JOURNAL OF ECONOMIC THEORY 1, 99–103 (1969)

# A Simple Remark on the Second Optimality Theorem of Welfare Economics

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1. As is well known, the assumptions required to establish the Second Optimality Theorem are much stronger than those required for the First.[2] Aside from the non-externality assumptions common to both, the only assumption required to establish that a competitive equilibrium is a Pareto optimum is that the given competitive equilibrium is a point of local nonsatiation for each consumer. But to establish that a given Pareto optimum is a competitive equilibrium, convexity assumptions are needed to assure that the aggregate supply set and the aggregate "no-worse-than" set can be separated by a hyperplane. The present note generalizes the Second Optimality Theorem in a very simple way. The assumptions of the revised Second Theorem are satisfied in cases where consumption externalities make the First Theorem inapplicable.

2. Notation here follows for the most part that of Debreu [2], whose statement of the Second Theorem will now be generalized. Briefly, there are $m$ consumers and $n$ producers, $X_i$ is the $i$th consumer's consumption set, completely preordered by $\underset{i}{\gtrsim}$ $Y_j$, is the $j$th producer's production set, $Y = \Sigma\, Y_j$ is the total production set. Debreu proves the following ([2], Theorem 6.4(1), p. 95):

Let $E$ be an economy such that: for every $i$,

(a) $X_i$ is convex;

(b.1) for every $x_i'$ in $X_i$, the sets $\{x_i \in X_i | x_i \underset{i}{\gtrsim} x_i'\}$ and $\{x_i \in X_i | x_i \underset{i}{\lesssim} x_i'\}$ are closed in $X_i$,

(b.2) if $x_i^1$ and $x_i^2$ are two points of $X_i$ and if $t$ is a real number, $0 < t < 1$, then $x_i^2 \underset{i}{\succ} x_i^1$ implies $tx_i^2 + (1-t)x_i^1 \underset{i}{\succ} x_i^1$;

[1] This note was written while the author was visiting professor at the University of California, San Diego. An earlier conversation with K. J. Arrow, in which a Remark became a Simple Remark, is gratefully acknowledged.

[2] The handy numerical titles are due to K. J. Arrow [1].

(c) $Y$ is convex.
Given an optimum $((x_i^*), (y_j^*))$ where some $x_i^*$ is not a satiation consumption, there is a price system $p$ different from 0 such that:
($\alpha$) $x_i^*$ minimizes $p \cdot x_i$ on $\{x_i \in X_i | x_i \underset{i}{\succsim} x_i^*\}$, for every $i$,
($\beta$) $y_j^*$ maximizes $p \cdot y_j$ on $Y_j$, for every $j$.
If, in addition, there is for every $i$ an $x_i \in X_i$ such that $p \cdot x_i < p \cdot x_i^*$, then $((x_i^*), (y_j^*))$ is an equilibrium relative to $p$.

Let $S = \prod_i X_i$, the Cartesian product of the consumption sets, with $s$ as a typical element. For every consumer $i$, let there be given a complete preordering $\overset{\sigma}{\underset{i}{\succsim}}$ of $S$ by $i$. And, as before, let $X_i$ be completely preordered by $\underset{i}{\succsim}$. We must now distinguish between (Pareto) optima relative to preorderings $\left(\overset{\sigma}{\underset{i}{\succsim}}\right)$ and optima relative to preorderings $\left(\underset{i}{\succsim}\right)$; both of these sets of consumer preorderings induce partial preorderings of $S$ according to the Pareto principle. If

$$s = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)$$

is an element of $S$, denote by $s_i'$ the vector $s$ with $x_i$ replaced by $x_i'$, and by $s'$ the vector $(x_1', \ldots, x_m')$. To Debreu's assumptions on consumers, add the following:

(b.3) For every $s \in S$, $s' \in S$,

$$x_i' \underset{i}{\succ} x_i \text{ implies } s_i' \overset{\sigma}{\underset{i}{\succ}} s,$$

and

$$x_i' \underset{i}{\succsim} x_i \text{ implies } s_i' \overset{\sigma}{\underset{k}{\succsim}} s, \quad \text{for } k = 1 \ldots m.$$

**3.** *Simple Remark.* The above theorem holds when its assumptions are amended by inclusion of (b.3) and its conclusion is amended by inserting the parenthetical clarification "$\left(\text{relative to } \left(\overset{\sigma}{\underset{i}{\succsim}}\right)\right)$" after the word "optimum".

To establish the validity of this remark, it suffices to note that an optimum relative to $\left(\overset{\sigma}{\underset{i}{\succsim}}\right)$ is an optimum relative to $\left(\underset{i}{\succsim}\right)$, hence a price system sustaining the latter also sustains the former.[3] Suppose, therefore,

[3] Note that the $(\underset{i}{\succsim})$ preferences are still the relevant ones for the purpose of defining equilibrium relative to a price system. Though each consumer's satisfaction may depend on the consumption of all, each is free to choose only his own.

a given attainable state with consumption plans represented by $s^* \in S$ is an optimum relative to $\left(\overset{\sigma}{\underset{i}{\succsim}}\right)$. And suppose, contrary to the assertion just made, that there is another attainable state involving consumptions $x_i'$, which satisfies $x_i' \underset{i}{\succsim} x_i^*$ for all $i$, with strict preference holding at least once. By changing one consumer's consumption vector at a time, we transform the list $s^*$ of consumption vectors into the list $s'$. At each step, the consumer whose vector is changed regards himself as no worse off in the $\left(\underset{i}{\succsim}\right)$ sense, and by assumption (b.3), all consumers concur in considering the new list at least as desirable as the old in the $\left(\overset{\sigma}{\underset{i}{\succsim}}\right)$ sense. Therefore, by transitivity, $s' \overset{\sigma}{\underset{i}{\succsim}} s^*$ for all $i$. Any consumer for whom $x_i' \underset{i}{\succ} x_i^*$ strictly prefers $s'$ to $s^*$, since he enjoys an actual improvement in well being in the $\left(\overset{\sigma}{\underset{i}{\succsim}}\right)$ sense when his consumption vector is changed. This implies that $s^*$ was not an optimum relative to $\left(\overset{\sigma}{\underset{i}{\succsim}}\right)$, a contradiction which establishes that $s^*$ is an optimum relative to $\left(\underset{i}{\succsim}\right)$.

**4.** Note that the usual statement of the Second Optimality Theorem is contained in the amended theorem as the special case in which each consumer's preferences among elements of $S$ actually depend only on his own consumption vector, $x_i$. An easily understood case covered by the amended theorem, but not by the original formulation, is that in which each consumer's preferences among elements of $S$ are representable by a Bergson welfare function of individualist type, $W^i(u^1(x_1), \ldots, u^m(x_m))$. Here, of course, the functions $u^i$ represent the own-consumption preferences of the various consumers, $\left(\underset{i}{\succsim}\right)$, and the functions $W^i$ are monotone increasing in all arguments. If this last condition is altered to require only that $W^i$ be monotone increasing in $u^i$, and nondecreasing in its other arguments, a class of preference structures is characterized which includes both the egoistic-individualist and the altruistic-individualist extremes.

The amended theorem points up the fact that there is more scope for reliance upon the price mechanism in a community of men of good will than in a community of men of ill will—provided that the good will is accompanied by respect for each other's tastes. If I am pleased when others become better off, according to their own tastes, I can delegate to others the task of deciding how to spend their incomes, and concern myself only with trying to bring about an equitable distribution of income. On the other hand, if I enjoy seeing people suffer, I will be frustrated by the "inefficiency" of any system which leaves people free to determine the spending of their incomes, however limited. Not content with attempts to redistribute income in favor of myself, I will support sumptuary legislation to deny to others the consumption of the things they like best.

**5.** It might be asked whether the inclusion of gifts among the possible uses of an individual's income would make it possible to amend the First Optimality Theorem in a similar way. That is, given a competitive equilibrium in which no individual would prefer to transfer any portion of his own income to anyone else, is this situation necessarily a Pareto optimum in the $\left(\underset{i}{\overset{\sigma}{\gtrsim}}\right)$ sense?[4] The answer is no, as the following example illustrates in a rather stark fashion.

Consider an economy in which all consumers have identical consumption sets and convex, representable own-consumption preferences, and each consumer $i$ has the following preference structure on $S$: There is an indifference curve in the common consumption set which defines the boundary of "poverty", consumption vectors dispreferred to those on this indifference curve represent poverty, those on or above it do not. Any element in $S$ such that no consumer is in poverty is strictly preferred to any element such that at least one consumer is in poverty. Within the two subsets "no one in poverty" and "someone in poverty", consumer $i$'s $\underset{i}{\overset{\sigma}{\gtrsim}}$ preferences simply reflect the standing of his consumption $x_i$ according to the commonly held $\gtrsim$ preferences. Clearly, this preference system satisfies (b.3).

If this economy has an attainable state in which no one is in poverty, then a Pareto optimum relative to $\left(\underset{i}{\overset{\sigma}{\gtrsim}}\right)$ can only be a position in which no one is in poverty. On the other hand, there may easily be competitive equilibria which, in the absence of gifts, would leave some individuals in poverty. Does the possibility of gifts make a difference? Only if some single individual can, by redistributing *his own income*, lift everyone out of poverty (without going into poverty himself). Otherwise, no single individual can improve his situation in any degree by giving any part of his income away.

[4] It is assumed here, of course, that (b.3) is satisfied, the economy is otherwise externality free, and each consumer is locally nonsatiated in his own consumption set.

## REFERENCES

*1.* ARROW, K. J. Uncertainty and the welfare economics of medical care. *American Economic Review* **53** (1963), 941–73.

*2.* DEBREU, G. *Theory of Value*, John Wiley and Sons, Inc., New York, 1959.

RECEIVED: October 21, 1968.