*Large Deflections of Multisandwich Shells of Arbitrary Shape**

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ABSTRACT: *A system of nonlinear differential equations governing the statical behavior of multisandwich shells built up of stiff and weak layers is derived in this contribution. The stiff layers are assumed to be elastic, isotropic, and obeying the Kirchhoff–Love hypothesis. The weak layers are assumed to be elastic, orthotropic, and deformable in tangential directions. The thickness of the shell is small compared to its radii of curvature. The shell may be of arbitrary shape. The derived system of equations is specialized to cylindrical shells and compared with the equations of Kurshin for sandwich shells and the equations of Bolotin for multisandwich plates with infinitesimal deflections.*

Nomenclature

x, y, z	Shell coordinates based on curvature lines. z is the normal distance outward from a reference surface
K, k	Principal curvatures of middle surface of a layer; $K = 1/R$, $k = 1/r$
R, r	Principal radii of curvature of middle surface of a layer
α, β, γ	Lamé coefficients for curvilinear orthogonal coordinates; $\gamma = 1$ for shell coordinates
A	$\alpha(x, y, 0)$, $B = \beta(x, y, 0)$
h	Thickness of a layer
b	Normal distance between middle surfaces of two neighboring stiff layers, $b_j = \frac{1}{2}(h_{j+1} + 2h_j + h_{j-1})$
i	$1, 3, \ldots, N$. Index i is used as either subscript or superscript on a letter to indicate a stiff layer
j	$2, 4, \ldots, N-1$. Index j is used as either subscript or superscript to indicate a weak layer

A comma before a subscript indicates partial derivative with respect to that subscript. Other symbols will be defined in the text wherever thay occur first.

Introduction

A typical multisandwich shell consists of N layers of different thicknesses and different properties of materials. $\frac{1}{2}(N+1)$ of these layers are stiff and $\frac{1}{2}(N-1)$ layers are weak. In the case of the common sandwich shell ($N = 3$) the stiff layers are usually referred to as facings and the single weak layer as core.

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Multisandwich shells and plates find application in the construction of self-sealing satellites to provide puncture protection for astronauts against micrometeoroids ("space nails"). In this application the number of layers would normally be five or seven. In general, multisandwich shells may be used with advantage whenever several different substances (e.g. a thermal insulator, a sealant, and a radiation shield) must be separated from each other within an integrated skin-like structure, such as a space ship, a fuel tank, a structure on the moon.

The literature concerned with the theory and analysis of multisandwich plates and shells is not extensive. Bolotin (**1**) gives a rather thorough discussion of multisandwich plates with transversely rigid weak layers and flexing stiff layers; (**2**) is mainly a heuristic, limited study of five-layer plates. Kao (**3**) treats the simple problem of axisymmetrically deformed cylindrical multisandwich shells composed of two transversely rigid weak cores and three membrane layers. Liaw and Little (**4**) develop the governing equations for multisandwich plates built up of membrane layers and orthotropic shear cores. These four contributions present linear analyses only. Wong and Salama (**5**) discuss the overall buckling of multisandwich plates composed of membrane layers and orthotropic, transversely rigid weak layers. Vasek (**6**) develops a general nonlinear theory of multisandwich plates in the sense of Föppl–von Kármán, assuming transversely compressible orthotropic weak layers.

For a survey of the literature on regular three-layer sandwich plates and shells, see (**7**, **18**).

The present contribution is concerned with the formulation of a rather general nonlinear theory of multisandwich shells of arbitrary shape. This theory is based on the following assumptions:

(1) The stiff layers are linearly elastic, homogeneous, and isotropic. The weak layers are elastic, homogeneous, and orthotropic in the directions of the coordinate lines.

(2) The stiff layers obey the Kirchhoff hypothesis and behave nonlinearly in the sense of Föppl–von Kármán. The weak layers cannot transmit stresses in tangential directions and are inextensional in the transverse direction.

(3) The shell is thin in the sense that its thickness is very small compared to the radii of curvature. It is not "shallow", however.

(4) The thicknesses and the materials of layers may be different.

(5) The layer-to-layer bonds are strong enough so that under all loadings no bond failure will occur.

Some Results of Geometry of Surfaces: Shell Coordinates

A surface may be defined by the equations $X = X(x,y)$, $Y = Y(x,y)$, $Z = Z(x,y)$ in which X, Y, Z are rectangular coordinates and x, y are parameters referred to as "surface coordinates".

Let the surface under consideration be the middle surface of a layer so that x, y are the curvilinear coordinates on this surface. Now measure a distance

z along one of the radii of principal curvature to the middle surface, away from the end of this radius. Then x, y, z are space coordinates. If we hold y and z constant, for example, then x may vary along a curve. This curve is called an x-coordinate line. Hence, we may speak of x-, y-, or z-coordinate lines. The z-coordinate line is always straight in our shell-coordinate system x, y, z. If x-, y-, z-coordinate lines are normal to each other, we have an "orthogonal" coordinate system. We assume here that x-, y-coordinate lines coincide with the lines of principal curvature of the surface on which they are situated. Such an orthogonal coordinate system is used in the present work.

The distance $\mathrm{d}s_s$ between two neighboring points is determined, in the orthogonal coordinates, by the equation

$$\mathrm{d}s_s^2 = \alpha^2\,\mathrm{d}x^2 + \beta^2\,\mathrm{d}y^2 + \gamma^2\,\mathrm{d}z^2, \tag{1}$$

where α, β, γ are the Lamé coefficients. For shell coordinates $\gamma = 1$.

In the shell coordinates the distance $\mathrm{d}s$ between two neighboring points on the middle surface of a layer is determined by the equation

$$\mathrm{d}s^2 = A^2\,\mathrm{d}x^2 + B^2\,\mathrm{d}y^2. \tag{2}$$

It is easy to show that

$$\alpha = A(1+Kz), \quad \beta = B(1+kz), \tag{3}$$

wherein K, k are the principal curvatures of the surface and A, B are functions of x and y only. We also note that $K = K(x,y)$, $\quad k = k(x,y)$.

It is often advantageous to make use of Codazzi's relations (**8**)

$$kA_{,y} = (KA)_{,y}, \quad KB_{,x} = (kB)_{,x} \tag{4}$$

from which

$$AK_{,y} = (k-K)\,A_{,y}, \quad Bk_{,x} = (K-k)\,B_{,x}, \tag{5}$$

where a comma before a subscript indicates partial derivative with respect to that subscript, e.g. $(KA)_{,y} = \partial/\partial y\,(KA)$.

We also observe that

$$\left.\begin{aligned} A\beta_{,x} &= \alpha B_{,x}, \quad B\alpha_{,y} = \beta A_{,y}, \\ \alpha_{,z} &= KA, \quad \beta_{,z} = kB \end{aligned}\right\} \tag{6}$$

in view of Eqs. 3 and 4.

Some Results of Continuum Mechanics

In the shell coordinate system the expressions for strain components assume the following form (**9**, **10**):

$$\varepsilon_x = e_x + \tfrac{1}{2}[e_x^2 + (\tfrac{1}{2}\,e_{xy} + \omega_z)^2 + (\tfrac{1}{2}\,e_{xz} - \omega_y)^2], \tag{7a}$$

$$\varepsilon_y = e_y + \tfrac{1}{2}[e_y^2 + (\tfrac{1}{2}\,e_{xy} - \omega_z)^2 + (\tfrac{1}{2}\,e_{yz} + \omega_x)^2], \tag{7b}$$

$$\varepsilon_z = e_z + \tfrac{1}{2}[e_z^2 + (\tfrac{1}{2}\,e_{xz} + \omega_y)^2 + (\tfrac{1}{2}\,e_{yz} - \omega_x)^2], \tag{7c}$$

$$\gamma_{xy} = e_{xy} + e_x(\tfrac{1}{2}e_{xy} - \omega_z) + e_y(\tfrac{1}{2}e_{xy} + \omega_z)$$
$$+ (\tfrac{1}{2}e_{xz} - \omega_y)(\tfrac{1}{2}e_{yz} + \omega_x), \tag{7d}$$

$$\gamma_{xz} = e_{xz} + e_x(\tfrac{1}{2}e_{xz} + \omega_y) + e_z(\tfrac{1}{2}e_{xz} - \omega_y)$$
$$+ (\tfrac{1}{2}e_{xy} + \omega_z)(\tfrac{1}{2}e_{yz} - \omega_x), \tag{7e}$$

$$\gamma_{yz} = e_{yz} + e_y(\tfrac{1}{2}e_{yz} - \omega_x) + e_z(\tfrac{1}{2}e_{yz} + \omega_x)$$
$$+ (\tfrac{1}{2}e_{xy} - \omega_z)(\tfrac{1}{2}e_{xz} + \omega_y), \tag{7f}$$

wherein

$$e_x = \frac{1}{\alpha}\left(U_{,x} + A_{,y}\frac{V}{B} + KAW\right), \tag{8a}$$

$$e_y = \frac{1}{\beta}\left(V_{,y} + B_{,x}\frac{U}{A} + kBW\right), \tag{8b}$$

$$e_z = W_{,z}, \tag{8c}$$

$$e_{xy} = \frac{\beta}{\alpha}\left(\frac{V}{\beta}\right)_{,x} + \frac{\alpha}{\beta}\left(\frac{U}{\alpha}\right)_{,y}, \tag{8d}$$

$$e_{xz} = \alpha\left(\frac{U}{\alpha}\right)_{,z} + \frac{1}{\alpha}W_{,x}, \tag{8e}$$

$$e_{yz} = \beta\left(\frac{V}{\beta}\right)_{,z} + \frac{1}{\beta}W_{,y} \tag{8f}$$

are the strain expressions of the linear theory, and

$$2\omega_x = \frac{1}{\beta}[W_{,y} - (\beta V)_{,z}], \tag{9a}$$

$$2\omega_y = \frac{1}{\alpha}[(\alpha U)_{,z} - W_{,x}], \tag{9b}$$

$$2\omega_z = \frac{1}{\alpha\beta}[(\beta V)_{,x} - (\alpha U)_{,y}] \tag{9c}$$

are the well-known expressions for the components of a curl.

In the case that strains and rotations are small compared to unity (but not infinitesimal), the equations of equilibrium, in the absence of body forces, may be written in the form (**9**):

$$(\beta f_{xx})_{,x} + (\alpha f_{yx})_{,y} + (\alpha\beta f_{zx})_{,z} + \frac{\beta}{B}A_{,y}f_{xy} + KA\beta f_{xz} - \frac{\alpha}{A}B_{,x}f_{yy} = 0, \tag{10a}$$

$$(\beta f_{xy})_{,x} + (\alpha f_{yy})_{,y} + (\alpha\beta f_{zy})_{,z} + kB\alpha f_{yz} + \frac{\alpha}{A}B_{,x}f_{yx} - \frac{\beta}{B}A_{,y}f_{xx} = 0, \tag{10b}$$

$$(\beta f_{xz})_{,x} + (\alpha f_{yz})_{,y} + (\alpha\beta f_{zz})_{,z} - KA\beta f_{xx} - kB\alpha f_{yy} = 0, \tag{10c}$$

where

$$f_{xx} = \sigma_x - \omega_z \tau_{xy} + \omega_y \tau_{xz} \tag{11a}$$

$$f_{xy} = \tau_{xy} + \omega_z \sigma_x - \omega_x \tau_{xz}, \tag{11b}$$

$$f_{xz} = \tau_{xz} + \omega_x \tau_{xy} - \omega_y \sigma_x, \tag{11c}$$

$$f_{yx} = \tau_{xy} - \omega_z \sigma_y + \omega_y \tau_{yz}, \tag{11d}$$

$$f_{yy} = \sigma_y + \omega_z \tau_{xy} - \omega_x \tau_{yz}, \tag{11e}$$

$$f_{yz} = \tau_{yz} + \omega_x \sigma_y - \omega_y \tau_{xy}, \tag{11f}$$

$$f_{zx} = \tau_{xz} - \omega_z \tau_{yz} + \omega_y \sigma_z, \tag{11g}$$

$$f_{zy} = \tau_{yz} + \omega_z \tau_{xz} - \omega_x \sigma_z, \tag{11h}$$

$$f_{zz} = \sigma_z + \omega_x \tau_{yz} - \omega_y \tau_{xz}, \tag{11i}$$

in which σ's and τ's denote the "engineering" components of stress, e.g. σ_x is the normal stress in the x-direction, τ_{xy} is the shearing stress in the y-direction on an element of area normal to the x-coordinate line.

In writing the foregoing equations free use was made of Eqs. 6.

Weak Layers

The layers of the shell are numbered 1, 2, 3, . . . , N. Thus, since the weak layers are always sandwiched in between stiff layers, they are denoted by $j = 2, 4, \ldots, N-1$, j always being an even integer, i.e. as an index, letter j always designates the jth (weak) layer.

For simplification, indexes will not be used under this heading. It is understood that all entities pertain to the jth (weak) layer and its middle surface.

For a weak layer it is customary to assume

$$\sigma_x = \sigma_y = \tau_{xy} = 0 \quad \text{and} \quad \varepsilon_z = 0. \tag{12}$$

Since the Föppl–von Kármán nonlinearity is, to a significant extent, due to these stresses, we may use equations of the linear theory for the weak layers. Thus, the three equations of equilibrium 10 become

$$\frac{\partial}{\partial z}(\alpha\beta\tau_{xz}) + KA\beta\tau_{xz} = 0 \quad \text{or} \quad \frac{\partial}{\partial z}(\alpha^2\beta\tau_{xz}) = 0, \tag{13a}$$

$$\frac{\partial}{\partial z}(\alpha\beta\tau_{yz}) + kB\alpha\tau_{yz} = 0 \quad \text{or} \quad \frac{\partial}{\partial z}(\alpha\beta^2\tau_{yz}) = 0, \tag{13b}$$

$$\frac{\partial}{\partial x}(\beta\tau_{xz}) + \frac{\partial}{\partial y}(\alpha\tau_{yz}) + \frac{\partial}{\partial z}(\alpha\beta\sigma_z) = 0. \tag{13c}$$

The first two equations are easily integrated to yield

$$\alpha^2\beta\tau_{xz} = A^2B\tau_x, \quad \alpha\beta^2\tau_{yz} = AB^2\tau_y, \tag{14}$$

where τ_x, τ_y are functions of x and y only. In view of assumptions (12), and the observation that further steps will involve integrations, we may assume, for the weak layers, that

$$\alpha = A(1+Kz) \approx A, \quad \beta = B(1+kz) \approx B$$

in the foregoing equations. Then

$$\tau_{xz} = \tau_x, \quad \tau_{yz} = \tau_y, \tag{15}$$

i.e. the transverse shearing stresses do not vary across the thickness of a weak layer. Also, by simple integration of Eq. 13c,

$$\sigma_z = \sigma(x,y) - zT(x,y), \tag{16}$$

wherein

$$T = (1/AB)\left[(B\tau_x)_{,x} + (A\tau_y)_{,y}\right] \tag{17}$$

and σ is a function of integration.

The stress–strain relations for the orthotropic weak layers, in view of the basic assumptions (12), simplify to the usual form (**11**):

$$\tau_x = G_x \gamma_{xz}, \quad \tau_y = G_y \gamma_{yz}, \tag{18}$$

where G_x and G_y are moduli of rigidity of the weak layer in the transverse direction.

Using Eqs. 7, 8, 15, and 18, together with the assumption of linear behavior and negligibility of Kz and kz in comparison to unity, we obtain the following expressions for the displacement components of a weak layer in the directions of the coordinate lines on an undeformed reference surface (middle surface in this paper):

$$W = w(x,y), \tag{19a}$$

$$U = u(x,y) + z\left(\frac{\tau_x}{G_x} - \frac{1}{A} w_{,x}\right), \tag{19b}$$

$$V = v(x,y) + z\left(\frac{\tau_y}{G_y} - \frac{1}{B} w_{,y}\right), \tag{19c}$$

where u, v, w are functions of integration whose meaning is obvious. Thus, in order to describe the behavior of a weak layer, we must determine the five arbitrary functions of x and y, namely, τ_x, τ_y, u, v, w. This is possible by using the conditions of continuity of displacements and stresses at the interfaces between layers, and the equations of equilibrium for the stiff layers.

Note that $w = w_i = w_j$ according to the assumption that all layers are inextensional in the transverse direction.

Stiff Layers

To denote the stiff layers we use odd integers as indexes. Any one of them will be represented by the letter $i = 1, 3, \ldots, N$. Under the present heading indexes will not be used, in order to simplify printing. It is understood that all entities pertain to the ith (stiff) layer and its middle surface.

We use the Kirchhoff hypothesis to describe the behavior of each of the $\frac{1}{2}(N+1)$ stiff layers. According to this hypothesis, we may take

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0. \tag{20}$$

With these conditions, Eqs. 7c, e, f may be integrated for U, V, W (**12**). Keeping only those terms that are required by the Föppl–von Kármán concept of nonlinearity (theory of plates and shells with moderately large deflections), we obtain

$$U = u + z\omega_y, \quad V = v - z\omega_x, \quad W = w, \tag{21}$$

where u, v, w are functions of x and y, and

$$\omega_x = (1/B)\, w_{,y} - kv, \quad \omega_y = Ku - (1/A)\, w_{,x}. \tag{22}$$

With these results, expressions for strains, Eqs. 7a, b, d, become

$$\varepsilon_x = e_x + \tfrac{1}{2}\omega_y^2 - \frac{z}{A}\left(\frac{1}{B} A_{,y}\, \omega_x - \omega_{y,x}\right), \tag{23a}$$

$$\varepsilon_y = e_y + \tfrac{1}{2}\omega_x^2 - \frac{z}{B}\left(\omega_{x,y} - \frac{1}{A} B_{,x}\, \omega_y\right), \tag{23b}$$

$$\gamma_{xy} = e_{xy} - \omega_x\, \omega_y - z\left[\frac{1}{A}\left(\omega_{x,x} + \frac{1}{B} A_{,y}\, \omega_y\right) - \frac{1}{B}\left(\omega_{y,y} + \frac{1}{A} B_{,x}\, \omega_x\right)\right], \tag{23c}$$

after neglecting several small-order terms and using the symbols

$$e_x = \frac{1}{A} u_{,x} + \frac{v}{AB} A_{,y} + Kw, \tag{24a}$$

$$e_y = \frac{1}{B} v_{,y} + \frac{u}{AB} B_{,x} + kw, \tag{24b}$$

$$e_{xy} = \frac{B}{A} \frac{\partial}{\partial x}\left(\frac{v}{B}\right) + \frac{A}{B} \frac{\partial}{\partial y}\left(\frac{u}{A}\right). \tag{24c}$$

The stresses are determined by using Hooke's law and Kirchhoff's hypothesis (**8**). They are given by

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y), \tag{25a}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \tag{25b}$$

$$\tau_{xy} = G\gamma_{xy}, \tag{25c}$$

where

$$G = \frac{E}{2(1+\nu)}. \tag{26}$$

We now direct attention to the equilibrium conditions for the stiff layers. We strive to eliminate the independent variable z from the differential equations of equilibrium 10. For this purpose it is convenient to introduce the so-called "stress resultants". They are defined as follows:

$$N_x = \int \sigma_x \,\mathrm{d}z, \tag{27a}$$

$$N_y = \int \sigma_y \,\mathrm{d}z, \tag{27b}$$

$$S = \int \tau_{xy} \,\mathrm{d}z \tag{27c}$$

$$Q_x = \int \tau_{xz} \,\mathrm{d}z, \tag{27d}$$

$$Q_y = \int \tau_{yz} \,\mathrm{d}z, \tag{27e}$$

$$M_x = \int z\sigma_x \,\mathrm{d}z, \tag{27f}$$

$$M_y = \int z\sigma_y \,\mathrm{d}z, \tag{27g}$$

$$H = \int z\tau_{xy} \,\mathrm{d}z, \tag{27h}$$

where all the integrals are definite integrals which must be evaluated between the lower limit $z = z_i = -\frac{1}{2}h$ and the upper limit $z = z_i = \frac{1}{2}h$. These definitions are not "exact" in the sense that we assumed Kz, kz to be much smaller than unity. We refer to the quantities N_x, N_y, S as forces (per unit length); Q_x, Q_y as shearing forces; M_x, M_y as bending couples; and H as twisting couple.

Using Eqs. 22–25 we can determine the expressions for the stress resultants N_x, N_y, S, M_x, M_y, H in terms of the displacement components u, v, w of the middle surface of a stiff layer. Thus

$$N_x = \frac{Eh}{1-\nu^2}[e_x + \tfrac{1}{2}\omega_y^2 + \nu(e_y + \tfrac{1}{2}\omega_x^2)], \tag{28a}$$

$$N_y = \frac{Eh}{1-\nu^2}[e_y + \tfrac{1}{2}\omega_x^2 + \nu(e_x + \tfrac{1}{2}\omega_y^2)], \tag{28b}$$

$$S = \frac{Eh}{2(1+\nu)}(e_{xy} - \omega_x \omega_y), \tag{28c}$$

$$M_x = -D\left[\frac{1}{A}\left(\frac{\omega_x}{B}A_{,y} - \omega_{y,x}\right) + \frac{\nu}{B}\left(\omega_{x,y} - \frac{\omega_y}{A}B_{,x}\right)\right], \tag{28d}$$

$$M_y = -D\left[\frac{1}{B}\left(\omega_{x,y} - \frac{\omega_y}{A}B_{,x}\right) + \frac{\nu}{A}\left(\frac{\omega_x}{B}A_{,y} - \omega_{y\,x}\right)\right], \tag{28e}$$

$$H = -\frac{1-\nu}{2}D\left[\frac{1}{A}\left(\omega_{x,x} + \frac{\omega_y}{B}A_{,y}\right) - \frac{1}{B}\left(\omega_{y,y} + \frac{\omega_x}{A}B_{,x}\right)\right], \tag{28f}$$

in which e_x, e_y, e_{xy} and ω_x, ω_y are given by expressions 24 and 22, respectively.

And
$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{29}$$

Next we simplify Eqs. 11 by noting that, for stiff shells, "rotation" ω_z is much smaller than ω_x or ω_y, stress σ_z is much smaller than the other five stresses, and that the products of the "rotations" and stresses are much smaller than stresses σ_x, σ_y, τ_{xy}, but not σ_z, τ_{xz}, τ_{yz}. Hence, we may take

$$\left.\begin{aligned} f_{xx} &= \sigma_x, & f_{xy} &= \tau_{xy}, & f_{xz} &= \tau_{xz} + \omega_x \tau_{xy} - \omega_y \sigma_x, \\ f_{yx} &= \tau_{xy}, & f_{yy} &= \sigma_y, & f_{yz} &= \tau_{yz} + \omega_x \sigma_y - \omega_y \tau_{xy}, \\ f_{zx} &= \tau_{xz}, & f_{zy} &= \tau_{yz}, & f_{zz} &= \sigma_z + \omega_x \tau_{yz} - \omega_y \tau_{xz}, \end{aligned}\right\} \tag{30}$$

where ω_x, ω_y are given by expressions 22 with a sufficient degree of accuracy. Note that ω_x, ω_y are functions of x and y only.

Substituting expressions 30 in Eqs. 10 and integrating these through the thickness of a stiff layer we obtain three equilibrium equations in terms of forces. Two moment equations are established by multiplying the first two of Eqs. 10 by z and then integrating these equations through the thickness h. The third moment equation that can be obtained is an identity (8).

In this fashion we get

$$\begin{aligned} &\frac{\partial}{\partial x}(BN_x) + \frac{1}{A}\frac{\partial}{\partial y}(A^2 S) - B_{,x} N_y + ABKQ_x \\ &\qquad + AB\left[\tau_{xz}\left(x, y, \frac{h}{2}\right) - \tau_{xz}\left(x, y, -\frac{h}{2}\right)\right] = 0, \end{aligned} \tag{31a}$$

$$\begin{aligned} &\frac{\partial}{\partial y}(AN_y) + \frac{1}{B}\frac{\partial}{\partial x}(B^2 S) - A_{,y} N_x + ABkQ_y \\ &\qquad + AB\left[\tau_{yz}\left(x, y, \frac{h}{2}\right) - \tau_{yz}\left(x, y, -\frac{h}{2}\right)\right] = 0, \end{aligned} \tag{31b}$$

$$\begin{aligned} &\frac{\partial}{\partial x}(BQ_x) + \frac{\partial}{\partial y}(AQ_y) - AB(KN_x + kN_y) + \frac{\partial}{\partial x}(\omega_x BS - \omega_y BN_x) \\ &\qquad + \frac{\partial}{\partial y}(\omega_x AN_y - \omega_y AS) + AB\left[\sigma_z\left(x, y, \frac{h}{2}\right) - \sigma_z\left(x, y, -\frac{h}{2}\right)\right] \\ &\qquad + AB\omega_x\left[\tau_{yz}\left(x, y, \frac{h}{2}\right) - \tau_{yz}\left(x, y, -\frac{h}{2}\right)\right] \\ &\qquad - AB\omega_y\left[\tau_{xz}\left(x, y, \frac{h}{2}\right) - \tau_{xz}\left(x, y, -\frac{h}{2}\right)\right] = 0, \end{aligned} \tag{31c}$$

$$\begin{aligned} &\frac{\partial}{\partial x}(BM_x) + \frac{1}{A}\frac{\partial}{\partial y}(A^2 H) - B_{,x} M_y - ABQ_x \\ &\qquad + \tfrac{1}{2}hAB\left[\tau_{xz}\left(x, y, \frac{h}{2}\right) + \tau_{xz}\left(x, y, -\frac{h}{2}\right)\right] = 0, \end{aligned} \tag{31d}$$

$$\begin{aligned} &\frac{\partial}{\partial y}(AM_y) + \frac{1}{B}\frac{\partial}{\partial x}(B^2 H) - A_{,y} M_x - ABQ_y \\ &\qquad + \tfrac{1}{2}hAB\left[\tau_{yz}\left(x, y, \frac{h}{2}\right) + \tau_{yz}\left(x, y, -\frac{h}{2}\right)\right] = 0. \end{aligned} \tag{31e}$$

We note that

$$\left.\begin{aligned} \tau_{xz}(x,y,\tfrac{1}{2}h_N) &= X_N, \quad \tau_{xz}(x,y,-\tfrac{1}{2}h_1) = X_1, \\ \tau_{yz}(x,y,\tfrac{1}{2}h_N) &= Y_N, \quad \tau_{yz}(x,y,-\tfrac{1}{2}h_1) = Y_1, \\ \sigma_z(x,y,\tfrac{1}{2}h_N) &= Z_N, \quad \sigma_z(x,y,-\tfrac{1}{2}h_1) = Z_1, \end{aligned}\right\} \tag{32}$$

are the distributed load components on the exposed surfaces of the multisandwich shell, and

$$\left.\begin{aligned} \tau_{xz}(x,y,\tfrac{1}{2}h_i) &= \tau_x^{i+1}, \quad \tau_{xz}(x,y,-\tfrac{1}{2}h_i) = \tau_x^{i-1}, \\ \tau_{yz}(x,y,\tfrac{1}{2}h_i) &= \tau_y^{i+1}, \quad \tau_{yz}(x,y,-\tfrac{1}{2}h_i) = \tau_y^{i-1}, \\ \sigma_z(x,y,\tfrac{1}{2}h_i) &= \sigma^{i+1} + \tfrac{1}{2}h_{i+1}\,T^{i+1}, \\ \sigma_z(x,y,-\tfrac{1}{2}h_i) &= \sigma^{i-1} - \tfrac{1}{2}h_{i-1}\,T^{i-1} \end{aligned}\right\} \tag{33}$$

follow from the continuity of stresses at the interfaces between the stiff layer i and the adjacent weak layers $i+1$ and $i-1$ (see Eqs. 16 and 17). Note that the placement of indexes as either subscripts or superscripts is governed by convenience.

Continuity of Displacements at Interfaces

Since the layers are bonded rigidly to each other, we must have continuity of displacements at the interfaces. The continuity conditions have the form:

$$u_i + \tfrac{1}{2}h_i\,\omega_y^i = u_{i+1} - \tfrac{1}{2}h_{i+1}\left(\frac{\tau_x^{i+1}}{G_x^{i+1}} - \frac{1}{A}w_{,x}\right), \tag{34a}$$

$$u_i - \tfrac{1}{2}h_i\,\omega_y^i = u_{i-1} + \tfrac{1}{2}h_{i-1}\left(\frac{\tau_x^{i-1}}{G_x^{i-1}} - \frac{1}{A}w_{,x}\right), \tag{34b}$$

$$v_i - \tfrac{1}{2}h_i\,\omega_x^i = v_{i+1} - \tfrac{1}{2}h_{i+1}\left(\frac{\tau_y^{i+1}}{G_y^{i+1}} - \frac{1}{B}w_{,y}\right), \tag{34c}$$

$$v_i + \tfrac{1}{2}h_i\,\omega_x^i = v_{i-1} + \tfrac{1}{2}h_{i-1}\left(\frac{\tau_y^{i-1}}{G_y^{i-1}} - \frac{1}{B}w_{,y}\right), \tag{34d}$$

since $w_i = w_{i+1} = w_{i-1} = w$, according to assumptions 12 and 20, and

$$A_i \approx A_{i+1} \approx A_{i-1} \approx A,$$
$$B_i \approx B_{i+1} \approx B_{i-1} \approx B$$

for thin multisandwich shells.

From Eqs. 34 by simple subtractions, we obtain the following useful formulas:

$$\frac{\tau_x^j}{G_y^j} = \frac{1}{h_j}(u_{j+1} - u_{j-1}) + \frac{b_j}{Ah_j}w_{,x}, \tag{35a}$$

$$\frac{\tau_y^j}{G_y^j} = \frac{1}{h_j}(v_{j+1} - v_{j-1}) + \frac{b_j}{Bh_j}w_{,y} \tag{35b}$$

after neglecting Kh and kh terms as small compared to unity.

$$b_j = \tfrac{1}{2}(h_{j+1} + 2h_j + h_{j-1}), \quad j = 2, 4, \ldots, N-1.$$

System of Governing Equations

If the displacement components u, v, w of the middle surfaces of the $\frac{1}{2}(N+1)$ stiff layers are chosen as the unknown functions, the system of governing differential equations to be solved simultaneously consists of $N+2$ equations, since $w_i = w_j = w$. All the remaining quantities of interest can be calculated from explicit relationships. There is no unique way of presenting this system of governing equations. Therefore, compactness will govern in what follows.

The system of equations to be solved may be presented in the form:

$$(BN_x^i)_{,x} + \frac{1}{A}(A^2 S_i)_{,y} - B_{,x} N_y^i + K\left[(BM_x^i)_{,x} + \frac{1}{A}(A^2 H_i)_{,y} - B_{,x} M_y^i\right] + AB(\tau_x^{i+1} - \tau_x^{i-1}) = 0, \tag{36a}$$

$$(AN_y^i)_{,y} + \frac{1}{B}(B^2 S_i)_{,x} - A_{,y} N_x^i + k\left[(AM_y^i)_{,y} + \frac{1}{B}(B^2 H_i)_{,x} - A_{,y} M_x^i\right] + AB(\tau_y^{i+1} - \tau_y^{i-1}) = 0, \tag{36b}$$

$$\begin{aligned}\sum_{i=1,3,\ldots}^{N} \Bigg\{ & \frac{\partial}{\partial x}\left[\frac{1}{A}\frac{\partial}{\partial x}(BM_x^i) + \frac{1}{A^2}\frac{\partial}{\partial y}(A^2 H_i) - \frac{B_{,x}}{A} M_y^i\right] \\ & + \frac{\partial}{\partial y}\left[\frac{1}{B}\frac{\partial}{\partial y}(AM_y^i) + \frac{1}{B^2}\frac{\partial}{\partial x}(B^2 H_i) - \frac{A_{,y}}{B} M_x^i\right] \\ & - (ABK - \omega_x^i A_{,y} + \omega_{y,x}^i B) N_x^i \\ & + \left[B^2 \frac{\partial}{\partial x}\left(\frac{\omega_x^i}{B}\right) - A^2 \frac{\partial}{\partial y}\left(\frac{\omega_y^i}{A}\right)\right] S_i \\ & - (ABk + \omega_y^i B_{,x} - \omega_{x,y}^i A) N_y^i \Bigg\} \\ & + \sum_{j=2,4,\ldots}^{N-1} b_j[(B\tau_x^j)_{,x} + (A\tau_y^j)_{,y}] \\ = AB\Bigg\{ & Z_1 - Z_N - \frac{h_1}{2}[(BX_1)_{,x} + (AY_1)_{,y}] - \frac{h_N}{2}[(BX_N)_{,x} + (AY_N)_{,y}]\Bigg\},\end{aligned} \tag{36c}$$

where

$$\tau_x^j = \frac{G_x^j}{h_j}\left(u_{j+1} - u_{j-1} + \frac{b_j}{A} w_{,x}\right), \tag{37a}$$

$$\tau_y^i = \frac{G_y^j}{h_j}\left(v_{j+1} - v_{j-1} + \frac{b_j}{B} w_{,y}\right), \tag{37b}$$

$$b_j = \tfrac{1}{2}(h_{j+1} + 2h_j + h_{j-1}), \quad j = 2, 4, \ldots, N-1. \tag{38}$$

The stress resultants and the "rotations" are expressed in terms of the displacement components u, v, w by Eqs. 28 and 22.

Equations 36 are the equilibrium equations, and Eqs. 37 are obtained from the conditions of continuity of displacements at the interfaces between the layers, Eqs. 35.

Boundary Conditions

The natural boundary conditions along the edge x = constant may be written in the form

$$N_x^i = 0, \tag{39a}$$

$$S_i + kH_i = 0, \tag{39b}$$

$$\sum_{i=1,3,\dots}^{N} M_x^i = 0, \tag{39c}$$

$$\sum_{i=1,3,\dots}^{N} \left(Q_x^i + \frac{1}{B} H_{,y}^i\right) + \sum_{j=2,4,\dots}^{N-1} \tau_x^j h_j = 0, \tag{39d}$$

wherein Q_x^i should be calculated from the moment equilibrium, Eq. 31d. The imposed boundary conditions present no great difficulties.

Cylindrical Shells

We now reduce the developed general equations to the rather practical special case of circular cylindrical multisandwich shells. Using the cylindrical coordinates x, $y = a\theta$, $\rho = a + z$, where a is the radius of a suitable reference surface of the thin shell, we have $R = \infty$, $r = a$, $K = 0$, $k = 1/a$, $A = B = 1$, so that the system of governing Eqs. 36 assumes the form:

$$N_{x,x}^i + S_{,y}^i + \tau_x^{i+1} - \tau_x^{i-1} = 0, \tag{40a}$$

$$N_{y,y}^i + S_{,x}^i + \frac{1}{a}(M_{y,y}^i + H_{,x}^i) + \tau_y^{i+1} - \tau_y^{i-1} = 0, \tag{40b}$$

$$\begin{aligned} \sum_{i=1,3,\dots}^{N} \Big[& M_{x,xx}^i + 2H_{,xy}^i + M_{y,yy}^i + N_x^i w_{,xx} \\ & + S_i \left(2w_{,xy} - \frac{1}{a} v_{,x}^i\right) + N_y^i \left(w_{,yy} - \frac{1}{a} v_{,y}^i - \frac{1}{a}\right)\Big] \\ & + \sum_{j=2\,4,\dots}^{N-1} b_j(\tau_{x,x}^j + \tau_{y,y}^j) \\ & = Z_1 - Z_N - \frac{h_1}{2}(X_{1,x} + Y_{1,y}) - \frac{h_N}{2}(X_{N,x} + Y_{N,y}), \end{aligned} \tag{40c}$$

in which

$$\tau_x^j = \frac{G_x^j}{h_j}(u_{j+1} - u_{j-1} + b_j w_{,x}), \tag{41a}$$

$$\tau_y^i = \frac{G_y^j}{h_j}(v_{j+1} - v_{j-1} + b_j w_{,y}), \tag{41b}$$

and

$$M_{x,xx}^i + 2H_{,xy}^i + M_{y,yy}^i = -D_i\left(\nabla^4 w - \frac{1}{a}\nabla^2 v_{,y}^i\right),$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{and} \quad \nabla^4 = \nabla^2 \nabla^2.$$

For the case of regular sandwich shells, i.e. $N = 3$, with identical internal and external facings, the governing equations may be written in the form:

$$\overset{+}{N}_{x,x}+\overset{+}{S}_{,y} = \tfrac{1}{2}(X_1-X_N), \tag{42a}$$

$$\overset{+}{N}_{y,y}+\overset{+}{S}_{,x}+\frac{1}{a}(\overset{+}{M}_{y,y}+\overset{+}{H}_{,x}) = \tfrac{1}{2}(Y_1-Y_N), \tag{42b}$$

$$\begin{aligned} D\nabla^4 w - \frac{D}{a}\nabla^2 \overset{+}{v}_{,y} - \overset{+}{N}_x w_{,xx} - 2\overset{+}{S}\left(w_{,xy}-\frac{1}{2a}\overset{+}{v}_{,x}\right) \\ +\frac{1}{a}\bar{S}\bar{v}_{,x} - \overset{+}{N}_y\left(w_{,yy}-\frac{1}{a}-\frac{1}{a}\overset{+}{v}_{,y}\right)+\frac{1}{a}\bar{N}_y\bar{v}_{,y} \\ -\frac{bG_x}{h_2}\left(\bar{u}_{,x}+\frac{b}{2}w_{,xx}\right)-\frac{bG_y}{h_2}\left(\bar{v}_{,y}+\frac{b}{2}w_{,yy}\right) \\ = \tfrac{1}{2}(Z_N-Z_1)+\frac{h}{4}(X_{1,x}+X_{N,x}+Y_{1,y}+Y_{N,y}), \end{aligned} \tag{42c}$$

$$\bar{N}_{x,x}+\bar{S}_{,y}-\frac{2G_x}{h_2}\left(\bar{u}+\frac{b}{2}w_{,x}\right) = -\tfrac{1}{2}(X_1+X_N), \tag{42d}$$

$$\bar{N}_{y,y}+\bar{S}_{,x}+\frac{1}{a}(\bar{M}_{y,y}+\bar{H}_{,x})-\frac{2G_y}{h_2}\left(\bar{v}+\frac{b}{2}w_{,y}\right) = -\tfrac{1}{2}(Y_1+Y_N), \tag{42e}$$

wherein

$$\overset{+}{N}_x = \tfrac{1}{2}[N_x^{(3)}+N_x^{(1)}], \quad \bar{N}_x = \tfrac{1}{2}[N_x^{(3)}-N_x^{(1)}], \tag{43}$$

and so on, and

$$h = h_1 = h_3, \quad D = D_1 = D_3,$$
$$G_x = G_x^{(2)}, \quad G_y = G_y^{(2)}.$$

Discussion

The system of $N+2$ differential equations, (36), together with the appropriate boundary conditions, may be solved for the displacement components of the middle surfaces of the stiff layers. Having obtained these, the remaining quantities of interest can be easily calculated by means of explicit relationships.

Of course it is not easy to solve the present system of equations for an arbitrary shell. However, there is a number of practical shell and plate problems whose solution can be obtained by means of well-known mathematical methods (e.g. linear and nonlinear bending of rectangular plates and cylindrical shells, axisymmetrically loaded circular plates and some shells of revolution).

The governing equations, (36), may be simplified in several ways.

First, they may be linearized by omitting products of displacements, of their derivatives and of forces. This would result in a very general linear theory of thin multisandwich shells. No such theory is in existence at present.

Second, by limiting the number of layers to three, a nonlinear theory of sandwich shells would be obtained. If linearized, this theory would be more general (as it is not for "shallow" shells only) than that in (**13**).

Third, some of the terms multiplied by the principal curvatures may sometimes be neglected as small compared to other terms in some of the equations.

Fourth, by setting $D_i = 0$, membrane characteristics could be assigned to the ith (stiff) layer.

The governing equations of this contribution, if linearized and specialized to multisandwich plates with isotropic weak layers, agree with Bolotin's theory (**1**). If specialized to cylindrical sandwich shells, the present theory is similar to Kurshin's (**14**) except that Kurshin does not consider tangential loads and neglects the terms v/a as compared to $w_{,y}$. This does not mean that Kurshin's theory is inferior for all possible problems, for there exists the fact of two errors canceling each other. However, if the critical hydrostatic pressure acting on an infinitely long regular cylindrical tube ($N = 1$) is calculated, Kurshin's theory yields $p_{cr} = 4D/a^3$ and the present theory, $p_{cr} = 3D/a^3$, which is the correct result (**15**). However, neither theory, despite Kurshin's presentation of a system of buckling equations, may be relied upon when it comes to calculating buckling loads. A dependable theory of buckling is under development by the author. At this point it may be stated that a sandwich or multisandwich shell may not be assumed as "thin" in the derivation of a dependable buckling theory.

References

(**1**) V. V. Bolotin, "Theory of Layered Plates" (in Russian), *Izvestiya Akademii Nauk SSSR, O.T.N., Mekhanika i Mashinostroenie*, No. 3, pp. 65–72, 1963.

(**2**) R. Schmidt, "Flexure of Five-ply Sandwich Plates", Rep. Eng. Res. Lab., Univ. of Arizona, Tucson, Arizona, 1963, pp. 21.

(**3**) J. S. Kao, "Axisymmetrical Deformation of Multi-layer Circular Sandwich Cylindrical Shells", *J. Franklin Inst.*, Vol. 282, No. 1, pp. 31–41, 1966.

(**4**) B.-D. Liaw and R. W. Little, "Theory of Bending of Multilayer Sandwich Plates", *AIAA Journal*, Vol. 5, No. 2, pp. 301–304, 1967.

(**5**) J. P. Wong and A. E. Salama, "Stability of Multilayer Sandwich Plates", *J. Eng. Mech. Div., ASCE*, Vol. 93, No. EM3, pp. 19–31, 1967.

(**6**) A. G. Vasek, "Nonlinear Theory of Multi-layered Sandwich Plates", M.S. thesis, Univ. of Detroit, 1966, pp. 51.

(**7**) L. M. Habip, "A Survey of Modern Developments in the Analysis of Sandwich Structures", *Appl. Mech. Rev.*, Vol. 18, No. 2, pp. 93–98, 1965.

(**8**) V. V. Novozhilov, "The Theory of Thin Shells", Groningen, The Netherlands, P. Noordhoff Ltd., 1959.

(**9**) V. V. Novozhilov, "Foundations of the Nonlinear Theory of Elasticity", Rochester, N.Y., Graylock Press, 1953.

(**10**) C. H. Tsao, "Stress-displacement Relations in Large Displacement Theory of Shells", *AIAA Journal*, Vol. 2, No. 11, pp. 2060–2062, 1964. [This note corrects errors in (**9**).]

(**11**) M. E. Raville, "Deflection and Stresses in a Uniformly Loaded, Simply Supported, Rectangular Sandwich Plate", Forest Prod. Lab. Rep. No. 1847, 1955.

(**12**) H. L. Langhaar and A. P. Boresi, "Strain Energy and Equilibrium of a Shell Subjected to Arbitrary Temperature Distribution", *Proc. Third U.S. Nat. Cong. of Appl. Mech., ASME*, New York, pp. 393–399, 1958.

(**13**) R. Schmidt, "Sandwich Shells of Arbitrary Shape", *Journal of Applied Mechanics*, Vol. 31, *Trans. ASME*, Vol. 86, *Series E*, pp. 239–244, 1964.

(**14**) L. M. Kurshin, "Equations of Three-layered Cylindrical Shells" (in Russian), *Izvestiya Akademii Nauk SSSR, Otd. Tekh. Nauk*, No. 3, pp. 142–144, 1958.

(**15**) F. J. Plantema, "Sandwich Construction", New York, John Wiley & Sons, Inc., p. 196, 1966.

(**16**) E. I. Grigolyuk, "Equations of Three-layered Shell with Light Core" (in Russian), *Izvestiya Akademii Nauk SSSR, O.T.N.*, No. 1, pp. 77–84, 1957.

(**17**) E. Reissner, "Small Bending and Stretching of Sandwich-type Shells", *NACA TN* 1832, 1949.

(**18**) W. A. Nash, "Recent Advances in the Buckling of Thin Shells", *Appl. Mech. Rev.*, Vol. 13, No. 3, pp. 161–164, 1960.