1)}(\alpha', \beta') + U^{(0)} \tag{4.31}$$

where

$$\int_V dv' = \int_0^{2\pi} d\phi' \int_0^{\pi} d\alpha' \int_{-\infty}^{\infty} d\beta' \, \frac{c^3 \sin\alpha'}{(\cosh\beta' - \cos\alpha')^3} \quad .$$

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APPENDIX
ORTHOGONALITY AND THE NORMALIZING FACTORS FOR THE SET $\left\{P^m_{s_j}\right\}$

In the usual manner we obtain

$$(s_i - s_j)(s_i + s_j + 1)\int_{\xi_1}^{1} P^m_{s_i}(\xi)P^m_{s_j}(\xi)\,d\xi = \left\{(1-\xi^2)\left[P^m_{s_i}\frac{d}{d\xi}P^m_{s_j} - P^m_{s_j}\frac{d}{d\xi}P^m_{s_i}\right]\right\}_{\xi_1}^{1} . \tag{A.1}$$

Since $P^m_{s_i}(\cos\alpha_1) = P^m_{s_j}(\cos\alpha_1) = 0$, we conclude that

$$\int_{\xi_1}^{1} P^m_{s_i}P^m_{s_j}\,d\xi = 0 \qquad \text{if} \quad s_i \neq s_j \quad \text{for} \quad m = 0, 1, 2, \ldots$$

To evaluate the integral for the case $s_i = s_j$, we consider the Taylor expansion of P_{s_j} around P_{s_i}, substitute the result into (A.1) and let $s_i \rightarrow s_j$ to obtain

$$\int_{\xi_1}^{1}\left[P_{s_i}(\xi)\right]^2 d\xi = -\frac{1-\xi_1^2}{2s_i+1}\left[\frac{d}{ds_i}P_{s_i}(\xi_1)\right]\left[\frac{d}{d\xi_1}P_{s_i}(\xi_1)\right] . \tag{A.2}$$

Then we observe that

$$P^{m+1}_{s_i}(\xi) = (1-\xi^2)^{1/2}\frac{d}{d\xi}P^m_{s_i}(\xi) + m\xi(1-\xi^2)^{-1/2}P^m_{s_i}(\xi) . \tag{A.3}$$

Squaring (A.3) and integrating,

$$\int_{\xi_1}^{1}\left[P^{m+1}_{s_i}(\xi)\right]^2 d\xi = \int_{\xi_1}^{1}\left[(1-\xi^2)\left\{\frac{dP^m_{s_i}(\xi)}{d\xi}\right\}^2 + 2m\xi\,\frac{dP^m_{s_i}(\xi)}{d\xi}\;\zeta\;\frac{m^2\xi^2}{1-\xi^2}\left\{P^m_{s_i}(\xi)\right\}^2\right]d\xi. \tag{A.4}$$

Integrating the first two terms by parts,

$$\int_{\xi_1}^{1} \left[P_{s_i}^{m+1}(\xi)\right]^2 d\xi = - \int_{\xi_1}^{1} P_{s_i}^{m}(\xi) \frac{d}{d\xi} \left\{(1-\xi^2) \frac{dP_{s_i}^{m}(\xi)}{d\xi}\right\} d\xi -$$

$$- m \int_{\xi_1}^{1} \left\{P_{s_i}^{m}(\xi)\right\}^2 d\xi + \int_{\xi_1}^{1} \frac{m^2\xi^2}{1-\xi^2} \left\{P_{s_i}^{m}(\xi)\right\}^2 d\xi \ . \qquad (A.5)$$

Substituting the value of the first integral on the right from the differential equation, we obtain

$$\int_{\xi_1}^{1} \left[P_{s_i}^{m+1}(\xi)\right]^2 d\xi = (s_i - m)(s_i + m + 1) \int_{\xi_1}^{1} \left[P_{s_i}^{m}(\xi)\right]^2 d\xi \ . \qquad (A.6)$$

Now we iterate (A.6) to obtain

$$\int_{\xi_1}^{1} \left[P_{s_i}^{m}(\xi)\right]^2 d\xi = \frac{\Gamma(s_i + m + 1)}{\Gamma(s_i - m + 1)} \int_{\xi_1}^{1} \left[P_{s_i}(\xi)\right]^2 d\xi \ . \qquad (A.7)$$

Finally, using the result (A.2) we have

$$\int_{\xi_1}^{1} P_{s_i}^{m}(\xi) P_{s_j}^{m}(\xi)\, d\xi = \begin{cases} 0, & s_i \neq s_j \\ -\dfrac{1-\xi_1^2}{2s_i + 1} \dfrac{dP_{s_i}(\xi_1)}{ds_i} \dfrac{dP_{s_i}(\xi_1)}{d\xi_1} \dfrac{\Gamma(s_i + m + 1)}{\Gamma(s_i - m + 1)} & s_i = s_j \end{cases} \qquad (A.8)$$

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13. ABSTRACT

By means of a method developed by R.E. Kleinman of the Radiation Laboratory, it is now possible to solve iteratively the Dirichlet problem for the scalar Helmholtz equation in the regions exterior to a non-separable body imbedded in the Euclidean 3-space provided (1) k, the complex wave number, is sufficiently small, and (2) the solution of the potential Dirichlet problem for the body in question is known. In the present work we consider the low frequency scattering of a plane wave at nose-on incidence from an ogive.

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