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HALF-SPACE GENERAL MULTIGROUP TRANSPORT THEORY*

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Abstract—A method for solving various half-space multigroup transport problems for the case of a general transfer matrix is explained. A non-linear integral equation for the emergent distribution of the albedo problem is derived. Then, by using the full-range completeness of the infinite medium eigenfunctions, the distribution inside the half-space is obtained from the emergent distribution. Finally, the Milne problem and the half-space Green's function problem are solved in terms of the emergent distribution of the albedo problem and the infinite medium eigenfunctions.

One advantage of this method is that it readily yields numerical results for emergent distributions of half-space problems. Even for the numerical evaluation of a particular solution inside the half-space, this technique is felt to be superior to other exact half-space methods since only full-range eigenfunction expansions are employed rather than the more difficult half-range expansion used in other techniques.

1. INTRODUCTION

THE MULTIGROUP approximation has become a widely used method for treating the energy dependence of the transport equation. However, in practical reactor problems one is generally unable to solve the multigroup transport equation, and as a consequence, additional approximations must be introduced, e.g. multigroup diffusion theory. Many of these approximations to the transport equation have been quite successful. Nevertheless, there is a very definite need for exact solutions of the multigroup transport equation since such solutions can be used to check the accuracy of the solutions obtained from the various approximate methods.

Only in the past few years have exact solutions to several multigroup problems been obtained. By applying the singular eigenfunction approach of CASE and ZWEIFEL (1967), the solution of the infinite medium Green's function problem has been obtained for the two-group (SIEWERT and SHIEH, 1967) and the N-group cases (YOSHIMURA, 1968). Until now, the solution of half-space problems has been restricted to special cases. Several two-group problems have been investigated (ZELAZNY and KUSZELL, 1961; METCALF, 1968). In a paper on radiative transfer Siewert and Zweifel treated the N-group case with the specific limitation that the determinants of the transfer matrix, **C**, and all its minors vanish (SIEWERT and ZWEIFEL, 1966). Finally, for the case of symmetric transfer (of which the two group case is a special example (PAHOR and SHULTIS, 1969)) LEONARD and FERZIGER (1966) showed how solutions to half-space problems can, in principle, be obtained by solving Fredholm equations.

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All of the above investigations are restricted in their applicability to practical situations either by the small number of permissible groups or by a special form of the transfer matrix. The purpose of this paper, therefore, is to consider half-space problems of the N-group isotropic transport equation with a completely arbitrary transfer matrix. However all of the previous half-space techniques have depended critically upon a 'half-range completeness' theorem whereby the solution of any half-space problem can be expanded uniquely in terms of only half of the infinite medium eigenfunctions of the transport equation. For the case of general transfer no such half-range theorem has been found and another approach must be used.

Recently an approach which circumvents this half-range difficulty has been used by PAHOR (1966, 1967) in the thermal neutron degenerate kernel case. It is this approach which will be used in this paper. First the emergent distribution for the problem is found. Then, once the angular flux is completely known at the surface, the full-range completeness property of the N-group infinite medium eigenfunctions (YOSHIMURA, 1968) can be used to obtain the complete solution inside the half-space.

The main problem then is the calculation of the emergent distributions. To this end, several different methods can be used. Case has obtained a Fredholm equation for the emergent flux in terms of the infinite medium Green's function (CASE, 1969). Secondly, from an eigenfunction expansion of the problem, a Fredholm equation for the emergent distribution involving only eigenfunctions can be obtained (PAHOR and SHULTIS, 1969; CASE and ZWEIFEL, 1967). However, both these methods are very difficult to evaluate numerically. A third approach, which readily yields numerical results for the emergent distribution, will be used in this paper. By applying the invariance principles of AMBARZUMIAN (1942) and CHANDRASEKHAR (1960), a nonlinear integral equation with a simple kernel can be easily derived for the general N-group albedo problem emergent distribution.

The plan of this paper is as follows: Section 2 reviews the known results and properties of the infinite medium eigenfunctions which will be needed later. The next section shows how the emergent distribution to the half-space albedo problem can be obtained in terms of two fundamental matrix functions, $\mathbf{V}(\mu)$ and $\mathbf{U}(\mu)$. These functions satisfy a pair of coupled nonlinear integral equations which are quite amenable to solution by numerical means. In Section 3 we demonstrate how the emergent distributions for the Milne problem and the half-space Green's function can also be expressed in terms of these U and V matrices. Then by applying the full-range completeness theorem of the infinite medium eigenfunctions, the complete solutions to these problems may be obtained.

2. EIGENFUNCTIONS OF THE MULTIGROUP TRANSPORT EQUATION

The linear Boltzmann equation for N energy groups in plane geometry and with isotropic scattering and fission can be written in the form

$$\mu \frac{\partial}{\partial x} \boldsymbol{\psi}(x, \mu) + \boldsymbol{\Sigma} \boldsymbol{\psi}(x, \mu) = \mathbf{C} \int_{-1}^{1} \mathrm{d}\mu' \boldsymbol{\psi}(x, \mu'). \tag{2.1}$$

The vector $\boldsymbol{\psi}(x, \mu)$ is an N-component vector, of which the ith component, $\psi_i(x, \mu)$, is the angular flux for the ith group. The components of the matrix $\sum$ are given by $\sigma_i \delta_{ij}$, σ_i being the total interaction cross section for the ith group. Finally, the elements, c_{ij}, of the transfer matrix $\mathbf{C}$ describe the transfer of neutrons from the jth

group to the ith group. For an isotropically scattering and fissioning medium the c_{ij} are given by

$$c_{ij} = \tfrac{1}{2}[\sigma^s_{j\to i} + \chi_i \nu_j \sigma_j^f] \tag{2.2}$$

where $\sigma^s_{j\to i}$ is the scattering cross section for the transfer of neutrons from the jth group to the ith group, σ_j^f is the fission cross section for the jth group, ν_j the number of fission neutrons produced by an incident jth group neutron, and χ_i is the fission spectrum fraction of the ith group.

It is always possible to order the groups such that (PAHOR and SHULTIS, 1969)

$$\sigma_1 > \sigma_2 > \ldots > \sigma_N, \tag{2.3}$$

and by dividing equation (2.1) by σ_N and measuring distance in units of the largest mean free path, σ_N^{-1}, one may set $\sigma_N = 1$ (SIEWERT and ZWEIFEL, 1966).

Using the analogy of the one-speed problem (CASE and ZWEIFEL, 1967), a set of eigenfunctions, $\psi(\nu, x, \mu)$ to equation (2.1) of the form

$$\psi(\nu, x, \mu) = \mathrm{e}^{-x/\nu}\boldsymbol{\phi}(\nu, \mu) \tag{2.4}$$

is sought. Substituting this ansatz in equation (2.1), the following equation for the eigenvectors $\boldsymbol{\phi}(\nu, \mu)$ is obtained:

$$\left(\boldsymbol{\Sigma} - \frac{\mu}{\nu}\mathbf{E}\right)\boldsymbol{\phi}(\nu, \mu) = \mathbf{C}\int_{-1}^{1} \mathrm{d}\mu'\, \boldsymbol{\phi}(\nu, \mu'), \tag{2.5}$$

where $\mathbf{E}$ is the unit matrix. The explicit form of these eigenvectors and their properties have been investigated by several authors (SIEWERT and SHIEH, 1967; YOSHIMURA, 1968; ZELAZNY and KUSZELL, 1961, 1962; LEONARD and FERZIGER, 1966). In order to establish notation, the basic form and properties of these eigenfunctions will be briefly quoted. We will use, with slight changes, the notation of YOSHIMURA (1968).

The eigenvectors can be written in the form

$$\boldsymbol{\phi}(\nu, \mu) = P\mathbf{F}(\nu, \mu)\mathbf{b}(\nu) + \mathbf{G}(\nu, \mu)\boldsymbol{\lambda}(\nu), \tag{2.6}$$

where P denotes the Cauchy principal value. The matrices $\mathbf{F}(\nu, \mu)$ and $\mathbf{G}(\nu, \mu)$ are defined as

$$[\mathbf{F}(z, \mu)]_{ij} = \frac{z}{\sigma_i z - \mu}\,\delta_{ij}, \tag{2.7}$$

and

$$[\mathbf{G}(z, \mu)]_{ij} = \delta(\sigma_i z - \mu)\delta_{ij}. \tag{2.8}$$

A simultaneous equation for the unknown $\mathbf{b}(\nu)$, which satisfies

$$\mathbf{b}(\nu) = \mathbf{C}\int_{-1}^{1} \mathrm{d}\mu'\, \boldsymbol{\phi}(\nu, \mu') = \mathbf{C}\mathbf{a}(\nu), \tag{2.9}$$

and the unknown vector $\boldsymbol{\lambda}(\nu)$ is obtained by substituting equation (2.6) into equation (2.9), namely

$$\boldsymbol{\Omega}(\nu)\mathbf{b}(\nu) = \int_{-1}^{1} \mathrm{d}\mu \mathbf{G}(\nu, \mu)\boldsymbol{\lambda}(\nu) \tag{2.10}$$

where

$$\boldsymbol{\Omega}(z) = \mathbf{C}^{-1} - P\int_{-1}^{1} \mathbf{F}(z, \mu)\, \mathrm{d}\mu. \tag{2.11}$$

To solve for $\mathbf{b}(\nu)$ and $\boldsymbol{\lambda}(\nu)$ the eigenvalue spectrum is divided into two regions.

(a) *Region* 1: $\nu \notin (-1, 1)$

In this region there may exist an even number, say $2M$, of discrete eigenvectors, which in component form are written as

$$[\boldsymbol{\phi}(\nu_{0s}, \mu)]_i = \frac{\nu_{0s}}{\sigma_i \nu_{0s} - \mu} [\mathbf{b}(\nu_{0s})]_i, \quad i = 1 \ldots N, \quad s = 1 \ldots 2M, \tag{2.12}$$

where $\mathbf{b}(\nu_{0s})$ is a well defined vector (YOSHIMURA, 1968). The discrete eigenvalues, ν_{0s}, $s = 1 \ldots 2M$, are solutions of the dispersion relation

$$\det \boldsymbol{\Omega}(\nu_{0s}) = 0. \tag{2.13}$$

It can be shown that if ν_{0s} is a solution of equation (2.13) then also $-\nu_{0s}$ and ν_{0s}^* (complex conjugate) are eigenvalues with

$$\mathbf{b}(\nu_{0s}) = \mathbf{b}(-\nu_{0s}) = \mathbf{b}^*(\nu_{0s}). \tag{2.14}$$

For a symmetric transfer matrix, the discrete eigenvalues, if they exist, are either real or imaginary (PAHOR and SHULTIS, 1969). For a general system, on the other hand, there does not appear to be any *a priori* reason to expect that the discrete eigenvalues are not complex. However, it may be argued on physical grounds that a subcritical medium must have a real dominant eigenvalue (defined as the eigenvalue with the largest real part), SHULTIS (1968)

(b) *Region* 2: $\nu \varepsilon (-1, 1)$

This region is divided into N subintervals, ν_j, $j = 1 \ldots N$, such that for $\nu \varepsilon \nu_j$, $(1/\sigma_{j-1}) < |\nu| \leq (1/\sigma_j)$ where $\sigma_0^{-1} = 0$. For the jth sub-interval, there are $(N - j + 1)$ linearly independent eigenvectors, $\boldsymbol{\phi}_j^m(\nu, \mu)$, where the ith component has the form

$$[\boldsymbol{\phi}_j^m(\nu, \mu)]_i = P \frac{\nu}{\sigma_i \nu - \mu} [\mathbf{b}_j^m(\nu)]_i + \delta(\sigma_i \nu - \mu)[\boldsymbol{\lambda}_j^m(\nu)]_i, \quad m = j \ldots N, \quad j = 1 \ldots N, \tag{2.15}$$

where P indicates the Cauchy principal value. The vectors $\mathbf{b}_j^m(\nu)$ and $\boldsymbol{\lambda}_j^m(\nu)$ are also defined by YOSHIMURA (1968).

The eigenvectors of both regions, $\boldsymbol{\phi}(\nu, \mu)$, depend parametrically upon $\mathbf{C}$. If we denote by $\boldsymbol{\phi}^\dagger(\nu, \mu)$ the eigenfunctions with $\mathbf{C}$ replaced by $\tilde{\mathbf{C}}$ (tilde denoting the transpose), we see from equation (2.5)

$$\boldsymbol{\phi}^\dagger(\nu, \mu, \mathbf{C}) = \boldsymbol{\phi}(\nu, \mu, \tilde{\mathbf{C}}). \tag{2.16}$$

In passing, it should be noted that $\mathbf{b}(\nu)$ and $\boldsymbol{\lambda}(\nu)$ are even functions of ν and hence the eigenvectors have the property

$$\boldsymbol{\phi}(\nu, -\mu) = \boldsymbol{\phi}(-\nu, \mu). \tag{2.17}$$

From the eigenvalue equation (2.5), one finds that the eigenvectors are orthogonal in the following sense:

$$\int_{-1}^{1} d\mu \, \mu \tilde{\boldsymbol{\phi}}^\dagger(\nu' \mu) \boldsymbol{\phi}(\nu, \mu) = 0 \quad \text{if} \quad \nu' \neq \nu. \tag{2.18}$$

Moreover, it is possible to choose particular linear combinations of eigenvectors for the independent eigenvectors of each subinterval, ν_j, such that all the 'continuum' eigenvectors are mutually orthogonal (YOSHIMURA, 1968), i.e.

$$\int_{-1}^{1} \mathrm{d}\mu\, \mu \tilde{\boldsymbol{\phi}}_j^{\dagger m'}(\pm\nu, \mu)\boldsymbol{\phi}_j{}^{m}(\pm\nu', \mu) = \pm N_j{}^{m}(\nu)\, \delta_{mm'}\, \delta(\nu - \nu'), \quad \nu, \nu' \varepsilon \nu_j. \tag{2.19}$$

Similarly for the 'discrete' eigenvectors, we have

$$\int_{-1}^{1} \mathrm{d}\mu\, \mu \tilde{\boldsymbol{\phi}}^{\dagger}(\pm\nu_{0s}, \mu)\boldsymbol{\phi}(\pm\nu_{0s'}, \mu) = \pm N_s\, \delta_{ss'}, \; s = 1 \ldots M. \tag{2.20}$$

The normalization functions $N_j{}^{m}(\nu)$ and N_s are given by YOSHIMURA (1968).

The eigenvectors, $\boldsymbol{\phi}(\nu, \mu)$, of equation (2.5) have the very useful property that they are 'full-range complete' (YOSHIMURA, 1968; ZELAZNY and KUSZELL, 1962). This property may be stated in terms of the following theorem.

Theorem. The set of functions $\boldsymbol{\phi}(\nu, \mu)$, $\nu \varepsilon [-1, 1]$ or $\nu = \pm\nu_{0s}$, $s = 1 \ldots M$, is complete in the sense that an arbitrary vector function $\boldsymbol{\psi}(\mu)$ defined for $\mu \varepsilon [-1, 1]$ can be expanded in the form

$$\boldsymbol{\psi}(\mu) = \sum_{s=1}^{M} \alpha(\nu_{0s})\boldsymbol{\phi}(\nu_{0s}, \mu) + \sum_{s=1}^{M} \alpha(-\nu_{0s})\boldsymbol{\phi}(-\nu_{0s}, \mu) + \sum_{j=1}^{N} \int_{\nu_j} \mathrm{d}\nu \left\{ \sum_{m=j}^{N} A_j{}^{m}(\nu)\boldsymbol{\phi}_j{}^{m}(\nu, \mu) \right\}, \tag{2.21}$$

where $\alpha(\nu_{0s})$, $\alpha(-\nu_{0s})$, and $A_j{}^{m}(\nu)$ are uniquely determined expansion coefficients.

3. EMERGENT DISTRIBUTION OF THE ALBEDO PROBLEM

In many problems in half-space transport theory, only the angular flux at the surface of the medium is needed. To this end, the emergent distribution of the half-space albedo problem will be considered in this section. The emergent distribution of this particular problem turns out to be of fundamental importance in determining the emergent distributions of all other half-space problems.

Consider an albedo problem for which the incident neutron beam belongs to the ith energy group. The angular flux of this 'ith albedo problem', $\boldsymbol{\psi}_i(0, \mu_0;\ x, \mu)$, is the solution of equation (2.1) with the boundary conditions

$$\text{(i)}\ \boldsymbol{\psi}_i(0, \mu_0;\ 0, \mu) = \mathbf{e}_i\, \delta(\mu - \mu_0), \; \mu > 0, \; \mu_0 > 0, \tag{3.1}$$

$$\text{(ii)}\ \lim_{x \to \infty} \boldsymbol{\psi}(0, \mu_0;\ x, \mu) = 0, \tag{3.2}$$

where $\mathbf{e}_i$ is a vector all of whose components are zero except the ith one which is unity. The N distinct albedo problems (one for each group) can be handled collectively by introducing the 'albedo matrix' $\boldsymbol{\Psi}(0, \mu_0;\ x, \mu)$ defined as

$$\boldsymbol{\Psi}(0, \mu_0;\ x, \mu) = [\boldsymbol{\psi}_1(0, \mu_0;\ x, \mu), \boldsymbol{\psi}_2(0, \mu_0;\ x, \mu), \ldots, \boldsymbol{\psi}_N(0, \mu_0;\ x, \mu)]. \tag{3.3}$$

This matrix is the solution of the transport equation

$$\left(\mu \frac{\partial}{\partial x}\mathbf{E} + \boldsymbol{\Sigma}\right)\boldsymbol{\Psi}(0, \mu_0;\ x, \mu) = \mathbf{C}\int_{-1}^{1} \mathrm{d}\mu'\, \boldsymbol{\Psi}(0, \mu_0;\ x, \mu'), \tag{3.4}$$

with the boundary conditions

$$\text{(i)}\ \boldsymbol{\Psi}(0, \mu_0;\ 0, \mu) = \mathbf{E}\,\delta(\mu - \mu_0),\ \mu > 0,\ \mu_0 > 0, \tag{3.5}$$

$$\text{(ii)}\ \lim_{x\to\infty} \boldsymbol{\Psi}(0, \mu_0;\ x, \mu) = 0. \tag{3.6}$$

From the definition of the albedo matrix, the quantity $\boldsymbol{\Psi}(0, \mu_0;\ 0, \mu)$ gives the emergent distribution $\psi^{\text{em}}(-\mu)$, $\mu > 0$, for any arbitrary incident $\psi^{\text{inc}}(\mu)$, $\mu > 0$ as

$$\psi^{\text{em}}(-\mu) = \int_0^1 \mathrm{d}\mu' \boldsymbol{\Psi}(0, \mu';\ 0, -\mu)\psi^{\text{inc}}(\mu'). \tag{3.7}$$

In fact the emergent distribution of any half-space problem can be expressed in terms of the matrix $\boldsymbol{\Psi}(0, \mu_0;\ 0, -\mu)$ $\mu_0 > 0$, $\mu > 0$. There are several methods for obtaining this portion of the albedo matrix without first solving the complete albedo problem (PAHOR and SHULTIS, 1969; PAHOR, 1966; CASE, 1969; CHANDRASEKHAR, 1960). Here we will obtain from 'the principle of invariance' (CHANDRASEKHAR, 1960) a nonlinear integral equation for this reflected flux.

The principle of invariance states that the reflected flux from a half-space is unchanged by the addition (or subtraction) of layers of arbitrary thickness to (or from) the medium (CHANDRASEKHAR, 1960). Thus if $\psi(x, \mu)$ is the angular flux at a distance x inside the half-space, the outwardly moving flux can then be expressed from equation (3.7) in terms of the inward flux as

$$\psi(x, -\mu) = \int_0^1 \mathrm{d}\mu' \boldsymbol{\Psi}(0, \mu';\ 0, -\mu)\psi(x, \mu'),\ \mu > 0. \tag{3.8}$$

In particular, equation (3.8) gives for the ith albedo problem

$$\psi_i(0, \mu_0;\ x, -\mu) = \int_0^1 \mathrm{d}\mu' \boldsymbol{\Psi}(0, \mu';\ 0, -\mu)\psi_i(0, \mu_0;\ x, \mu'). \tag{3.9}$$

Finally if we treat the N albedo problems collectively, the above vector equation yields the following relation for the albedo matrix:

$$\boldsymbol{\Psi}(0, \mu_0;\ x, -\mu) = \int_0^1 \mathrm{d}\mu' \boldsymbol{\Psi}(0, \mu';\ 0, -\mu)\boldsymbol{\Psi}(0, \mu_0;\ x, \mu'). \tag{3.10}$$

To obtain a nonlinear integral equation for $\boldsymbol{\Psi}(0, \mu';\ 0, -\mu)$, first differentiate equation (3.10) with respect to x and set $x = 0$. Then using the transport equation (3.4) to evaluate the derivatives, and employing the boundary condition (3.5) one obtains

$$\frac{1}{\mu}\boldsymbol{\Sigma}\,\mathbf{S}(\mu_0, \mu) + \frac{1}{\mu_0}\mathbf{S}(\mu_0, \mu)\boldsymbol{\Sigma} = \left[\mathbf{E} + \int_0^1 \frac{\mathrm{d}\mu'}{\mu'}\mathbf{S}(\mu', \mu)\right]\mathbf{C}\left[\mathbf{E} + \int_0^1 \frac{\mathrm{d}\mu'}{\mu'}\mathbf{S}(\mu_0, \mu')\right], \tag{3.11}$$

where the generalized $\mathbf{S}$ matrix is defined as

$$\mathbf{S}(\mu_0, \mu) = \mu\boldsymbol{\Psi}(0, \mu_0;\ 0, -\mu),\ \mu > 0. \tag{3.12}$$

Since each bracket on the right hand side of equation (3.11) is a function of only one angular variable, this equation may be written as

$$\frac{1}{\mu}\,\boldsymbol{\Sigma}\,\mathbf{S}(\mu_0, \mu) + \frac{1}{\mu_0}\,\mathbf{S}(\mu_0, \mu)\boldsymbol{\Sigma} = \mathbf{U}(\mu)\mathbf{C}\mathbf{V}(\mu_0), \tag{3.13}$$

with

$$\mathbf{U}(\mu) = \mathbf{E} + \int_0^1 \frac{\mathrm{d}\mu'}{\mu'}\,\mathbf{S}(\mu', \mu), \tag{3.14}$$

$$\mathbf{V}(\mu) = \mathbf{E} + \int_0^1 \frac{\mathrm{d}\mu'}{\mu'}\,\mathbf{S}(\mu, \mu'). \tag{3.15}$$

Unfortunately $\boldsymbol{\Sigma}$ and $\mathbf{S}(\mu, \mu_0)$ do not commute, and to obtain an equation more amenable to numerical solution, the definition of a 'term-wise matrix product' is introduced. If $\mathbf{D}$ is the term-wise product (denoted by *) of $\mathbf{A}$ and $\mathbf{B}$, i.e. $\mathbf{D} = \mathbf{A}*\mathbf{B}$, then in component form we have

$$[\mathbf{D}]_{ij} = [\mathbf{A}]_{ij}[\mathbf{B}]_{ij},\ i, j = 1 \ldots N. \tag{3.16}$$

It should be noted that the term-wise product operator is neither associative nor distributive with the conventional matrix product.

To obtain a system of equations equivalent to equation (3.11) involving only $\mathbf{U}$ and $\mathbf{V}$, integrate equation (3.11) first with respect to μ_0. Then use the definition of $\mathbf{U}(\mu)$, equation (3.14) and the term-wise matrix product notation to obtain

$$\mathbf{U}(\mu) = \mathbf{E} + \mu \int_0^1 \mathrm{d}\mu' \mathbf{A}(\mu, \mu')*[\mathbf{U}(\mu)\mathbf{C}\mathbf{V}(\mu')], \tag{3.17}$$

where the matrix $\mathbf{A}(\mu, \mu')$ is defined as

$$[\mathbf{A}(\mu, \mu')]_{ij} = \frac{1}{\sigma_i\mu' + \sigma_j\mu}\,. \tag{3.18}$$

Similarly by integrating equation (3.11) with respect to μ an expression for $\mathbf{V}(\mu)$ is obtained:

$$\mathbf{V}(\mu) = \mathbf{E} + \mu \int_0^1 \mathrm{d}\mu' \mathbf{A}(\mu', \mu)*[\mathbf{U}(\mu')\mathbf{C}\mathbf{V}(\mu)]. \tag{3.19}$$

Equations (3.17) and (3.19) are two simultaneous nonlinear integral equations for the $\mathbf{U}$ and $\mathbf{V}$ matrix functions. For $N = 1$, they correspond to CHANDRASEKHAR's (1960) one-speed nonlinear H function equation with $U \equiv V \equiv H$.

The equations for $\mathbf{U}$ and $\mathbf{V}$, as they stand, can be solved numerically by the method of successive iterations (SHULTIS, 1968). However it is possible to cast the $\mathbf{U}$ and $\mathbf{V}$ equations into a different form which does not involve the unconventional term-wise matrix product and whose iterative convergence has been found to be much better than that of equations (3.17) and (3.19).

In equation (3.19) it is not possible to factor the term $\mathbf{V}(\mu)$ outside of the integral because of the term-wise product. However, it is possible to transform this matrix equation into a system of vector equations in which such a factorization can be accomplished.

Consider first the integrand in equation (3.19). Substituting explicitly for $\mathbf{A}(\mu', \mu)$, and denoting summation by the repeated index notation (where a repeated lower case Greek subscript signifies summation from 1 to N), this integrand can be written in component form as

$$[\mathbf{A}(\mu', \mu)^*\{\mathbf{U}(\mu')\mathbf{C}\mathbf{V}(\mu)\}]_{ij} \equiv \frac{U_{iv}(\mu')c_{v\eta}N_{\eta j}(\mu)}{\sigma_i\mu + \sigma_j\mu'}. \tag{3.20}$$

Now define the matrix $\mathbf{V}_k(\mu)$ all of whose elements are zero except the kth column which equals the kth column of the $\mathbf{V}(\mu)$ matrix, i.e.

$$[\mathbf{V}_k(\mu)]_{ij} = [\mathbf{V}(\mu)]_{ij}\,\delta_{jk}. \tag{3.21}$$

With this notation the integrand (3.20) becomes

$$\mathbf{D}_\eta(\mu, \mu')\mathbf{U}(\mu')\mathbf{C}\mathbf{V}_\eta(\mu) \tag{3.22}$$

where the diagonal matrix $\mathbf{D}_k(\mu, \mu')$ is defined as

$$[\mathbf{D}_k(\mu, \mu')]_{ij} = \frac{1}{\sigma_i\mu + \sigma_k\mu'}\,\delta_{ij}. \tag{3.23}$$

Using this notation, equation (3.19) can be written in the form

$$\mathbf{V}(\mu) = \sum_{i=1}^{N} \mathbf{V}_i(\mu) = \mathbf{E} + \mu \int_0^1 \mathrm{d}\mu'\mathbf{D}_\eta(\mu, \mu')\mathbf{U}(\mu')\mathbf{C}\mathbf{V}_\eta(\mu). \tag{3.24}$$

Similarly the transpose of equation (3.17) may be written as

$$\tilde{\mathbf{U}}(\mu) = \sum_{i=1}^{N} \tilde{\mathbf{U}}_k(\mu) = \mathbf{E} + \mu \int_0^1 \mathrm{d}\mu'\mathbf{D}_\eta(\mu, \mu')\tilde{\mathbf{V}}(\mu')\tilde{\mathbf{C}}\tilde{\mathbf{U}}_\eta(\mu), \tag{3.25}$$

where

$$[\tilde{\mathbf{U}}_k(\mu)]_{ij} = [\mathbf{U}(\mu)]_{ji}\,\delta_{jk}. \tag{3.26}$$

These two new matrix equations for $\mathbf{U}(\mu)$ and $\mathbf{V}(\mu)$, equations (3.24) and (3.25) can be reduced to systems of N vector equations. This is possible because of the particularly simple forms of the matrices $\mathbf{V}_k(\mu)$ and $\mathbf{U}_k(\mu)$. If one defines the vectors $\mathbf{v}_i(\mu)$ and $\mathbf{u}_i(\mu)$ as

$$\mathbf{v}_i(\mu) = \begin{bmatrix} V_{1i}(\mu) \\ V_{2i}(\mu) \\ \cdot \\ \cdot \\ \cdot \\ V_{Ni}(\mu) \end{bmatrix} \quad \text{and} \quad \mathbf{u}_i(\mu) = \begin{bmatrix} U_{i1}(\mu) \\ U_{i2}(\mu) \\ \cdot \\ \cdot \\ \cdot \\ U_{iN}(\mu) \end{bmatrix}, \quad i = 1 \ldots N, \tag{3.27}$$

then equations (3.24) and (3.25) become

$$\mathbf{v}_i(\mu) = \mathbf{e}_i + \mu \int_0^1 \mathrm{d}\mu'\mathbf{D}_i(\mu, \mu')\mathbf{U}(\mu')\mathbf{C}\mathbf{v}_i(\mu), \quad i = 1 \ldots N, \tag{3.28}$$

and

$$\mathbf{u}_i(\mu) = \mathbf{e}_i + \mu \int_0^1 \mathrm{d}\mu'\mathbf{D}_i(\mu, \mu')\tilde{\mathbf{V}}(\mu')\tilde{\mathbf{C}}\mathbf{u}_i(\mu), \quad i = 1 \ldots N. \tag{3.29}$$

This system of vector equations is exactly equivalent to the **U** and **V** matrix equations which involved direct products (equations (3.17) and (3.19)). However now we can factor $\mathbf{v}_i(\mu)$ or $\mathbf{u}_i(\mu)$ from the integrands to obtain

$$\mathbf{v}_i(\mu) = \left[\mathbf{E} - \mu \int_0^1 \mathrm{d}\mu' \mathbf{D}_i(\mu, \mu')\mathbf{U}(\mu')\mathbf{C}\right]^{-1} \mathbf{e}_i \tag{3.30}$$

and

$$\mathbf{u}_i(\mu) = \left[\mathbf{E} - \mu \int_0^1 \mathrm{d}\mu' \mathbf{D}_i(\mu, \mu')\tilde{\mathbf{V}}(\mu')\tilde{\mathbf{C}}\right]^{-1} \mathbf{e}_i. \tag{3.31}$$

It has been found that this system of equations is also solved readily by the method of successive iterations. However, the convergence rate is significantly better than that of the iterative solution of equations (3.17) and (3.19) (SHULTIS, 1968).

Once the **U** and **V** matrices have been calculated, the generalized $\mathbf{S}(\mu_0, \mu)$ function is readily obtained from equation (3.13); and thus the emergent distribution for the ith albedo problem is in view of equations (3.1), (3.3), (3.7) and (3.12)

$$\boldsymbol{\psi}_i(0, \mu_0;\ 0, -\mu) = \mu^{-1}\mathbf{S}(\mu_0, \mu)\mathbf{e}_i = \mu_0\{\mathbf{A}(\mu, \mu_0)^*[\mathbf{U}(\mu)\mathbf{C}\mathbf{V}(\mu')]\}\mathbf{e}_i. \tag{3.32}$$

At the end of this section it should be mentioned that in solving the albedo problem for a given transfer matrix **C**, we have, in fact, also solved the albedo problem for the transposed transfer matrix $\tilde{\mathbf{C}}$.

To show this, let us consider the solution $\boldsymbol{\psi}_j^\dagger(0, \mu_1;\ x, \mu)$ of the transport equation

$$\left(\mu \frac{\partial}{\partial x}\mathbf{E} + \boldsymbol{\Sigma}\right)\boldsymbol{\psi}_j^\dagger(0, \mu_1;\ x, \mu) = \tilde{\mathbf{C}} \int_1^1 \boldsymbol{\psi}_j^\dagger(0, \mu_1;\ x, \mu')\,\mathrm{d}\mu' \tag{3.33}$$

which satisfies the boundary conditions

$$\text{(i)}\ \boldsymbol{\psi}_j^\dagger(0, \mu_1;\ 0, \mu) = \mathbf{e}_j\,\delta(\mu_1 - \mu),\ \mu_1 > 0,\ \mu > 0, \tag{3.34}$$

$$\text{(ii)}\ \lim_{x\to\infty} \boldsymbol{\psi}_j^\dagger(0, \mu_1;\ x, \mu) = 0. \tag{3.35}$$

To find the relationship between the emergent distribution $\boldsymbol{\psi}_i(0, \mu_0;\ 0, -\mu)$, defined by equations (2.1), (3.1) and (3.2), and $\boldsymbol{\psi}_j^\dagger(0, \mu_1;\ 0, -\mu)$ first multiply equation (2.1) from the left by $\tilde{\boldsymbol{\psi}}_j^\dagger(0, \mu_1;\ x, -\mu)$; then multiply the transpose of equation (3.33), with μ replaced by $-\mu$, from the right by $\boldsymbol{\psi}_i(0, \mu_0;\ x, \mu)$. Subtraction of these two results and integration over μ from -1 to 1, and over x from 0 to ∞, yields the identity

$$\int_{-1}^1 \mathrm{d}\mu\,\mu \int_0^\infty \mathrm{d}x \frac{\partial}{\partial x}[\tilde{\boldsymbol{\psi}}_j^\dagger(0, \mu_1;\ x, -\mu)\boldsymbol{\psi}_i(0, \mu_0;\ x, \mu)] = 0. \tag{3.36}$$

Use of the boundary conditions (3.1), (3.2), (3.34) and (3.35) and integration by parts of the above equation gives the relation

$$\mu_0\tilde{\boldsymbol{\psi}}_j^\dagger(0, \mu_1;\ 0, -\mu_0)\mathbf{e}_i = \mu_1\tilde{\mathbf{e}}_j\boldsymbol{\psi}_i(0, \mu_0;\ 0, -\mu_1). \tag{3.37}$$

As before we introduce an albedo matrix $\boldsymbol{\Psi}^\dagger(0, \mu_1;\ x, -\mu)$ defined as

$$\boldsymbol{\Psi}^\dagger(0, \mu_1;\ x, \mu) = [\boldsymbol{\psi}_1^\dagger(0, \mu_1;\ x, \mu), \boldsymbol{\psi}_2^\dagger(0, \mu_1;\ x, \mu) \ldots \boldsymbol{\psi}_N^\dagger(0, \mu_1;\ x, \mu)], \tag{3.38}$$

and the generalized $\mathbf{S}^{\dagger}(\mu_1, \mu)$ matrix

$$\mathbf{S}^{\dagger}(\mu_1, \mu) = \mu\boldsymbol{\Psi}^{\dagger}(0, \mu_1;\ 0, -\mu). \tag{3.39}$$

Then it follows from equations (3.3), (3.12), (3.37), (3.38) and (3.39) that $\mathbf{S}(\mu_0, \mu_1)$ and $\mathbf{S}^{\dagger}(\mu_1, \mu_0)$ are related by the equation

$$\mathbf{S}^{\dagger}(\mu_1, \mu_0) = \tilde{\mathbf{S}}(\mu_0, \mu_1). \tag{3.40}$$

Finally, by defining

$$\mathbf{U}^{\dagger}(\mu) = \mathbf{E} + \int_0^1 \frac{\mathrm{d}\mu'}{\mu'}\, \mathbf{S}^{\dagger}(\mu', \mu), \tag{3.41}$$

and

$$\mathbf{V}^{\dagger}(\mu) = \mathbf{E} + \int_0^1 \frac{\mathrm{d}\mu'}{\mu'}\, \mathbf{S}^{\dagger}(\mu, \mu'), \tag{3.42}$$

it follows that

$$\mathbf{U}^{\dagger}(\mu) = \tilde{\mathbf{V}}(\mu), \tag{3.43}$$

and

$$\mathbf{V}^{\dagger}(\mu) = \tilde{\mathbf{U}}(\mu). \tag{3.44}$$

4. EMERGENT DISTRIBUTIONS OF OTHER HALF-SPACE PROBLEMS

In this section it is shown that the emergent distributions for the generalized Milne problem and the half-space Green's function problem can be expressed in terms of the generalized **S** function or the **U** and **V** matrices of the previous section.

(a) *The generalized Milne problem*

For every positive eigenvalue $\nu\varepsilon(0, 1)$ or $\nu = \nu_{0s}$, $s = 1 \ldots M$, a Milne problem can be defined. Denoting its solution by $\psi_{\nu}(x, \mu)$, it is defined as the solution of the transport equation, equation (2,1), with the following boundary conditions:

$$\text{(i)}\ \psi_{\nu}(0, \mu) = 0, \mu > 0, \tag{4.1}$$

$$\text{(ii)}\ \lim_{x\to\infty} \psi_{\nu}(x, \mu) = \boldsymbol{\phi}(-\nu, \mu)\mathrm{e}^{x/\nu}, \nu > 0, \tag{4.2}$$

where $\boldsymbol{\phi}(-\nu, \mu)$ may be any of the eigenvectors—regular or singular.

The first step in obtaining the solution is to find the emergent distribution, $\psi_{\nu}(0, -\mu)$, $\mu > 0$. Consider a solution of the transport equation, $\psi(x, \mu)$, defined as

$$\psi(x, \mu) = \psi_{\nu}(x, \mu) + \psi_a(x, \mu) \tag{4.3}$$

where $\psi_a(x, \mu)$ is an albedo problem solution with the boundary conditions

$$\text{(i)}\ \psi_a(0, \mu) = \boldsymbol{\phi}(-\nu, \mu), \mu > 0, \tag{4.4}$$

$$\text{(ii)}\ \lim_{x\to\infty} \psi_a(x, \mu) = 0. \tag{4.5}$$

Hence from equation (4.3), $\psi(x, \mu)$ must have the boundary conditions:

$$\text{(i)}\ \psi(0, \mu) = \boldsymbol{\phi}(-\nu, \mu), \mu > 0, \tag{4.6}$$

$$\text{(ii)}\ \lim_{x\to\infty} \psi(x, \mu) = \boldsymbol{\phi}(-\nu, \mu)\mathrm{e}^{x/\nu}. \tag{4.7}$$

Clearly the unique solution for $\psi(x, \mu)$ is

$$\psi(x, \mu) = \phi(-\nu, \mu)e^{x/\nu}. \tag{4.8}$$

Equations (4.3) and (4.8) then yield for the emergent Milne distribution

$$\psi_\nu(0, -\mu) = \phi(-\nu, -\mu) - \psi_a(0, -\mu), \mu > 0. \tag{4.9}$$

The emergent albedo distribution, $\psi_a(0, -\mu)$, can be expressed in terms of the S function. From equations (3.7), (3.12) and (4.4)

$$\psi_a(0, -\mu) = \frac{1}{\mu}\int_0^1 d\mu' \mathbf{S}(\mu', \mu)\phi(-\nu, \mu'); \tag{4.10}$$

and hence the emergent Milne distribution in terms of the S function is

$$\psi_\nu(0, -\mu) = \phi(\nu, \mu) - \frac{1}{\mu}\int_0^1 d\mu' \mathbf{S}(\mu', \mu)\phi(-\nu, \mu'). \tag{4.11}$$

In the same way we obtain the emergent distribution, $\psi_\nu^\dagger(0, -\mu)$ for the transposed transfer matrix $\tilde{\mathbf{C}}$

$$\psi_\nu^\dagger(0, -\mu) = \phi^\dagger(\nu, \mu) - \frac{1}{\mu}\int_0^1 d\mu' \mathbf{S}^\dagger(\mu', \mu)\phi^\dagger(-\nu, \mu'). \tag{4.12}$$

Once the S function has been determined, these equations could be used to obtain numerical values for the emergent Milne distribution. However, rather than calculating the S function from the U and V functions, and then substituting into equations (4.11) and (4.12), the emergent distribution can be expressed directly in terms of the U and V functions. This reduction of equation (4.11) leads to a far simpler expression for numerical evaluation.

From equations (3.13), (3.23), (3.25) and (3.26) it can be shown that the $\mathbf{S}(\mu_0, \mu)$ matrix may be written in the form

$$\mathbf{S}(\mu', \mu) = \mu\mu'\mathbf{U}_\eta(\mu)\mathbf{C}\mathbf{V}(\mu')\mathbf{D}_\eta(\mu, \mu'), \tag{4.13}$$

where the double index notation is again used to denote summation. Recall also that the eigenvector, $\phi(\nu, -\mu')$, $\nu > 0$, $\mu' > 0$, in view of equations (2.6) and (2.9), can be expressed as

$$\phi(\nu, -\mu') = \mathbf{F}(\nu, -\mu')\mathbf{C}\mathbf{a}(\nu). \tag{4.14}$$

If the diagonal matrix $\mathbf{M}_k(\nu, \mu, \mu_0)$ is defined as

$$\mathbf{M}_k(\nu, \mu, \mu') = \mathbf{D}_k(\mu, \mu')\mathbf{F}(\nu, -\mu'), \tag{4.15}$$

then the integrand of (4.11) is

$$\frac{1}{\mu}\mathbf{S}(\mu', \mu)\phi(\nu, -\mu) = \mu'\mathbf{U}_\eta(\mu)\mathbf{C}\mathbf{V}(\mu')\mathbf{M}_\eta(\nu, \mu, \mu')\mathbf{C}\mathbf{a}(\nu). \tag{4.16}$$

This expression can be considerably simplified by considering the explicit form of $\mathbf{M}_k(\nu, \mu, \mu')$. Substitution of $\mathbf{F}$ and $\mathbf{D}_k$ from equations (2.7) and (3.23) yields (in component form)

$$[\mathbf{M}_k(\nu, \mu, \mu')]_{ij} = \frac{\nu}{(\sigma_i\nu + \mu')(\sigma_i\mu + \sigma_k\mu')}\delta_{ij}. \tag{4.17}$$

The identity

$$\frac{1}{(\sigma_i\mu + \sigma_k\mu')(\sigma_i\nu + \mu')} = \frac{1}{\sigma_i}\frac{1}{\sigma_k\nu - \mu}\left[\frac{\sigma_k}{\sigma_i\mu + \sigma_k\mu'} - \frac{1}{\sigma_i\nu + \mu'}\right] \tag{4.18}$$

may be written as

$$\frac{1}{(\sigma_i\mu + \sigma_k\mu')(\sigma_i\nu + \mu')} = \frac{1}{\sigma_i}\frac{P}{\sigma_k\nu - \mu}\frac{\sigma_k}{\sigma_i\mu + \sigma_k\mu'} - \frac{1}{\sigma_i}\frac{P}{\sigma_k\nu - \mu}\frac{1}{\sigma_i\nu + \mu'}. \tag{4.19}$$

This result transforms equation (4.17) to

$$[\mathbf{M}_k(\nu, \mu, \mu')]_{ij} = \frac{1}{\sigma_k\nu - \mu}\left\{\frac{\sigma_k}{\sigma_i}\frac{1}{\sigma_i\mu + \sigma_k\mu'}\delta_{ij} - \frac{1}{\sigma_i}\frac{1}{\sigma_i\nu + \mu'}\delta_{ij}\right\}; \tag{4.20}$$

and since

$$\frac{\mu'}{b\mu + d\mu'} = \frac{1}{d}\left[1 - b\frac{\mu}{b\mu + d\mu'}\right], \tag{4.21}$$

for any real non-zero b and d, equation (4.20) yields

$$\mu'[\mathbf{M}_k(\nu, \mu, \mu')]_{ij} = \frac{1}{\sigma_k\nu - \mu}\left\{\frac{\nu}{\sigma_i\nu + \mu'}\delta_{ij} - \mu[\mathbf{D}_k(\mu, \mu')]_{ij}\right\}. \tag{4.22}$$

Substitution of this result into (4.16) and use of (4.14) gives for the emergent distribution

$$\boldsymbol{\psi}_\nu(0, -\mu) = \boldsymbol{\phi}(\nu, \mu) - \frac{P\nu}{\sigma_\eta\nu - \mu}\mathbf{U}_\eta(\mu)\mathbf{C}\int_0^1 \mathrm{d}\mu'\mathbf{V}(\mu')\boldsymbol{\phi}(\nu, -\mu') + \frac{P\mu\nu}{\sigma_\eta\nu - \mu}\mathbf{U}_\eta(\mu)\mathbf{C}\int_0^1 \mathrm{d}\mu'\mathbf{V}(\mu')\mathbf{D}_\eta(\mu, \mu')\mathbf{Ca}(\nu). \tag{4.23}$$

This last term may be further simplified by considering the nonlinear integral equation for the **U**-function. The transpose of equation (3.26) is

$$\mathbf{U}(\mu) = \mathbf{E} + \mu\mathbf{U}_\eta(\mu)\mathbf{C}\int_0^1 \mathrm{d}\mu'\mathbf{V}(\mu')\mathbf{D}_\eta(\mu, \mu'), \tag{4.24}$$

or solely in terms of $\mathbf{U}_k(\mu)$

$$\mathbf{U}_k(\mu) = \mathbf{E}_k + \mu\mathbf{U}_k(\mu)\mathbf{C}\int_0^1 \mathrm{d}\mu'\mathbf{V}(\mu')\mathbf{D}_k(\mu, \mu'), \tag{4.25}$$

where $[\mathbf{E}_k]_{ij} = \delta_{ij}\delta_{ik}$.

Hence the emergent distribution is

$$\boldsymbol{\psi}_\nu(0, -\mu) = \boldsymbol{\phi}(\nu, \mu) - \frac{P\nu}{\sigma_\eta\nu - \mu}\mathbf{U}_\eta(\mu)\mathbf{C}\int_0^1 \mathrm{d}\mu'\mathbf{V}(\mu')\boldsymbol{\phi}(\nu, -\mu') + \frac{P\nu}{\sigma_\eta\nu - \mu}(\mathbf{U}_\eta(\mu) - \mathbf{E}_\eta)\mathbf{Ca}(\nu). \tag{4.26}$$

However, from equation (2.6)

$$\boldsymbol{\phi}(\nu, \mu) = \frac{P\nu}{\sigma_\eta\nu - \mu}\mathbf{E}_\eta\mathbf{Ca}(\nu) + \mathbf{G}(\nu, \mu)\boldsymbol{\lambda}(\nu). \tag{4.27}$$

Combining the last term in equation (4.26) with $\boldsymbol{\phi}(\nu, \mu)$ the emergent distribution simplifies to

$$\boldsymbol{\psi}_\nu(0, -\mu) = \mathbf{G}(\nu, \mu)\boldsymbol{\lambda}(\nu) + \frac{P\nu}{\sigma_\eta \nu - \mu}\mathbf{U}_\eta(\mu)\mathbf{C}\left[\mathbf{E} - \int_0^1 \mathrm{d}\mu' \mathbf{V}(\mu')\mathbf{F}(\nu, -\mu')\mathbf{C}\right]\mathbf{a}(\nu). \tag{4.28}$$

Finally writing this equation completely in terms of the matrix $\mathbf{U}(\mu)$, the emergent distribution of the generalized Milne problem is given by the very simple equation

$$\boldsymbol{\psi}_\nu(0, -\mu) = \mathbf{G}(\nu, \mu)\boldsymbol{\lambda}(\nu) + P\mathbf{F}(\nu, \mu)\mathbf{U}(\mu)\mathbf{h}(\nu), \ \nu > 0, \ \mu > 0, \tag{4.29}$$

where the constant vector $\mathbf{h}(\nu)$ is

$$\mathbf{h}(\nu) = \mathbf{C}\left\{\mathbf{E} - \int_0^1 \mathrm{d}\mu' \mathbf{V}(\mu')\mathbf{F}(\nu, -\mu')\mathbf{C}\right\}\mathbf{a}(\nu). \tag{4.30}$$

Using the above expression the emergent distribution for a multigroup Milne problem is easily evaluated. Such calculations have been performed for various cases of borated water and for enriched uranium systems (SHULTIS, 1968).

(b) *Half-space Green's function*

As a final example of the use of the generalized **S** function technique, the half-space Green's function problem will be solved. The half-space Green's function, with the source neutrons belonging solely to the ith energy group, $\mathbf{g}_i(x_0, \mu_0; x, \mu)$, is defined by the equation

$$\left(\mu \frac{\partial}{\partial x}\mathbf{E} + \boldsymbol{\Sigma}\right)\mathbf{g}_i(x_0, \mu_0; x, \mu) = \mathbf{C}\int_0^1 \mathrm{d}\mu' \mathbf{g}_i(x_0, \mu_0; x, \mu') + \delta(\mu - \mu_0)\delta(x - x_0)\mathbf{e}_i, \quad x_0 > 0, \tag{4.31}$$

with the boundary conditions

$$\text{(i)} \ \mathbf{g}_i(x_0, \mu_0; 0, \mu) = 0, \ \mu > 0, \tag{4.32}$$

$$\text{(ii)} \lim_{x \to \infty} \mathbf{g}_i(x_0, \mu_0; x, \mu) = 0. \tag{4.33}$$

The first step towards obtaining the solution, is to determine the emergent distribution, $\mathbf{g}_i(x_0, \mu_0; 0, -\mu)$, $\mu > 0$. Consider the Green's function to be composed of two parts:

$$\mathbf{g}_i(x_0, \mu_0; x, \mu) = \mathbf{g}_i{}^\infty(x_0, \mu_0; x, \mu) + \boldsymbol{\psi}_a(x, \mu), \ x_0 > 0, \tag{4.34}$$

where $\mathbf{g}_i{}^\infty(x_0, \mu_0; x, \mu)$ is the known infinite medium Green's function (YOSHIMURA, 1968). The albedo problem solution $\boldsymbol{\psi}_a(x, \mu)$, satisfies the homogeneous transport equation with the boundary conditions

$$\text{(i)} \ \boldsymbol{\psi}_a(0, \mu) = -\mathbf{g}_i{}^\infty(x_0, \mu_0; 0, \mu), \ \mu > 0, \tag{4.35}$$

$$\text{(ii)} \lim_{x \to \infty} \boldsymbol{\psi}_a(x, \mu) = 0. \tag{4.36}$$

Clearly $\mathbf{g}_i(x_0, \mu_0; x, \mu)$ defined by equation (4.34) satisfies equation (4.31) and has the required boundary conditions.

The emergent distribution of the albedo solution, $\psi_a(0, -\mu)$, can be expressed in terms of its incident distribution from equation (3.7). Hence from equation (4.34) the emergent distribution $\mathbf{g}_i(x_0, \mu_0; 0, -\mu)$ is

$$\mathbf{g}_i(x_0, \mu_0; 0, -\mu) = \mathbf{g}_i^{\infty}(x_0, \mu_0; 0, -\mu) - \frac{1}{\mu}\int_0^1 d\mu' \mathbf{S}(\mu', \mu)\mathbf{g}_i^{\infty}(x_0, \mu_0; 0, \mu'). \tag{4.37}$$

5. COMPLETE SOLUTIONS TO THE HALF-SPACE PROBLEMS

Once the emergent distributions of half-space problems are known, the use of *full-range* completeness and orthogonality of the eigenvectors readily yield the coefficients of an eigenfunction expansion of the flux inside the medium. In the following the complete solution to the half-space albedo, Milne and Green's function problems will be obtained.

(a) *Albedo problem*

First, we seek the complete solution for the ith albedo problem in the form

$$\psi_i(0, \mu_0; x, \mu) = \sum_{s=1}^{M} \alpha(\nu_{0s})\phi(\nu_{0s}, \mu)e^{-x/\nu_{0s}} + \sum_{s=1}^{M} \alpha(-\nu_{0s})\phi(-\nu_{0s}, \mu)e^{x/\nu_{0s}}$$
$$+ \sum_{j=1}^{N}\int_{\eta_{j-1}}^{\eta_j} \sum_{m=j}^{N} \{A_j^m(\nu)\phi_j^m(\nu, \mu)e^{-x/\nu} + A_j^m(-\nu)\phi_j^m(-\nu, \mu)e^{x/\nu}\}\, d\nu,$$
$$\mu\varepsilon(-1, 1). \tag{5.1}$$

where

$$\eta_j \equiv \sigma_j^{-1} \quad \text{and} \quad \eta_0 \equiv 0.$$

In view of the full-range completeness of the eigenvectors $\phi(\pm\nu, \mu)$, $\nu\varepsilon(0, 1)$, $\nu = \pm\nu_{0s}$, we can now determine the expansion coefficients in equation (5.1) so that the above equation with $x = 0$, equals the known surface distribution.

The expansion coefficients $\alpha(\pm\nu_{0s})$ and $A_j^m(\pm\nu)$ are readily obtained by applying full-range orthogonality relations (equations (2.19) and (2.20)) and equations (3.1) and (3.32). Explicitly

$$\alpha(\pm\nu_{0s}) = \pm\frac{\mu_0}{N_s}\tilde{\phi}^{\dagger}(\pm\nu_{0s}, \mu_0)\mathbf{e}_i \mp \frac{1}{N_s}\int_0^1 d\mu\, \tilde{\phi}^{\dagger}(\pm\nu_{0s}, -\mu)\mathbf{S}(\mu_0, \mu)\mathbf{e}_i, \tag{5.2}$$

and

$$A_j^m(\pm\nu) = \pm\frac{\mu_0}{N_j^m(\nu)}\tilde{\phi}_j^{m\dagger}(\pm\nu, \mu)\mathbf{e}_i \mp \frac{1}{N_j^m(\nu)}\int_0^1 d\mu\, \tilde{\phi}_j^{m\dagger}(\pm\nu, -\mu)\mathbf{S}(\mu_0, \mu)\mathbf{e}_i. \tag{5.3}$$

In Appendix A it is shown that the $\mathbf{S}(\mu_0, \mu)$ function satisfies certain relationships with the eigenvectors. From equations (A.5) and (3.40) it is seen at once that $\alpha(-\nu_{0s})$ and $A_j^m(-\nu)$, $\nu > 0$, are identically zero! Thus we see that the boundary condition at infinity, equation (3.6), is satisfied. Therefore, the expansion equation (5.1) with $\alpha(-\nu_{0s})$ and $A_j^m(-\nu)$ set equal to zero represents the complete solution of the problem. This result in turn implies the half-range completeness of the eigenvectors $\phi(\nu, \mu)$, $\nu\varepsilon(0, 1)$, $\nu = \nu_{0s}$.

(b) *Milne problem*

Since half-space albedo problems can always be expanded in terms of only the decaying eigenfunctions, equation (4.3) shows that the solution for the generalized Milne problem can be written as

$$\boldsymbol{\psi}_\nu(x, \mu) = \boldsymbol{\phi}(-\nu, \mu)e^{x/\nu} + \sum_{s=1}^{M} \alpha(\nu_{0s})\boldsymbol{\phi}(\nu_{0s}, \mu)e^{-x/\nu_{0s}} + \sum_{j=1}^{N} \int_{\eta_{j-1}}^{\eta_j} d\nu' \left\{ \sum_{m=j}^{N} \mathbf{A}_j{}^m(\nu')\boldsymbol{\phi}_j{}^m(\nu', \mu) \right\} e^{-x/\nu'}. \quad (5.4)$$

Setting $x = 0$ in this equation and using equations (4.1) and (4.29) one finds from the full-range orthogonality relations that

$$\alpha(\nu_{0s}) = -\frac{1}{N_s} \int_0^1 d\mu\mu\tilde{\boldsymbol{\phi}}^\dagger(\nu_{0s}, \mu)\boldsymbol{\psi}_\nu(0, -\mu)$$

$$= -\frac{1}{N_s} \int_0^1 d\mu\mu\tilde{\boldsymbol{\phi}}^\dagger(\nu_{0s}, \mu)\{\mathbf{G}(\nu, \mu)\boldsymbol{\lambda}(\nu) + P\mathbf{F}(\nu, \mu)\mathbf{U}(\mu)\mathbf{h}(\nu)\} \quad (5.5)$$

and

$$\mathbf{A}_j{}^m(\nu') = -\frac{1}{N_j{}^m(\nu')} \int_0^1 d\mu\mu\tilde{\boldsymbol{\phi}}_j{}^{\dagger m}(\nu', \mu)\{\mathbf{G}(\nu, \mu)\boldsymbol{\lambda}(\nu) + P\mathbf{F}(\nu, \mu)\mathbf{U}(\mu)\mathbf{h}(\nu)\}. \quad (5.6)$$

Often the Milne problem of most interest is the one related to the largest discrete eigenvalue, ν_l. The asymptotic behavior of this particular problem for large x is

$$\boldsymbol{\psi}_l{}^{as}(x, \mu) = \boldsymbol{\phi}(-\nu_l, \mu)e^{x/\nu_l} + \alpha(\nu_l)\boldsymbol{\phi}(\nu_l, \mu)e^{-x/\nu_l}. \quad (5.7)$$

A quantity of interest for this problem is the extrapolated end point, x_0, defined such that

$$\rho^{as}(x_0) = \int_{-1}^{1} d\mu\boldsymbol{\psi}_{\nu l}{}^{as}(x_0, \mu) = \mathbf{a}(\nu_l)e^{x_0/\nu_l} + \alpha(\nu_l)\mathbf{a}(\nu_l)e^{-x_0/\nu_l} = 0. \quad (5.8)$$

Solving for x_0 and substituting for $\alpha(\nu_l)$ from equation (5.5), the extrapolated end point is

$$x_0 = -\frac{\nu_l}{2} \ln\left\{ \frac{1}{N_l} \int_0^1 d\mu\mu\tilde{\boldsymbol{\phi}}^\dagger(\nu_l, \mu)\boldsymbol{\psi}_\nu(0, -\mu) \right\}, \quad (5.9)$$

or in terms of the $\mathbf{U}(\mu)$ matrix

$$x_0 = -\frac{\nu_l}{2} \ln\left\{ \frac{1}{N_l} \int_0^1 d\mu\mu\tilde{\mathbf{a}}^\dagger(\nu_l)\mathbf{CF}^2(\nu_l, \mu)\mathbf{U}(\mu)\mathbf{h}(\nu_l) \right\}. \quad (5.10)$$

(c) *Green's function*

From equation (4.34) we see that the complete solution of the ith half-space Green's function can be written as

$$\mathbf{g}_i(x_0, \mu_0; x, \mu) = \mathbf{g}_i{}^\infty(x_0, \mu_0; x, \mu) + \sum_{s=1}^{M} \alpha(\nu_{0s})\boldsymbol{\phi}(\nu_{0s}, \mu)e^{-x/\nu_{0s}} + \sum_{j=1}^{N} \int_{\eta_{j-1}}^{\eta_j} d\nu \left\{ \sum_{m=j}^{N} A_j{}^m(\nu)\boldsymbol{\phi}_j{}^m(\nu, \mu) \right\} e^{-x/\nu}. \quad (5.11)$$

Using full-range orthogonality relations and equations (4.36) the expansion coefficients are readily found to be

$$\alpha(\nu_{0s}) = -\frac{1}{N_s}\int_0^1 d\mu\mu\tilde{\boldsymbol{\Phi}}^{\dagger}(\nu_{0s}.\ \mu) \times \left\{\mathbf{G}_i^{\infty}(x_0,\ \mu_0;\ 0,\ \mu) - \frac{1}{\mu}\int_0^1 d\mu' \mathbf{S}(\mu',\ \mu)\mathbf{G}_i^{\infty}(x_0,\ \mu_0;\ 0,\ \mu')\right\}. \quad (5.12)$$

and

$$A_j^m(\nu) = -\frac{1}{N_j^m(\nu)}\int_0^1 d\mu\mu\tilde{\boldsymbol{\Phi}}_j^{\dagger m}(\nu,\ \mu) \times \left\{\mathbf{G}_i^{\infty}(x_0,\ \mu_0;\ 0,\ \mu) - \frac{1}{\mu}\int_0^1 d\mu' \mathbf{S}(\mu',\ \mu)\mathbf{G}_i^{\infty}(x_0,\ \mu_0;\ 0,\ \mu')\right\}. \quad (5.13)$$

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APPENDIX A

As previously mentioned, equations (3.15) and (3.17) are the multigroup generalization of Chandrasekhar's one-speed H function nonlinear integral equation. Since this one-speed equation does not have a unique solution (MULLIKIN, 1964a,b; PAHOR and KUSCER, 1966), one suspects that the nonlinear integral equation for $\mathbf{S}(\mu_0,\ \mu)$ and therefore also for the equations for $\mathbf{U}(\mu)$ and $\mathbf{V}(\mu)$ are not uniquely soluble. Proceeding as in the one-speed case (PAHOR and KUSCER, 1966) or in the case of the degenerate kernal approximation (PAHOR, 1969) it can indeed be shown, that for each discrete root, ν_{0s}, equation (3.11) admits a 'non-physical' solution denoted by $\mathbf{S}_{0s}(\mu_0,\ \mu)$. It is equal to

$$\mathbf{S}_{0s}(\mu_0,\ \mu) = \mathbf{S}(\mu_0,\ \mu) + \frac{2\mu\ \mu_0}{\nu_{0s}\beta^2(\nu_{0s})}\psi\nu_{0s}(0,\ -\mu)\tilde{\boldsymbol{\Phi}}^{\dagger}_{\nu_{0s}}(0,\ -\mu_0), \quad (A.1)$$

where

$$\beta(\nu_{0s}) = \int_0^1 d\mu\tilde{\boldsymbol{\Phi}}^{\dagger}_{\nu_{0s}}(0,\ -\mu)\mathbf{C}\int_0^1 d\mu\boldsymbol{\Phi}_{\nu_{0s}}(0,\ -\mu). \quad (A.2)$$

It is possible, however, to give a set of conditions which must be satisfied by the physical solutions $\mathbf{S}(\mu_0, \mu)$ or $\mathbf{U}(\mu)$ and $\mathbf{V}(\mu)$. The eigenfunctions $\boldsymbol{\phi}(\nu, \mu)\, e^{-x/\nu}$, $\mathrm{Re}\{\nu\} > 0$, are solutions to the transport equation; and since they tend to zero for large x, they are solutions to half-space albedo problems with incident distributions given by $\boldsymbol{\phi}(\nu, \mu)$, $\mu > 0$. Thus from equations (3.7) and (3.12), the $\mathbf{S}(\mu_0, \mu)$ function must satisfy

$$\boldsymbol{\phi}(\nu, -\mu) = \frac{1}{\mu}\int_0^1 d\mu' \mathbf{S}(\mu', \mu)\boldsymbol{\phi}(\nu, \mu'), \mu > 0, Re\{\nu\} > 0. \tag{A.3}$$

Integrating this condition over μ from 0 to 1 and using equation (3.15) and (2.17), we obtain

$$\int_{-1}^1 d\mu \boldsymbol{\phi}(\nu, \mu) \equiv \mathbf{a}(\nu) = \int_0^1 d\mu' \mathbf{V}(\mu')\boldsymbol{\phi}(\nu, \mu'). \tag{A.4}$$

Similarly by considering $\boldsymbol{\phi}^\dagger(\nu, \mu)\, e^{-x/\nu}$, $Re\{\nu\} > 0$, as albedo problem solutions of the transport equation for a transposed transfer matrix, $\tilde{\mathbf{C}}$, from equations (A.3) and (2.17) one has the conditions on the $\mathbf{S}^\dagger(\mu_0, \mu)$ matrix

$$\boldsymbol{\phi}^\dagger(-\nu, \mu) = \mu\frac{1}{\mu}\int_0^1 d\mu' \mathbf{S}^\dagger(\mu', \mu)\boldsymbol{\phi}^\dagger(-\nu, -\mu'), Re\{\nu\} > 0. \tag{A.5}$$

Again integrating over μ from 0 to 1 and using equations (3.40), (3.14) and (2.17), equation (A.5) yields

$$\int_{-1}^1 d\mu \boldsymbol{\phi}^\dagger(-\nu, \mu) = \mathbf{a}^\dagger(\nu) \equiv \int_0^1 d\mu \tilde{\mathbf{U}}(\mu)\boldsymbol{\phi}^\dagger(-\nu, -\mu), Re\{\nu\} > 0. \tag{A.6}$$

Equations (A.4) and (A.6) for the discrete roots ν_{0s}, $s = 1 \ldots M$ are $2M$ conditions which the physical $\mathbf{U}(\mu)$ and $\mathbf{V}(\mu)$ functions must satisfy. In one-speed case these equations become identical, and it has been proved that they are a sufficient condition to uniquely specify the real physical H function given by the nonlinear integral equation (3.17), (Pahor, 1966; Pahor and Kuscer, 1966). Also for the degenerate kernal approximation, Pahor (1967), using a corresponding set of discrete eigenfunction conditions, proved that these conditions were sufficient for uniquely specifying his generalized S-function. Although it has not been possible to show that the discrete root conditions for the general multigroup case are a set of sufficient conditions, it is felt that they are a severe restriction on the possible solutions of equations (3.17) and (3.19), and in all likelihood they are sufficient. Therefore, in iterating equations (3.17) and (3.19), the conditions (A.4) and (A.6) must be used as a check. At the same time, an estimate of the accuracy of the iterations can be obtained from these conditions.