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Exterior Initial-Boundary Value Problems for Quasilinear Hyperbolic Equations in Time-Dependent Domains

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INTRODUCTION

Let Q be a region in Euclidean space R^{n+1} that can be mapped smoothly onto the exterior of an infinite right circular cylinder. Let

$$Q(t) = Q \cap ([0, t] \times R^n);$$

and let $B(t)$ be the lateral boundary of $Q(t)$. We denote $(t, x_1, \dots, x_n) \in Q$ by (t, x), and we write $D_i = \partial/\partial x_i$, $D_t = (\cdot)_t = D_0 = \partial/\partial t$. We adopt the usual convention that repeated indices are to be summed. In this paper we consider two specializations of the following general problem for quasi-linear hyperbolic equations in the time-dependent (noncylindrical) domain Q: for each $T > 0$ find a function u, defined in $Q(T)$, that satisfies (in a sense made precise in Section 2 below) the equations

$$Lu \equiv D_t^2 u - D_j(a_{ij} D_i u) + a_i D_i u + au = -f(t, x, u, D_t u, \dots, D_n u), \tag{1}$$

$$u \mid_{B(T)} = 0, \tag{2}$$

$$u(0, x) = U_0(x), \qquad u_t(0, x) = U_1(x), \tag{3}$$

where $a_{ij}(t, x)$, $a_i(t, x)$, $a(t, x)$, f, U_0, and U_1 are given functions.

One physical model of a problem of this kind is that of scattering of accoustical waves by a moving body in space that also changes its shape with time.

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Our main results are two nonequivalent existence theorems for the problem (1)-(3). Roughly speaking, our first theorem (Theorem 1 of Section 2) is that, under appropriate smoothness conditions on the coefficients of Lu, U_0, U_1, and f, if (i) the supports of f, U_0, and U_1 are compact, (ii) f is independent of the derivatives of u, and (iii) there exists an $M_T > 0$ such that for all $z \in L^2(Q(T))$

$$\int_{Q(T)} |f(t, x, z)|^2 \, dQ < M_T,$$

then there exists at least one weak solution to the problem (1)-(3). The main tool used in the proof is the Leray-Schauder fixed-point theorem. This existence theorem permits considerable nonlinearity in f, but the hypotheses are not strong enough to yield uniqueness. Our second theorem (Theorem 2 of Section 3) relaxes the hypotheses on f in one way and strengthens them in another. It does not require compactness of the initial data, and its hypotheses readily yield uniqueness. Roughly speaking, our second theorem is that, under appropriate smoothness conditions on the coefficients of Lu, f, U_0, and U_1, if f satisfies a Lipschitz condition in its last $n + 2$ arguments, then there exists a unique solution of the problem (1)-(3). The main tool used in the proof, as the Lipschitz condition suggests, is Picard iteration.

There is growing interest in the study of mixed initial boundary-value problems for both linear and nonlinear hyperbolic equations in noncylindrical domains [3], [4], [7]-[9]. In [9], Rogak proved existence and uniqueness of a weak solution to a mixed problem for a semilinear hyperbolic equation in a rather restrictive class of domains. In [6], Lions, using singular perturbation techniques, proved existence, but not uniqueness, for solutions of nonlinear hyperbolic equations in domains expanding with time. We deal here with more general domains, but the nonlinearities we permit are weaker than those permitted by Lions.

The organization of the paper is as follows. In Section 1 we introduce some definitions and state the problem we consider precisely. In Section 2 we state and prove our first existence theorem. In Section 3 we state and prove our second existence (and uniqueness) theorem. In a final section we mention some possible directions for future research.

1. The Problem

There are certain basic assumptions to which we adhere throughout the paper. These are:

(A-1) All coefficients of Lu lie in $C^2(Q)$ with bounded second partial derivatives.

(A-2) The matrix (a_{ij}) is symmetric and uniformly strongly elliptic; that is, there exists a $c > 0$ (c independent of t) such that for any real n-vector $\xi = (\xi_1, \ldots, \xi_n)$

$$\xi_i \xi_i \leqslant a_{ij}(t, x)\, \xi_i \xi_j \leqslant c \xi_i \xi_i .$$

(A-3) The surface B, the boundary of Q, is timelike.

Further hypotheses on f, U_0, and U_1 are more conveniently introduced later.

For any domain X in R^{n+1} or R^n we denote the inner product and norm in $L^2(X)$ by $(\quad,\quad)_X$ and $\|\quad\|_X$, respectively. We denote by $H^1(X)$ the Sobolev space of (classes of) functions in $L^2(X)$ all of whose first partial derivatives (in the sense of distributions) are also in $L^2(X)$. This space is a Hilbert space with norm, for $X \subset R^{n+1}$, given by

$$\| u \|_{H^1(X)} = \left\{ \int_X \left[u^2 + \sum_{i=0}^{n} (D_i u)^2 \right] dX \right\}^{1/2}.$$

By $H_0{}^1(X)$ we denote the closure of $\overset{\circ}{C}{}^\infty(X)$, the space of infinitely differentiable functions with compact support in X, in the norm of $H^1(X)$.

The Sobolev embedding theorems [1], [10], permit us to speak of values of functions $u \in H^1(X)$ restricted to the boundary, or a portion of the boundary, of X. The restriction map is the extension by continuity of the restriction map for C^∞-functions which are dense in $H^1(X)$. It is in this sense that we shall speak of the boundary values of functions in $H^1(X)$.

Let $\Omega(t_0)$ denote the section of Q cut by the hyperplane $t = t_0$. We can now introduce the first of our assumptions upon U_0 and U_1.

(A-4) $U_0 \in H_0{}^1(\Omega(0))$ and $U_1 \in L^2(\Omega(0))$.

We define a weak solution to the mixed initial boundary-value problem (1)-(3) in the usual way [5] by multiplying both sides of equation (1) by a suitable test function, integrating by parts, and seeking a solution to the resulting integral identity. We choose the class of test functions to be

$$\Phi = \{\phi \mid \phi \in C^\infty(Q(T)), \operatorname{supp} \phi \subset Q(T) \text{ and is compact},$$
$$D_t\phi(T, x) = \phi(T, x) = \phi \mid_{B(T)} = 0\}.$$

Definition 1. A *weak solution* to the mixed initial boundary-value problem (1)-(3) is a function $u \in H^1(Q(T))$ such that $u\,|_{B(T)} = 0$, $u\,|_{\Omega(0)} = U_0$, and

$$\int_{Q(T)} \{D_t u\, D_t\phi - (a_{ij}\, D_i u)\, D_j\phi - \phi(a_i\, D_i u + au)\}\, dQ$$

$$= \int_{Q(T)} f(t, x, u, D_t u, \ldots, D_n u)\, \phi\, dQ + \int_{\Omega(0)} U_1\, D_t\phi\, d\Omega \tag{4}$$

for each $\phi \in \Phi$.

For convenience, we shall write this integral identity as

$$(L^{1/2}u, L^{1/2}\phi)_{Q(T)} = (f(t, x, u, D_t u, \ldots, D_n u), \phi)_{Q(T)} + (U_1, D_t\phi)_{\Omega(0)}\,. \tag{5}$$

It is clear that if a weak solution u is actually smooth, then it is a classical solution.

It was proved by Lee [4] that the corresponding *linear* problem

$$Lu = h(t, x) \qquad \text{for} \qquad (t, x) \in Q(T),$$
$$u\,|_{B(T)} = 0,$$
$$u(0, x) = U_0(x), \qquad u_t(0, x) = U_1(x)$$

has a unique weak solution if $h \in L^2(Q(T))$ and hypotheses A-1 through A-4 are satisfied. Lee also obtained an energy inequality for weak solutions: For $t \in [0, T]$,

$$\int_{\Omega(t)} e(u)\, d\Omega \leqslant K \left[\int_{\Omega(0)} e(u)\, d\Omega + \int_{Q(t)} f^2\, dQ \right], \tag{6}$$

where

$$e(u) = u^2 + (D_t u)^2 + a_{ij}\, D_i u\, D_j u$$

and $K = K(T)$ is a constant independent of u and t.

REMARK. If u is a weak solution to the linear problem, and all the data have compact support in $Q(T)$, then u also has compact support in $Q(T)$, and the support of u depends only on the support of the data and T.

This follows directly from a sharper form of Lee's energy inequality; see [4; Theorem 3.1].

2. EXISTENCE OF WEAK SOLUTIONS

We next make some further assumptions on f, U_0, and U_1, the last three of which will be used in Theorem 1 only.

(A-5) For each i $(0 \leqslant i \leqslant n+1)$ the map

$$z_i \to f(t, x, z_0, \ldots, z_i, \ldots, z_{n+1}),$$

considered as a map from $L^2(Q(T))$ into itself, is continuous.

(A-6) For each positive T the map

$$(t, x) \to f(t, x, z_0) \in L^2(Q(T))$$

for all $z_0 \in L^2(Q(T))$ and has compact support in $Q(T)$.

(A-7) For each positive T there exists a positive M_T such that for all $z_0 \in L^2(Q(T))$

$$\int_{Q(T)} |f(t, x, z_0)|^2 \, dQ < M_T.$$

(A-8) The supports of U_0 and U_1 are compact in $\Omega(0)$.

If the hypotheses A-6 and A-8 hold, it follows from the remark succeeding (6) that, if u is a weak solution to the linear problem

$$(*) \qquad \begin{aligned} Lu &= -f(t, x, z_0), \qquad & u\,|_{B(T)} &= 0, \\ u(0, x) &= U_0(x), \qquad \text{and} \qquad & u_t(0, x) &= U_1(x), \end{aligned}$$

where z_0 is a fixed element of $L^2(Q(T))$, then supp u is a compact set depending only on the supports of U_0, U_1, and f [as a function of (t, x)]. We may thus truncate $Q(T)$ by an R^{n+1} sphere to a bounded domain $Q_1(T)$ that contains the supp u uniformly for all $z_0 \in L^2(Q(T))$ and the supports of U_0, U_1 and f. We choose the truncating sphere sufficiently large to ensure that $B(T)$ remains part of the lateral boundary of the truncated domain. We denote the section of $Q_1(T)$ at t_0 by $\Omega_1(t_0)$.

Definition 2.

$$\mathscr{V} = \{v \mid v \in H^1(Q_1(T)),\, v\,|_{B(T)} = 0,\, v\,|_{\Omega(0)} = U_0(x)\}.$$

Note

$$H_0^1(Q_1(T)) \subset \mathscr{V} \subset H^1(Q_1(T),$$

and $\mathscr{V}$ is a closed subspace of $H^1(Q_1(T))$.

Theorem 1. *Suppose that f is independent of the $D_i u$ and that assumptions* A-1 *through* A-8 *are satisfied, then there exists at least one weak solution to the* *problem* (1)-(3); *that is, there exists at least one function* u *such that* $u \in H^1(Q_1(T))$, supp $u \subset Q_1(T)$ *and is compact,* $u|_{\Omega(0)} = U_0$, $u|_{B(T)} = 0$, *and* u *satisfies*

$$(L^{1/2}u, L^{1/2}\phi)_{Q(T)} = (f(t, x, u), \phi)_{Q(T)} + (U_1, D_t\phi)_{\Omega(0)}$$

for each $\phi \in \Phi$.

PROOF OF THEOREM 1.

For $v \in \mathscr{V}$ and $\lambda \in [0, 1]$, $f(t, x, \lambda v) \in L^2(Q_1(T))$. Let

$$\mathscr{C} : \mathscr{V} \times [0, 1] \to \mathscr{V}$$

be defined by

$$\mathscr{C}(v, \lambda) = \mathscr{C}_\lambda(v) = u_\lambda - (1 - \lambda)\, u_0 ,$$

where $u_\lambda (0 \leqslant \lambda \leqslant 1)$ is the unique solution (whose existence is guaranteed by Lee's Theorem 4.4 of [4]) to the *linear* problem $(*)$ in which z_0 is chosen to be λv. We shall show that the map $\mathscr{C}$ satisfies the conditions of the Leray-Schauder fixed point theorem in the form given by Browder [2, Lemma 24] and thus conclude that $\mathscr{C}$ has a fixed point for $\lambda = 1$. Obviously, such a fixed point is a weak solution to our problem.

LEMMA 1. *Let* u *be the weak solution to* $(*)$. *If* E *is a bound for the initial energy of the system; that is, if*

$$\int_{\Omega_1(0)} e(u)\, d\Omega < E < \infty,$$

then

$$\| u \|^2_{H^1(Q_1(t))} \leqslant K \left[tE + \int_0^t \| f \|^2_{Q_1(\tau)}\, d\tau \right] \tag{7}$$

where $0 \leqslant \tau \leqslant t \leqslant T$.

PROOF. Lee's energy inequality takes the form

$$\begin{aligned} \int_{\Omega_1(t)} e(t, x)\, d\Omega &\leqslant K \left[\int_{\Omega_1(0)} e(0, x)\, d\Omega + \int_{Q_1(t)} f^2\, dQ \right] \\ &\leqslant K[E + \| f \|^2_{Q_1(t)}]. \end{aligned} \tag{8}$$

By the positive definiteness of (a_{ij}), the left-hand side of (8) dominates

$\| u \|^2_{H^1(\Omega_1(t))}$. Since $u \in H^1(Q_1(T))$, $e(u) \in L^1(Q_1(T))$. Thus, by Fubini's Theorem,

$$g(t) = \int_{\Omega_1(t)} e(u)\, d\Omega$$

is summable on $[0, T]$. Since $g(t) \leqslant KE + K \| f \|^2_{Q_1(T)}$, the desired inequality (7) is obtained from (8) by integration.

COROLLARY. *If u is the weak solution to $(*)$ with null initial data,*

$$\| u \|^2_{H^1(Q_1(T))} \leqslant KT \| f \|^2_{Q_1(T)} . \tag{9}$$

This follows immediately from the observation that for $t \in [0, T]$, $\| f \|^2_{Q_1(t)}$ is a monotone nondecreasing function of t.

REMARK. We note that the boundedness of $Q_1(T)$, which is necessary later on in our proof of Theorem 1, was not needed in Lemma 1 and that (7) and (9) actually hold for solutions to the linear problem $(*)$ even if f is also a function of $z_1 ,..., z_{n+1}$. This will be used in the proof of Theorem 2.

LEMMA 2. *$\mathscr{C}$ is a compact map from $\mathscr{V} \times [0, 1] \to \mathscr{V}$.*

PROOF. We first establish continuity and then compactness.

(i) $\mathscr{C}$ is continuous. Let $\lambda_j \to \lambda$ and let $v_j \to v$ in $\mathscr{V}$. Set $u_j = \mathscr{C}_{\lambda_j} v_j$ and $\mathscr{C}_\lambda v = u$.

We must prove $u_j \to u$ in $\mathscr{V}$.

By definition,

$$u_j = \bar{u}_j - (1 - \lambda_j)\, u_0 ,$$

where $\bar{u}_j$ is the solution to $(*)$ with z_0 replaced by $\lambda_j v_j$; and $u = u_\lambda - (1 - \lambda)\, u_0$. Since $\bar{u}_j - u_\lambda$ is the weak solution to the linear problem

$$L(\bar{u}_j - u_\lambda) = f(t, x, \lambda v) - f(t, x, \lambda_j v_j),$$
$$(\bar{u}_j - u_\lambda) \,|_{B(T)} = 0$$

with zero initial conditions, we may apply (9) to $\| \bar{u}_j - u_\lambda \|_{\mathscr{V}}$. Thus

$$\begin{aligned} \| u_j - u \|_{\mathscr{V}} &\leqslant \| \bar{u}_j - u_\lambda \|_{\mathscr{V}} + | \lambda_j - \lambda | \cdot \| u_0 \|_{\mathscr{V}} \\ &< (KT)^{1/2} \| f(t, x, \lambda v) - f(t, x, \lambda_j v_j) \|_{Q_1(T)} + | \lambda_j - \lambda | \cdot \| u_0 \|_{\mathscr{V}} . \end{aligned}$$

Since f is assumed to be continuous in its third argument, $u_j \to u$ in $\mathscr{V}$.

(ii) $\mathscr{C}$ is compact. Let $\{\lambda_j, v_j\}$ be a bounded sequence in $\mathscr{V} \times [0, 1]$, and let $\{u_j\}$ be the corresponding sequence of image points. It suffices to show there exists a subsequence of $\{u_j\}$ Cauchy in $\mathscr{V}$.

Now $\{v_j\}$ is bounded in $\mathscr{V}$. Since $Q_1(T)$ is bounded, the injection from $H^1(Q_1(T))$ to $L^2(Q_1(T))$ is a compact map by the Rellich-Kondrashoff Theorem [1, p. 32]. Thus there exists a subsequence $\{v_{j'}\}$ of $\{v_j\}$ Cauchy in $L^2(Q_1(T))$. Since $\{\lambda_{j'}\}$ is precompact, there exists a subsequence $\{\lambda_{j''}\}$ Cauchy in the reals, and $\{\lambda_{j''}v_{j''}\}$ is also Cauchy in $L^2(Q_1(T))$. Let us relabel the indices, drop the primes, and consider $\{\lambda_j v_j\}$ to be Cauchy in $L^2(Q_1(T))$.

But

$$\begin{aligned} \| u_j - u_k \|_{\mathscr{V}} &= \| \bar{u}_j - (1 - \lambda_j) u_0 - \bar{u}_k + (1 - \lambda_k) u_0 \|_{\mathscr{V}} \\ &\leqslant \| \bar{u}_j - \bar{u}_k \|_{\mathscr{V}} + | \lambda_j - \lambda_k | \cdot \| u_0 \|_{\mathscr{V}} . \end{aligned}$$

Since $\bar{u}_j - \bar{u}_k$ is the solution to a linear problem with zero initial conditions, we may again apply the energy inequality (9) and use the continuity of f in its third argument to conclude that $\{u_j\}$ is Cauchy in $\mathscr{V}$. This completes the proof of Lemma 2.

Continuing the proof of Theorem 1, we next define

$$\beta = \{v \mid v \in \mathscr{V}, \| v \|_{H^1(Q_1(T))} \leqslant 3N\},$$

where $N = [KT(M_T + E)]^{1/2}$. (See assumption A-7 and Lemma 1 for definitions of M_T and E.)

PROPOSITION. *$\mathscr{C}_\lambda(v) \neq v$ for all $v \in \partial\beta$, the boundary of β, and $\mathscr{C}_0(v) = 0$ for all $v \in \beta$.*

PROOF. If for some $v \in \partial\beta$, $\mathscr{C}_\lambda(v) = v$, then it would be that

$$v = u_\lambda - (1 - \lambda) u_0$$

where u_λ and u_0 are both weak solutions to linear problems of type $(*)$. But from (7) it follows that

$$\| v \|_{\mathscr{V}} \leqslant \| u_\lambda \|_{\mathscr{V}} + | 1 - \lambda | \cdot \| u_0 \|_{\mathscr{V}} \leqslant 2[KT(M_T + E)]^{1/2} = 2N.$$

This contradicts the hypothesis that $v \in \partial\beta$. The second assertion in the proposition is obvious.

We have now shown that all the conditions of the Leray-Schauder Theorem are satisfied by $\mathscr{C}$. Thus there exists at least one $v \in \beta \subset \mathscr{V}$ such that $\mathscr{C}(v, 1) = v$. That is,

$$(L^{1/2}v, L^{1/2}\phi)_{Q_1(T)} = (f(t, x, v), \phi)_{Q_1(T)} + (U_1, D_t\phi)_{\Omega_1(0)}$$

for all $\phi \in \Phi$, and

$$v\,|_{B(T)} = 0, \qquad \text{and} \qquad v\,|_{\Omega_1(0)} = U_0 .$$

Since the supports of v, f, U_0, and U_1 are all contained in $Q_1(T)$, the above integrals can be extended to the untruncated domains $Q(T)$ and $\Omega(0)$. This completes the proof of Theorem 1.

3. Existence and Uniqueness

In order to prove both existence and uniqueness of a solution to the problem (1)-(3), we find it necessary to require more of the function f in one way. On the other hand, we are enabled in doing this to relax other requirements such as compactness of the data and the independence of f from the derivatives of u. The restrictive assumption on f is

(A-9) The function f satisfies a Lipschitz condition: for each T there exists a positive number L, depending only on T, such that, for every $u, v \in H^1(Q(T))$,

$$\| f(t, x, u, D_tu,..., D_nu) - f(t, x, v, D_tv,..., D_nv) \|_{Q(T)}^2 \leqslant L \| u - v \|_{H^1(Q(T))}^2 . \tag{10}$$

Theorem 2. *Suppose that assumptions* A-1 *through* A-5 *and assumption* A-9 *are satisfied. Then there exists a unique weak solution u to the mixed initial boundary-value problem* (1)-(3). *This solution depends continuously on the initial data and f in the sense that*

$$\| u \|_{H^1(Q(T))}^2 \leqslant K(T) \left[T \int_{\Omega(0)} e(u)\, d\Omega + \int_0^T \| f \|_{Q(\tau)}^2\, d\tau \right] .$$

Proof. We execute the following iteration procedure. Let $u_0(t, x) = U_0(x)$. Let u_{k+1} be the unique element in $H^1(Q(T))$ for which

$$u_{k+1}(0, x) = U_0(x), \qquad u_{k+1}\,|_{B(T)} = 0,$$

and

$$(L^{1/2}u_{k+1}, L^{1/2}\phi)_{Q(T)} = (f(t, x, u_k, D_t u_k, \ldots, D_n u_k), \phi)_{Q(T)} + (U_1, D_t\phi)_{\Omega(0)} \tag{11}$$

for all $\phi \in \Phi$, $k = 0, 1, 2, \ldots$.

Subtracting the corresponding members of (11) for any two successive values of k ($k \geqslant 1$), we find

$$(L^{1/2}(u_{k+1} - u_k), L^{1/2}\phi)_{Q(T)} = (f_k - f_{k-1}, \phi)_{Q(T)}, \tag{12}$$

where $f_k = f(t, x, u_k, \ldots, D_n u_k)$. Note that $u_{k+1} - u_k$ is a weak solution to problem (1)-(3) with right-hand side $f_k - f_{k-1}$ and zero initial data. Lemma 1, with $Q_1(t)$ replaced by $Q(t)$, the identity (12), and assumption A-9 imply together that, for each $t \in [0, T]$,

$$\begin{aligned} \| u_{k+1} - u_k \|^2_{H^1(Q(t))} &\leqslant K \int_0^t \| f_k - f_{k-1} \|^2_{Q(\tau)} \, d\tau \\ &\leqslant KL \int_0^t \| u_k - u_{k-1} \|^2_{H^1(Q(\tau))} \, d\tau. \end{aligned} \tag{13}$$

From (13) a simple induction shows that, if

$$\chi(t) = \sup_{\tau \in [0, t]} \{KL, \| u_1 \|^2_{H^1(Q(\tau))}, 1\},$$

then

$$\| u_{k+1} - u_k \|^2_{H^1(Q(T))} \leqslant \frac{[\chi(T)]^{k+1} T^k}{k!}.$$

Thus $\{u_k\}$ is a Cauchy sequence in $H^1(Q(T))$. We denote the limit by u. Because $u_k \to u$ in $H^1(Q(T))$, we can pass to the limit in the left member of (11). For the same reason, as well as the fact that f is continuous in all but possibly its first two arguments, we can do the same in the right-hand member of (11). We obtain the identity

$$(L^{1/2}u, L^{1/2}\phi)_{Q(T)} = (f(t, x, u, D_t u, \ldots, D_n u), \phi)_{Q(T)} + (U_1, D_t\phi)_{\Omega(0)}.$$

for all $\phi \in \Phi$. Moreover, $u \mid_{B(T)} = 0$, and $u \mid_{\Omega(0)} = U_0$ since the set of functions in $H^1(Q(T))$ with these properties is closed in $H^1(Q(T))$.

Uniqueness of the weak solution u just found follows from Lemma 1 and A-9 by a standard argument. The continuous dependence of u on the data follows directly since each u_k satisfies the energy inequality of Lemma 1.

4. Further Problems

An outstanding example of a source function f not allowed by either of our two main theorems is $f = u^3$. It is desirable to weaken the growth conditions we have imposed upon f in any way possible.

Lions remarked in [5, p. 150] that it would be useful to extend theorems of the type given here to the setting of operator equations. The literature that has appeared on hyperbolic equations in noncylindrical domains since 1964 gives reason to hope this soon may be done.

In the case of cylindrical domains, the asymptotic behavior of u as $t \to \infty$ has been studied extensively. No results are known for the asymptotic behavior of u as $t \to \infty$ in noncylindrical domains. This kind of problem arises, for example, in the study of diffraction of radar waves by any pulsating and/or rapidly moving body, and hence is of high interest.

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