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THE QUARTIC ISOBARIC MULTIPLET MASS EQUATION

JOACHIM JÄNECKE
The University of Michigan, Ann Arbor †

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Abstract: The quadratic isobaric multiplet mass equation must be replaced by an expression which is quartic in T_z if the electrostatic interaction is treated in second-order perturbation theory. The coefficients of the cubic and quartic terms as well as the corrections to the constant, linear and quadratic terms are related to the off-diagonal reduced Coulomb matrix elements. A numerical analysis of the experimental Coulomb displacement energies in the 1p and 1d2s shell in terms of Hecht's (first-order) Coulomb energy equations showed systematic deviations for the $T = \frac{3}{2}$ multiplets, which were attributed to higher-order perturbations. For the $A = 9$ quadruplet the major perturbations appear to result from a lower-lying $T = \frac{1}{2}$ state (or states) which increases the excitation energies of the $T = \frac{3}{2}$ states in ^{9}B and ^{9}Be by about 105 and 93 keV, respectively, while the energies of ^{9}C and ^{9}Li are essentially unaffected. The constant and linear terms are changed very little, but the quadratic term is decreased by about 15 %. Small cubic and quartic terms are generated. It is concluded that the quadratic isobaric multiplet mass equation often works well, not because first-order perturbation theory is a good approximation, but because higher-order perturbations are mostly absorbed by the three coefficients. Particularly the coefficient of the quadratic term may be affected considerably.

1. Introduction

Recently, Hecht [1,2]) has derived several Coulomb energy equations for states of good isospin T. The equations were derived in first-order perturbation theory for simple shell-model states in two extreme coupling schemes. A comparison between the experimental and calculated Coulomb displacement energies in the 1p shell, the the 1d2s shell and the $1f_{\frac{7}{2}}$ shell showed very good agreement [3-5]) except for a few cases. These cases included the precisely measured [6,7]) $A = 9$ isobaric quadruplet. The agreement with the calculated energies was very poor. The attempt to interpret this discrepancy was the major motivation for the present study.

2. The quadratic isobaric multiplet mass equation

The energies of the $2T+1$ members of an isobaric multiplet are shifted relative to each other mostly because of the electrostatic interaction between the protons in the nucleus. For all charge-dependent two-body forces of tensorial rank of two or less, one obtains in first-order perturbation theory the well-known quadratic isobaric multiplet mass equation [8])

$$M(A, T, T_z) = a(A, T) + b(A, T)T_z + c(A, T)T_z^2. \tag{1}$$

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The T_z dependence is factored out with the use of the Wigner-Eckart theorem. The coefficients $a(A,T)$, $b(A,T)$ and $c(A,T)$ are related to the scalar, vector and tensor Coulomb energies $E^{(0)}(A, T)$, $E^{(1)}(A, T)$ and $E^{(2)}(A, T)$, which are simply related to the diagonal reduced matrix elements of the scalar, vector, and tensor Coulomb energy operators. Eq. (1) has been successfully applied to many isobaric multiplets [7]).

3. The quartic isobaric multiplet mass equation

In a second-order perturbation treatment of the electrostatic interaction (and of all other charge-dependent interactions of tensorial rank of two or less), the correction term

$$\Delta M(A, T, T_z) = \sum_{\substack{\alpha \\ T'=T,\, T\pm 1,\, T\pm 2}} \Delta M_{T'-T}(\alpha, T, T_z) = -\sum_{\substack{\alpha \\ T'=T,\, T\pm 1,\, T\pm 2}} \frac{\langle \alpha T' T_z | H_C | T T_z \rangle^2}{\varepsilon^{(0)}(\alpha, T') - \varepsilon^{(0)}(T)} \quad (2)$$

has to be added to eq. (1). The energy denominators represent the unperturbated energy separations between the perturbing states and the analogue state under consideration. It should be noted that the energy denominators are *independent* of T_z. The Coulomb energy operator H_C leads to a strong T_z dependence, but the coefficients of the T_z dependent terms are essentially independent of T and α. The quantity α stands for all quantum numbers other than T' and T_z of the perturbing states (which, of course, must have the same spin and parity as the members of the isobaric multiplet). The contributions come from states with the same isospin and from states which differ in isospin by one and by two units. States with $T' = T$ contribute through the isoscalar, isovector, and isotensor Coulomb energy operators, while states with $T' = T \pm 1$ contribute only through the isovector and isotensor operators, and states with $T' = T \pm 2$ contribute only through the isotensor operator. The T_z dependence in eq. (2) can be factored out with the use of the Wigner-Eckart theorem, and the following expressions for the five cases with $T' - T = -2, -1, 0, 1, 2$ are obtained:

$$\Delta M_{-2}(\alpha, T, T_z) = -\frac{[e^{(2)}_{-2}(\alpha, T)\sqrt{T^2 - T_z^2}\sqrt{(T-1)^2 - T_z^2}]^2}{\varepsilon^{(0)}(\alpha, T-2) - \varepsilon^{(0)}(T)}, \quad (3)$$

$$\Delta M_{-1}(\alpha, T, T_z) = -\frac{[e^{(1)}_{-1}(\alpha, T)\sqrt{T^2 - T_z^2} - e^{(2)}_{-1}(\alpha, T) T_z \sqrt{T^2 - T_z^2}]^2}{\varepsilon^{(0)}(\alpha, T-1) - \varepsilon^{(0)}(T)}, \quad (4)$$

$$\Delta M_0(\alpha, T, T_z) = -\frac{[e^{(0)}_0(\alpha, T) + e^{(1)}_0(\alpha, T) T_z + e^{(2)}_0(\alpha, T)(3T_z^2 - T(T+1))]^2}{\varepsilon^{(0)}(\alpha, T) - \varepsilon^{(0)}(T)}, \quad (5)$$

$$\Delta M_{+1}(\alpha, T, T_z) = -\frac{[e^{(1)}_{+1}(\alpha, T)\sqrt{(T+1)^2 - T_z^2} - e^{(2)}_{+1}(\alpha, T) T_z \sqrt{(T+1)^2 - T_z^2}]^2}{\varepsilon^{(0)}(\alpha, T+1) - \varepsilon^{(0)}(T)}, \quad (6)$$

$$\Delta M_{+2}(\alpha, T, T_z) = -\frac{[e^{(2)}_{+2}(\alpha, T)\sqrt{(T+2)^2 - T_z^2}\sqrt{(T+1)^2 - T_z^2}]^2}{\varepsilon^{(0)}(\alpha, T+2) - \varepsilon^{(0)}(T)}. \quad (7)$$

The quantities $e^{(0)}_{T'-T}(\alpha, T)$, $e^{(1)}_{T'-T}(\alpha, T)$ and $e^{(2)}_{T'-T}(\alpha, T)$ are related to the *off-diagonal* reduced matrix elements of the isoscalar, isovector and isotensor Coulomb energy operators, respectively. The relationships are similar to those between $E^{(0)}(A, T)$, $E^{(1)}(A, T)$ and $E^{(2)}(A, T)$ and the *diagonal* reduced matrix elements.

The combination of eqs. (3)–(7) and eq. (2) immediately leads to the result that in a second-order perturbation treatment the quadratic isobaric multiplet mass equation (1) has to be replaced by the quartic isobaric multiplet mass equation

$$M(A, T, T) = \tilde{a}(A, T)+\tilde{b}(A, T)T_z+\tilde{c}(A, T)T_z^2+\tilde{d}(A, T)T_z^3+\tilde{e}(A, T)T_z^4. \quad (8)$$

The coefficients of the cubic and quartic terms as well as the corrections to the coefficients of the constant, linear and quadratic terms ($\Delta a = \tilde{a}-a$, $\Delta b = \tilde{b}-b$, $\Delta c = \tilde{c}-c$) are related to off-diagonal reduced Coulomb matrix elements.

The main perturbing effect will result from states which are energetically close and, at the same time, have relatively large connecting Coulomb matrix elements. Only states with $T' = T-2$, $T' = T-1$ and possibly $T' = T$ can be energetically close. Of these, only states with $T' = T-1$ are likely to give rise to large matrix elements for the following reasons. States with $T' = T-2$ are connected only through off-diagonal matrix elements of the tensor Coulomb energy operator which are presumably much smaller than those of the vector Coulomb energy operator (in analogy to the relative magnitudes of $E^{(2)}(A, T)$ and $E^{(1)}(A, T)$). Nearby states with $T' = T$ cannot generate large matrix elements because the states are orthogonal in spin and space, and the Coulomb energy operator has only a weak spatial dependence. We have $\langle a|H_C|b\rangle \approx \text{const}\,\langle a|b\rangle = 0$. The conclusion therefore is that the most likely source for departures from the results of a first-order perturbation calculation are nearby states with the same J^π and with $T' = T-1$. Since $e^{(2)}_{-1}(\alpha, T)$ is probably much smaller than $e^{(1)}_{-1}(\alpha, T)$, one can see from eq. (4) that the predominant effect is likely to be a change in the quadratic term of the isobaric multiplet mass equation (increase or decrease for $\varepsilon^{(0)}(\alpha, T-1)-\varepsilon^{(0)}(T) \gtrless 0$, respectively) while large cubic or quartic terms are probably not generated †.

4. Analysis of the $A = 9$ isobaric quadruplet

It was mentioned in sect. 1 that a least-squares analysis [3–5]) of the experimental Coulomb displacement energies in the 1p shell, the 1d2s shell and the $1f_{\frac{7}{2}}$ shell in terms of Hecht's first-order Coulomb energy equations [1,2]) showed essentially very good agreement typically to within ± 10 keV. It was observed, though, that meaningful fits in the 1p shell were obtained only if the precise data [6,7]) for the $A = 9$ quadruplet were excluded from the analysis. Using the two-body Coulomb interaction energies obtained from a least-squares fit to the $T = \frac{1}{2}$ and $T = 1$ data only, the displacement energies as well as the various coefficients for the $A = 9$ quadruplet which are expected in first-order perturbation theory can be predicted. The depar-

† The rather large value obtained by Hensley [9]) for the cubic term of the $A = 21$ quadruplet is not in agreement with this expectation.

tures of the experimental from the calculated displacement energies are denoted by R_1, R_2, R_3 and $R_4 = R_1 + R_2 + R_3$. The residuals R_i are schematically indicated in fig. 1, and the numerical values are given in table 1. The energies of the $T = \frac{3}{2}$ states in ^{9}B and ^{9}Be are increased by about 100 keV, while the displacement energy between ^{9}C and ^{9}Li appears to be only little affected. The residuals will be related to the contributions obtained in a second-order perturbation treatment.

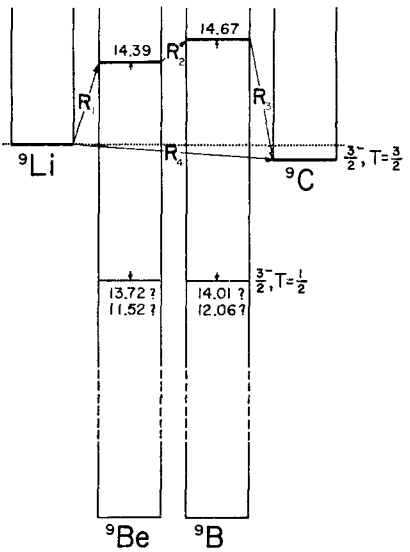

Fig. 1. Schematic representation of the departures of the experimental energies for the $A = 9$ isobaric quadruplet from the energies expected in first-order perturbation theory. (Read 11.82 instead of 11.52).

TABLE 1

Experimental and calculated residuals R_i and coefficients of the quadratic and quartic isobaric multiplet mass equation for the $A = 9$ isobaric quadruplet

	Exp. [a]	Calc. [a]
R_1	+ 81.2±20.6	+ 92.9 ±4.9
R_2	+ 10.6± 7.1	+ 11.7 ±4.2
R_3	−105.9± 7.2	− 104.6 ±5.1
R_4	− 14.1±25.0	0.0
a		3080 (approx.) [b]
b		−1321.6 ±5.0
c		+ 312.9 ±1.2
Δa		+ 111.0 ±4.5
Δb	− 12.5± 8.7	− 13.1 ±9.5
Δc		− 49.0 ±2.0
$\tilde{d}$	+ 7.2± 4.3	+ 5.8 ±4.2
$\tilde{e}$		− 0.17±0.18
$\Delta c + \frac{5}{2}\tilde{e}$	− 46.8± 5.8	− 49.5 ±2.0

[a]) All values in keV.

[b]) Contribution from Coulomb energy only.

The three residuals R_1, R_2 and R_3 make it possible to determine three combinations of the five corrections quantities Δa, Δb, Δc, $\tilde{d}$ and $\tilde{e}$ in the quartic eq. (8). The combinations are

$$\Delta b = -R_2 + \tfrac{1}{24}(R_3 - 2R_2 + R_1), \tag{9}$$

$$\Delta c + \tfrac{5}{2}\tilde{e} = +\tfrac{1}{4}(R_3 - R_1), \tag{10}$$

$$\tilde{d} = -\tfrac{1}{6}(R_3 - 2R_2 + R_1). \tag{11}$$

Table 1 shows in the column denoted by exp the numerical values of the derived quantities of eqs. (9)-(11). Additional information about the remaining quantities, the quartic coefficient for example, can only be obtained if the above information is supplemented by theoretical arguments.

It was pointed out at the end of sect. 3 that the most likely source for perturbations are states with $T = \frac{1}{2}$. If this is true, one can immediately see from eq. (4) that the residual R_4 must be zero. This is indeed the case within the experimental uncertainty. Therefore, the simplifying assumption will be made that the major contributions to the energy shifts as shown in fig. 1 are only caused by states with $J^\pi = \frac{3}{2}^-$ and $T = \frac{1}{2}$. Such an assumption is only reasonable if it leads to acceptable conclusions. Ultimately, a direct theoretical confirmation based on known wave functions of the perturbed and perturbing states is, of course, desirable.

It the above assumption is correct, the *three* residuals as well as all the coefficients can be expressed by *two* quantities only as will be shown below. Using the abbreviations $\Delta\varepsilon(\alpha) = \varepsilon^{(0)}(\alpha, \frac{1}{2}) - \varepsilon^{(0)}(\frac{3}{2})$ and $e^{(1)}_{-1}(\alpha) = e^{(1)}_{-1}(\alpha, \frac{3}{2})$, we have

$$R_1 = -\sum_\alpha \frac{2[e^{(1)}_{-1}(\alpha) - \frac{1}{2}e^{(2)}_{-1}(\alpha)]^2}{\Delta\varepsilon(\alpha)}, \tag{12}$$

$$R_2 = -\sum_\alpha \frac{4e^{(1)}_{-1}(\alpha)e^{(2)}_{-1}(\alpha)}{\Delta\varepsilon(\alpha)}, \tag{13}$$

$$R_3 = +\sum_\alpha \frac{2[e^{(1)}_{-1}(\alpha) + \frac{1}{2}e^{(2)}_{-1}(\alpha)]^2}{\Delta\varepsilon(\alpha)}, \tag{14}$$

$$\Delta a = -\sum_\alpha \frac{9[e^{(1)}_{-1}(\alpha)]^2}{4\Delta\varepsilon(\alpha)}, \tag{15}$$

$$\Delta b = +\sum_\alpha \frac{9e^{(1)}_{-1}(\alpha)e^{(2)}_{-1}(\alpha)}{\Delta\varepsilon(\alpha)}, \tag{16}$$

$$\Delta c = +\sum_\alpha \frac{[e^{(1)}_{-1}(\alpha)]^2}{\Delta\varepsilon(\alpha)} - \sum_\alpha \frac{9[e^{(2)}_{-1}(\alpha)]^2}{4\Delta\varepsilon(\alpha)}, \tag{17}$$

$$\tilde{d} = -\sum_\alpha \frac{2e^{(1)}_{-1}(\alpha)e^{(2)}_{-1}(\alpha)}{\Delta\varepsilon(\alpha)}, \tag{18}$$

$$\tilde{e} = + \sum_{\alpha} \frac{[e^{(2)}_{-1}(\alpha)]^2}{\Delta\varepsilon(\alpha)}, \tag{19}$$

$$\Delta c + \tfrac{5}{2}\tilde{e} = + \sum_{\alpha} \frac{[e^{(1)}_{-1}(\alpha)]^2}{\Delta\varepsilon(\alpha)} + \sum_{\alpha} \frac{[e^{(2)}_{-1}(\alpha)]^2}{4\Delta\varepsilon(\alpha)}. \tag{20}$$

The three residuals R_1, R_2 and R_3 can be reproduced exactly by properly adjusting the three quantities

$$\textstyle\sum_\alpha\{[e^{(1)}_{-1}(\alpha)]^2/\Delta\varepsilon(\alpha)\},\ \sum_\alpha\{[e^{(2)}_{-1}(\alpha)]^2/\Delta\varepsilon(\alpha)\} \text{ and } \sum_\alpha\{[e^{(1)}_{-1}(\alpha)e^{(2)}_{-1}(\alpha)]/\Delta\varepsilon(\alpha)\}.$$

The corrections Δa, Δb, Δc and the coefficients $\tilde{d}$ and $\tilde{e}$ for the quartic isobaric multiplet mass equation (8) can then be calculated. However, it is even sufficient to assume that the major perturbation results from a single state only. All sums in eqs. (12)–(20) are reduced to one term, and by properly adjusting the *two* quantities $e^{(1)}_{-1}(\alpha)/\sqrt{|\Delta\varepsilon(\alpha)|}$ and $e^{(2)}_{-1}(\alpha)/\sqrt{|\Delta\varepsilon(\alpha)|}$ with $\Delta\varepsilon(\alpha) < 0$, it is indeed possible to reproduce the three residuals R_1, R_2 and R_3 and again to calculate the corrections Δa, Δb, Δc and the coefficients $\tilde{d}$ and $\tilde{e}$ for the quartic isobaric multiplet mass equation (8). The comparison for this case is shown in table 1, and the agreement is very good. The calculated values show that the relative changes of the coefficients a (contribution from Coulomb energy only) and b are only about 3 % and 1 %, respectively. The coefficient c, however, is decreased by about 15 %. The cubic and quartic coefficients are quite small.

An inspection of the level schemes of ^{9}Be and ^{9}B shows that there are two pairs of states [10]) which are possible candidates for perturbing the $\frac{3}{2}^-$, $T = \frac{3}{2}$ analogue states at E_x = 14390 and 14670 keV, respectively. One pair of states with unknown

TABLE 2

Diagonal and off-diagonal matrix elements of the vector and tensor Coulomb energy operators for the members of the $A = 9$ isobaric quadruplet in the presence of perturbing $T = \frac{1}{2}$ states at lower excitation energy ($T_z = \pm\frac{1}{2}$)

		$\Delta\varepsilon = -500$	-2500
$E^{(1)}(T=\frac{3}{2}) =$	2104.1 ± 5.0	$e^{(1)}_{-1}(\alpha, T=\frac{3}{2}) =$	157.1 ± 4.5 $\quad 351.3\pm10.1$
$E^{(2)}(T=\frac{3}{2}) =$	104.3 ± 0.4	$e^{(2)}_{-1}(\alpha, T=\frac{3}{2}) =$	9.2 ± 6.7 $\quad 20.6\pm15.0$
$\langle\frac{3}{2}, \pm\frac{1}{2}\|H_C^{(1)}\|\frac{3}{2}, \pm\frac{1}{2}\rangle =$	$\pm1052.0\pm2.5$	$\langle\alpha, \frac{1}{2}, \pm\frac{1}{2}\|H_C^{(1)}\|\frac{3}{2}, \pm\frac{1}{2}\rangle =$	222.1 ± 6.3 $\quad 496.6\pm14.1$
$\langle\frac{3}{2}, \pm\frac{3}{2}\|H_C^{(1)}\|\frac{3}{2}, \pm\frac{3}{2}\rangle =$	$\mp3156.1\pm7.5$		
$\langle\frac{3}{2}, \pm\frac{1}{2}\|H_C^{(2)}\|\frac{3}{2}, \pm\frac{1}{2}\rangle =$	$-\ 312.9\pm1.2$	$\langle\alpha, \frac{1}{2}, \pm\frac{1}{2}\|H_C^{(2)}\|\frac{3}{2}, \pm\frac{1}{2}\rangle =$	$\mp6.6\pm4.7$ $\quad \mp14.8\pm10.5$
$\langle\frac{3}{2}, \pm\frac{3}{2}\|H_C^{(2)}\|\frac{3}{2}, \pm\frac{3}{2}\rangle =$	$+\ 312.9\pm1.2$		

T	T_z			
$\frac{3}{2}$	$+\frac{3}{2}$	^{9}Li	$\psi = 1.0\,\psi(T=\frac{3}{2})$	
$\frac{3}{2}$	$+\frac{1}{2}$	^{9}Be	$\psi = \alpha\psi(T=\frac{3}{2}) + \beta\psi(T=\frac{1}{2})$	$\beta^2 = 15.7\,\%$ $\quad 3.6\,\%$
$\frac{3}{2}$	$-\frac{1}{2}$	^{9}B	$\psi = \alpha'\psi(T=\frac{3}{2}) + \beta'\psi(T=\frac{1}{2})$	$\beta'^2 = 17.3\,\%$ $\quad 4.0\,\%$
$\frac{3}{2}$	$-\frac{3}{2}$	^{9}C	$\psi = 1.0\,\psi(T=\frac{3}{2})$	

All energies in keV.

spin and parities has excitation energies of $E_x = 13720$ and 14010 keV, respectively. Another pair of states with spin and parities of $\frac{1}{2}^-$ or $\frac{3}{2}^-$ has excitation energies of $E_x = 11820$ and 12060 keV, respectively. The latter state is populated by a β^+-decay branch from ^{9}C. We now assume unperturbed energy separations of about $\Delta\varepsilon = -500$ keV *or* -2500 keV and obtain from the values of table 1 the relevant off-diagonal matrix elements and the amount of isospin mixing. The results are given in table 2 with the corresponding diagonal matrix elements which were calculated from the quantities $b(A, T)$ and $c(A, T)$. The magnitude of the off-diagonal matrix elements seems to indicate that a second-order perturbation treatment may not be sufficient and an even higher-order treatment is needed. Both assumptions given above lead to reasonable results. If the states which are energetically closer ($\Delta\varepsilon = -500$ keV) cause the perturbation, the amount of isospin mixing exceeds 15 % which appears somewhat high. It should be possible experimentally to detect such a large amount of isospin mixing. If the other pair of states cause the perturbation, the amount of isospin mixing is only 4 %, but the off-diagonal matrix-element of $H_C^{(1)}$ of about 500 keV appears somewhat high. For both cases, the off-diagonal matrix elements of $H_C^{(1)}$ are of the order of the diagonal matrix elements of $H_C^{(2)}$, while the off-diagonal matrix elements of $H_C^{(2)}$ are considerably smaller. It would be very desirable to know more about the wave functions of the perturbing states and to hopefully confirm the above numbers by a direct calculation.

5. Analysis of other selected isobaric multiplets

Other isobaric quadruplets in the 1p shell and the 1d2s shell have been analysed similarly. A trend is apparent even though the experimental uncertainties are considerably larger. Perturbations do exist, and they affect mostly the coefficient of the quadratic term of the isobaric multiplet mass equation. Moreover, the quantity Δc is essentially independent of A and always negative. We have approximately $\Delta c \approx -50$ keV. The perturbations must therefore result mostly from one or several states with $T = \frac{1}{2}$ at lower excitation energies. This result is confirmed by fig. 2, which shows a plot of the experimental tensor Coulomb energies $E^{(2)}$ $(= \frac{1}{3}\tilde{c})$ for the isobaric quadruplets as a function of A. The A-dependence is smooth, and there are definitely no fluctuations of the order of ± 15 % or ± 17 keV. It is not quite clear why there is such a regular behavior. One might have expected that perturbing states with $T = \frac{1}{2}$ are sometimes lower and sometimes higher in excitation energy.

A possible explanation [11]) of the effect can be obtained in a simple-minded model. Using fourfold degenerate Nilsson-like or Hartree-Fock single-particle levels, the structure of the $T = \frac{3}{2}$ analogue states can be described as follows. If $A = 4k+1$ $(= 4k'+5)$, there are three nucleons in the first unfilled orbit and two in the second unfilled orbit. The former are coupled to $T = \frac{1}{2}$, the latter are coupled to $T = 1$. Both groups of nucleons then couple to $T = \frac{3}{2}$ to form the analogue state. Similarly, if $A = 4k+3$, there are two nucleons in the first unfilled orbit and one in the second

unfilled orbit. The former couple to $T = 1$, the latter nucleon has $T = \frac{1}{2}$. Again, the nucleons couple to $T = \frac{3}{2}$ to form the analogue state. It is clear from the above description that in both cases the nucleons of the two unfilled orbits may as well couple to $T = \frac{1}{2}$ with the same J^π as the $T = \frac{3}{2}$ state. These states are presumably somewhat lower in energy. They have a very similar space structure and may therefore lead to relatively large connecting matrix elements of the Coulomb energy operator.

The experimental displacement energies of several isobaric triplets also showed departures from the expectations of the first-order calculations. The departures essentially consisted of an energy shift by about 50–100 keV of the $T_z = 0$ members

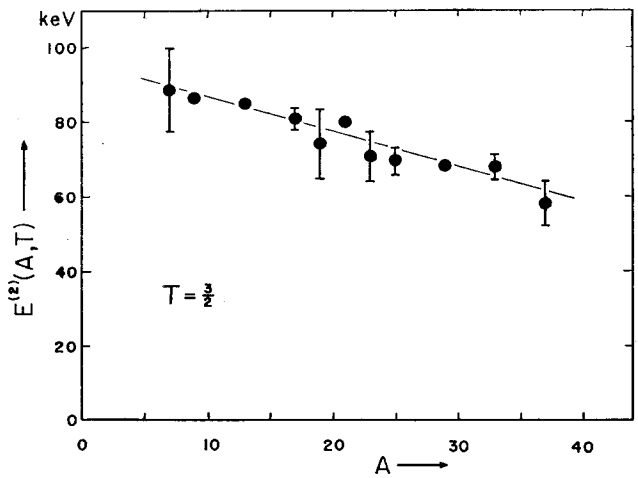

Fig. 2. Plot of the experimental tensor Coulomb energies $E^{(2)}$ ($= \frac{1}{3}\bar{c}$) for the isobaric quadruplets as a function of A. Experimental uncertainties are indicated if $\geqq 3$ keV. The straight line was drawn simply to show the trend of the data points and to show the absence of irregularities.

of certain isobaric triplets, namely in ^{8}Be, ^{18}F, ^{28}Si and ^{42}Sc. Eqs. (2)–(7) should be applicable if the departures are due to isospin mixing and if a second-order perturbation treatment is sufficient. Such a treatment is not adequate for the two 2^+ states [10]) in ^{8}Be at $E_x = 16628$ and 16923 keV, but it confirms that there is practically complete isospin mixing. The shift of the 3^+, $T = 1$ state [12]) at $E_x = 9319$ keV in ^{28}Si can probably be attributed to the presence of the 3^+, $T = 0$ state at $E_x = 8587$ keV. A connecting vector Coulomb energy matrix element of about 200 keV would account for the observed shift. There should be isospin admixtures of about 17 % in intensity. The 0^+, $T = 1$ states in ^{18}F and ^{42}Sc at $E_x = 1045$ and 0 keV, respectively, are shifted downward by about 40–50 keV for unknown reasons [3,4]). If the shifts result from the presence of some higher excited 0^+, $T = 0$ states, it would require matrix elements of about 200 keV $\sqrt{\Delta\varepsilon\ (\text{in MeV})}$, where $\Delta\varepsilon$ is the energy separation between the states. The amount of $T = 0$ admixtures would be about 4 %/$\Delta\varepsilon$ (in MeV) in intensity.

6. Conclusions

It has been shown that even in light nuclei a first-order perturbation treatment of the electrostatic interaction may not be sufficient to describe the experimental Coulomb displacement energies. The $A = 9$ isobaric quadruplet seems to be perturbed by the presence of a lower-lying $T = \frac{1}{2}$ state ($T_z = \pm\frac{1}{2}$) with the same spin and parity.

The quadratic isobaric multiplet mass equation has to be modified if a second-order perturbation treatment of the electrostatic interaction is indicated. The coefficients of the constant, linear and quadratic terms are affected, and cubic and quartic terms are generated. However, the quadratic isobaric multiplet mass equation often works very well not because first-order perturbation theory is a good approximation but because higher-order perturbations are mostly absorbed by the three coefficients. Particularly the coefficient of the quadratic term may be affected considerably.

The knowledge of good wave functions including proper radii is important to perform precise Coulomb energy calculations. If a second-order perturbation treatment is necessary, the analogue states are no longer connected by the isospin ladder operators $T_{\pm}$. This effect has also to be considered in the calculations.

Sometimes the hope is expressed that a detailed comparison between the experimental and calculated coefficients of the quadratic isobaric multiplet mass equation can yield information about charge-dependent nuclear forces. Here, too, one has to make certain that higher-order perturbations are small or properly taken into consideration.

Many stimulating discussions with K. T. Hecht are gratefully acknowledged.

Noted added in proof: A more precise measurement [13]) of the mass of ^{9}Li has been reported recently. The conclusions of the present analysis are not affected.

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