The logarithmic singularity in the specific heat is used to investigate the logarithmic corrections to the phenomenological theory of the \( \lambda \)-transition in liquid helium-4. A justification of the theory of Mamaladze is also given.

In a previous communication [1], a new phenomenological theory for the \( \lambda \)-transition of liquid helium, based on a scaling law analysis and the identification of the order parameter as the condensate density \( n_0 \), was presented. The critical exponents of various physical quantities predicted by this theory were in good agreement with experimental data, up to a point. The reservation expressed here arises from the inability of the scaling-law analysis to distinguish between a \( (\tau^x \ln \tau^{-1}) \) dependence and for example a \( (\tau^{-x} \ln \tau^{-1}) \) dependence, where \( x \) is some critical exponent and \( \tau = 1 - T/T_\lambda \). Because of this ambiguity in the scaling-law analysis, we use, in this note, the observed logarithmic singularity in the specific heat as a basis to investigate the logarithmic corrections to the theory of ref. 1. We also point out a possible justification of the somewhat ad hoc theory of Mamaladze [2, 3].

The phenomenological theory begins with the Gibbs free-energy density written in the form
\[
f(\Psi) = f_1(\tau) + A(\tau) \left| \nabla \Psi \right|^2 - B(\tau) \left| \Psi \right|^2 + \frac{1}{2} C(\tau) \left| \Psi \right|^4,
\]
(1)
where the notation of ref. 1 is followed. In the bulk sample at equilibrium, eq. (1) yields
\[
n_0 \sim \left| \Psi \right|^2 = B/C \quad \text{and} \quad f = f_1 - \frac{1}{2} B^2/C.
\]

In the presence of low-speed superflow, eq. (1) gives the following expression for the superfluid density [4]: \( \rho_s = (2m^2/k^2)AB/C \). Finally for the theory to be valid, we need the fluctuations in the order-parameter to be small, which can be expressed by condition:
\[
(\xi_\perp/e_\parallel) \approx (\sqrt{2}/8\pi\epsilon)(C/B^{1/2}A^{1/2}) k_BT_\lambda \ll 1.
\]

The conditions of ref. 1 can be summarized by \( A \sim \tau^{-1/3}, B \sim \tau, C \sim \text{const.} \), with \( f - f_1 \sim -\tau^2 \), \( \rho_s \sim \tau^{2/3} \) and \( (\xi_\perp/e_\parallel) \sim \text{const.} \).

From the observed logarithmic singularity in the specific heat \( C_\rho \) over 4 decades [5], we have that \( f - f_1 = f_0 \tau^2 \ln |\Lambda/\tau| \), where \( f_0 \approx (5/16)nkB_T\lambda \), \( n \) is the number density at \( T_\lambda \) and \( \Lambda = e^{11/2} \approx 245 \). From recent measurements of \( \rho_s \) over 3 decades [6,7], we see that \( \rho_s \sim \tau^{2/3} \).

Therefore it is desirable to construct a theory such that (I) \( B^2/C \sim \tau^2 \ln |\Lambda/\tau| \); (II) \( AB/C \sim \tau^{2/3} \), and (III) \( A^{1/2}B^{1/2}/C \sim \text{const.} \). Condition III expresses the expectation that the fluctuations are small right up to \( T_\lambda \). Unfortunately, these three conditions on the three coefficients are not compatible; and one of these conditions must be changed. From this point of view, the theory of ref. 1 represents the replacing of condition (I) by (I') \( B^2/C \sim \tau^2 \) in the spirit of the scaling-law analysis. However, a) as \( C_\rho \) is measured to a precision one order of magnitude greater than \( \rho_s \); b) as there exists an exact logarithmic singularity in the specific heat for the two-dimensional Ising model; and c) as \( \rho_s \) is not analogous to quantities which can be measured in other second-order transitions; we retain condition (I) in preference to (II). In place of condition (II), we use (II') \( B/C \sim \tau \), which is derived from the microscopic result [8, 9] \( n_0 \sim \tau \). The conditions (I), (II'), (III) determine unambiguously the following temperature dependences:
\[
A(\tau) = A_0(\tau^{-1} \ln |\Lambda/\tau|)^{1/2}, \quad B(\tau) = B_0 \tau^2 \ln |\Lambda/\tau|, \quad C(\tau) = C_0 \ln |\Lambda/\tau|.
\]
Note that \( \rho_s \) is now proportional to \( (\tau^2 \ln |\Lambda/\tau|)^{1/2} \).

To evaluate the coefficients, we observe that there are yet no experimental data on \( n_0(\tau) \).

Therefore, we introduce the change in variable
\( \Psi = (2mA)^{1/2} \Psi \) (such that \( \rho_0 = m \vert \nabla \Psi \vert^2 \)) into eq. (1) to obtain

\[
f(\Psi) = f_1(\tau) + (\hbar^2/2m) \nabla \Psi \cdot \nabla \Psi - \alpha(\tau) \vert \Psi \vert^2 + \frac{1}{2} \beta(\tau) \vert \Psi \vert^4,
\]

where

\[
\alpha(\tau) = \alpha_0 (\tau^2 \ln \vert A/\tau \vert)^{1/2},
\]

\[
\beta(\tau) = \beta_0 (\tau^2 \ln \vert A/\tau \vert)^{1/2},
\]

with \( \alpha_0 = (\hbar^2/2m) B_0/A_0 \) and \( \beta_0 = (\hbar^2/2m)^2 \times C_0/A_0^2 \). If the logarithmic corrections are omitted in eq. (3), then eqs. (2, 3) reproduce the somewhat ad hoc theory of Mamaladze [2, 3]. In particular, no \( \vert \Psi \vert^6 \) term is needed, contrary to the suggestion of Amit [10] and in agreement with experimental data [3]. Note that \( f \) is a functional of \( \Psi \), not \( \Psi \), so that the instability of eq. (2) with respect to \( \Psi \) is irrelevant. Finally, we remark that the data of \( \rho_0 \) [7] can be fitted well with \( \rho_0 = 1.05 \times 10^{-16} \) erg \( \tau^2 \ln \vert A/\tau \vert \). Using this result for \( \rho_0 \) and the expression for \( f \) derived from \( c_D \), we obtain \( \alpha_0 = 1.8 \times 10^{-16} \) erg cm\(^3\) and \( \beta_0 = 7.7 \times 10^{-39} \) erg cm\(^3\).

Similar logarithmic corrections apply to the phenomenological theory of critical points [11, 12]. For example, the corrected coefficients of the two-dimensional Ising model [11] are:

\[
A(\tau) \sim \tau^{-1/4}; \quad B(\tau) \sim \tau^{1/4} \ln \vert A/\tau \vert; \quad C_n(\tau) \sim \ln \vert A/\tau \vert; \quad \text{where} \ A = \xi/\eta.
\]

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References


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HOLOGRAPHIE UND INTERFERENZVERSUCHE MIT INHOMOGENEN OBERFLÄCHENWELLEN

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Holography with the inhomogeneous surface wave of total reflection, reconstruction of this surface wave, and interference of two surface waves are discussed.