JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 25, 622–627 (1969)

Duality and Bounds for the Matrix Riccati Equation*

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1. INTRODUCTION

Recently, Bellman [2] and Aoki [1] have presented methods for determining upper and lower bounds for the solution of the matrix Riccati equation. Their methods are based on guessing certain matrices and solving associated linear equations. In this paper, a duality relationship for a class of linear regulator problems in optimal control theory is defined. It turns out that the duality relationship, in the context of optimal control, makes quite clear the meaning of Bellman and Aoki's results. Finally a comment is made regarding improvement of the bounds. As indicated, the interpretations are made in an optimal control context; however, analogous interpretations can be given in the context of optimal linear filtering.

2. THE PRIMAL AND DUAL PROBLEMS

Vector and matrix notation is used throughout, with x an n-vector, u an r-vector, and y an m-vector. All matrices are time invariant and of appropriate size. A prime denotes matrix transpose.

Primal Problem

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

$$J = \tfrac{1}{2} \int_0^\infty (y'(t) Q y(t) + u'(t) R u(t))\, dt. \tag{3}$$

The primal problem is to determine $u(t)$ which makes J a minimum.

* This research was supported by the United States Air Force under Grant No. AF-AFOSR-814-66.

Dual Problem

$$\dot{x}(t) = -A'x(t) + C'y(t), \qquad x(0) = x_0 \tag{4}$$

$$u(t) = B'x(t) \tag{5}$$

$$J = \tfrac{1}{2}\int_0^\infty (u'(t)\,R^{-1}u(t) + y'(t)Q^{-1}y(t))\,dt. \tag{6}$$

The dual problem is to determine $y(t)$ which makes J a minimum.

For the primal problem, the following assumptions are made:

(a) A, B is completely controllable

(b) A, C is completely observable

(c) $Q > 0$

(d) $R > 0$.

(If F_1 and F_2 are square matrices, then $F_1 \geqslant F_2$ ($F_1 > F_2$) means $F_1 - F_2$ is positive semidefinite (definite)).

The assumptions for the dual problem are:

(e) $-A'$, C' is completely controllable

(f) $-A'$, B' is completely observable

(g) $Q^{-1} > 0$

(h) $R^{-1} > 0$.

Notice that the primal and dual optimal control problems, as defined above, are closely related. In the primal problem, the input is an r-vector and the output is an m-vector; in the dual problem, the input is an m-vector, the output is an r-vector. It is also clear that the assumptions for the primal problem and the assumptions for the dual problem are equivalent, and these assumptions guarantee the existence of an optimal control law for each problem with the property that the cost is finite and the resulting linear system is asymptotically stable.

It is well known that the optimal control for the primal problem is

$$u(t) = -R^{-1}B'Mx(t) \tag{7}$$

and the optimal cost is

$$J = \tfrac{1}{2}\,x_0'Mx_0\,, \tag{8}$$

where M is the positive definite symmetric matrix which satisfies

$$MA + A'M - MBR^{-1}B'M + C'QC = 0. \tag{9}$$

Similarly, the optimal control for the dual problem is

$$y(t) = -QCNx(t) \tag{10}$$

and the optimal cost is

$$J = \tfrac{1}{2}\, x_0' N x_0 \,, \tag{11}$$

where N is the positive definite symmetric matrix which satisfies

$$-NA' - AN - NC'QCN + BR^{-1}B' = 0. \tag{12}$$

The most significant aspect of the duality relationship is

THEOREM 1.

$$N = M^{-1}. \tag{13}$$

Thus, if one solves the primal (dual) optimal control problem, the solution to the dual (primal) optimal problem can be directly obtained.

3. Determination of Bounds

The problem posed by Bellman and Aoki is to determine upper and lower bounds, in a positive definite sense, for the matrix Riccati equation (9), or (12).

The following two lemmas throw some light on this problem.

LEMMA 1. *Let M_0 be an arbitrary positive definite symmetric matrix such that the matrix $A - BR^{-1}B'M_0$ has eigenvalues with negative real parts. Such an M_0 exists. Then if the control law*

$$u(t) = -R^{-1}B'M_0 x(t) \tag{14}$$

is used in the primal problem the corresponding cost is given by

$$J = \tfrac{1}{2}\, x_0' M_1 x_0 < \infty, \tag{15}$$

where M_1 is the unique positive definite symmetric matrix satisfying

$$M_1(A - BR^{-1}B'M_0) + (A - BR^{-1}B'M_0)'\, M_1 + M_0BR^{-1}B'M_0 + C'QC = 0 \tag{16}$$

Further, it follows that

$$\tfrac{1}{2}\, x_0' M x_0 \leqslant \tfrac{1}{2}\, x_0' M_1 x_0 \,. \tag{17}$$

The analogous result for the dual problem is given in

LEMMA 2. *Let N_0 be an arbitrary positive definite symmetric matrix such* *that the matrix* $-A' - C'QCN_0$ *has eigenvalues with negative real parts. Such an* N_0 *exists. Then if the control law*

$$y(t) = -QCN_0x(t) \tag{18}$$

is used in the dual problem the corresponding cost is given by

$$J = \tfrac{1}{2} x_0'N_1x_0 < \infty, \tag{19}$$

where N_1 *is the unique positive definite symmetric matrix satisfying*

$$N_1(-A' - C'QCN_0) + (-A' - C'QCN_0)' N_1 + N_0C'QCN_0 + BR^{-1}B' = 0. \tag{20}$$

Further, it follows that

$$\tfrac{1}{2} x_0'Nx_0 \leqslant \tfrac{1}{2} x_0'N_1x_0 . \tag{21}$$

It is clear that for any positive definite symmetric matrix M_0 satisfying the hypotheses of Lemma 1, the control law given in (14) is a feasible (or suboptimal) control for the primal problem in the sense that the resulting linear system is asymptotically stable and the corresponding cost function given by (15) is finite for any x_0. For arbitrary choice of M_0, the control (14) will not be optimal and thus the corresponding cost (15) will be no less than the cost using the optimal control, given in (8). Thus, we obtain the inequality in (17). Similarly, for arbitrary N_0, the control in (18) is a feasible suboptimal control for the dual problem and the inequality in (21) is obtained.

By using a suboptimal control for the primal problem and for the dual problem and determining the corresponding cost functions as indicated in Lemma 1 and Lemma 2, it follows that, as a consequence of Theorem 1, the optimal cost function for the primal problem satisfies

$$\tfrac{1}{2} x_0'N_1^{-1}x_0 \leqslant \tfrac{1}{2} x_0'Mx_0 \leqslant \tfrac{1}{2} x_0'M_1x_0 , \tag{22}$$

and the optimal cost function for the dual problem satisfies

$$\tfrac{1}{2} x_0'M_1^{-1}x_0 \leqslant \tfrac{1}{2} x_0'Nx_0 \leqslant \tfrac{1}{2} x_0'N_1x_0 . \tag{23}$$

The inequalities in (22) and (23), stated without regard to the control context, yield a result which is completely equivalent to the result which can be obtained using the method of quasilinearization as proposed by Bellman. For completeness, this result is stated.

THEOREM 2. *Let* M_0 *and* N_0 *be positive definite symmetric matrices chosen so that the matrices* $A - BR^{-1}B'M_0$ *and* $-A' - C'QCN_0$ *have eigen values* *with negative real parts. Then if M_1, N_1 and M are the positive definite solutions to* (16), (20), *and* (9) *respectively,*

$$N_1^{-1} \leqslant M \leqslant M_1 . \tag{24}$$

By guessing the matrices M_0 and N_0, it is possible to determine upper and lower bounds to the solution of the nonlinear equation (9) by solving the linear equations (16) and (20).

4. Improvement of the Bounds

The suboptimal controls generated in Lemma 1 and Lemma 2 can be improved, as indicated by Kleinman [3]. If the control

$$u(t) = -R^{-1}B'M_1x(t) \tag{25}$$

is used for the primal problem and the control

$$y(t) = -QCN_1x(t) \tag{26}$$

is used for the dual problem, it can be shown that the cost corresponding to (25) is less than the cost corresponding to (14) and the cost corresponding to (26) is less than the cost corresponding to (18), and this improvement holds for all x_0. In fact, the control laws can be improved iteratively by generating the sequence M_k, $k = 0, 1,...$ from

$$M_k(A - BR^{-1}B'M_{k-1}) + (A - BR^{-1}B'M_{k-1})' M_k + M_{k-1}BR^{-1}B'M_{k-1} + C'QC = 0$$

for the primal problem and the sequence N_k, $k = 0, 1,...$ from

$$N_k(-A' - C'QCN_{k-1}) + (-A' - C'QCN_{k-1})' N_k + N_{k-1}C'QCN_{k-1} + BR^{-1}B' = 0$$

for the dual problem. These sequences have the properties that

(a) $J = \frac{1}{2} x_0'M_kx_0 [J = \frac{1}{2} x_0'N_kx_0]$ is the cost corresponding to the control $u(t) = -R^{-1}B'M_{k-1}x(t)$ $[y(t) = -QCN_{k-1}x(t)]$ for the primal [dual] problem.

(b) $M \leqslant M_k \leqslant M_{k-1}$, $N \leqslant N_k \leqslant N_{k-1}$

(c) $\lim_{k\to\infty} M_k = M$, $\lim_{k\to\infty} N_k = N$.

The above algorithm is an effective way of improving the suboptimal controls for the primal and dual problems, and hence a way of improving the bounds on the Riccati equation, since

$$N_{k-1}^{-1} \leqslant N_k^{-1} \leqslant M \leqslant M_k \leqslant M_{k-1} .$$

Notice that properties (b) and (c) above imply that the improvement is monotonic, and further that the sequence N_k^{-1}, $k = 0, 1,...$ converges to M from below, and the sequence M_k, $k = 0, 1,...$ converges to M from above. It is interesting to note that the above algorithms can be obtained by application of Newton's method in function spaces. Thus convergence is extremely rapid as the solution is approached.

5. Conclusions

A duality relationship for a class of optimal control problems has been defined, and several implications of this duality have been considered. As a consequence of some known inequalities in optimal control theory, upper and lower bounds for the solution of a matrix Riccati equation can be determined. These bounds have been interpreted in terms of suboptimal control laws for the primal and the dual problem. It has also been shown how these bounds can be improved.

Only the time invariant, infinite time regulator optimal control problem has been considered. This is because the results are rather elegant and only a minimum of assumptions need be made. However, a duality relationship for time varying linear regulator problems can be defined by a direct extension of the definition given here. Under appropriate assumptions, most of the results obtained here can be carried over to the time varying case. In particular, analogous versions of Theorem 1, Lemma 1, Lemma 2, Theorem 2 can be shown to hold. Again the interpretation can be given that the determination of an upper bound for the solution of the Riccati equation (the one corresponding to the primal optimal control Problem) is equivalent to determination of a suboptimal control for the primal problem; determination of a lower bound is equivalent to the determination of a suboptimal control for the dual problem.

References

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