*Minimizing System Sensitivity Through Feedback**

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ABSTRACT: *A function space representation is used to examine the problem of reducing system sensitivity by means of feedback. It is shown that this question leads naturally to the problem of minimizing an abstract sensitivity index. In the course of the study earlier results of Cruz* (**3**) *and Zames* (**15**) *for stationary differential systems are extended to cover large classes of nonstationary discrete, distributive and composite systems.*

I. Introduction

The study of system sensitivity as initiated by Bode (**1**) and carried forward by numerous others [see for instance (**2, 3, 4, 5**)] is concerned with: (1) The definition of a *measure of the change* for some system characteristic (arising from a class of disturbances); and (2) The development of design procedures to minimize, with respect to this measure, fluctuations in the system characteristic. In an earlier paper (**6**) it is shown that a function space representation of linear systems could be used to advantage in the formulation of system sensitivity measures. An important feature of such an approach is that free and forced response sensitivity problems of discrete, continuous and composite systems can be treated within a common framework.

In the present paper the function space representation is used once more to examine design questions related to the reduction of system sensitivity. The analysis deals with linear systems subjected to input and output disturbances and (*not* necessarily linear) plant variations with emphasis on the question: *What are some of the fundamental limitations of the reduction of sensitivity by use of feedback?* To investigate this question efficiently, standard notation and terminology from the domain of functional analysis must be used. (Refs. (**7–9**) are introductory texts which contain all the necessary definitions.) The development of the paper, however, is guided by engineering reasoning as well as mathematical considerations.

II. System Equations

One of the classical areas of system analysis is the study of the use of feedback to reduce system sensitivity to component variations and other disturbances. The present analysis deals with the simple closed loop system of Fig. 1. In this figure the following transformations are evident: $F:B_1 \rightarrow B_2$ represents

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the internal response of a fixed plant, $G:B_1 \rightarrow B_1$ and $L:B_3 \rightarrow B_1$ are compensation transformations while $K:B_2 \rightarrow B_2$ and $J:B_2 \rightarrow B_3$ are output constraints. The symbols B_1, B_2 and B_3 denote Banach spaces. The system variables are as follows: u and x denote the system input and useful output respectively, e_4 denotes the observable output, e_3 denotes the internal (state function space) response, e_2 represents the plant input, while the elements ξ and η denote system disturbances.

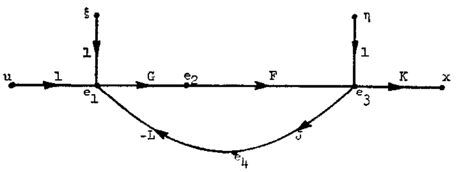

FIG. 1. Simple feedback system.

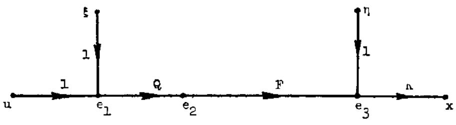

FIG. 2. Simple open loop system.

The defining equations for the system are evidently

$$e_3 = \eta + FGe_1$$
$$e_1 = u + \xi - LJe_3.$$

Letting $M = LJ$ these equations imply the set

$$\begin{aligned} e_3 &= \eta + FG[u + \xi - Me_3] \\ e_1 &= u + \xi - M[\eta + FGe_1] \end{aligned} \tag{1}$$

which implicitly determine the response variables $\{e_3, e_1\}$ in terms of the forcing functions $\{u, \xi, \eta\}$. For linear transformations such that $I + MFG$ and $I + FGM$ are invertable, Eqs. 1 simplify to the familiar forms

$$\begin{aligned} e_3 &= (I + FGM)^{-1}[FG(u + \xi) + \eta] \\ e_1 &= (I + MFG)^{-1}[u + \xi - M\eta]. \end{aligned} \tag{2}$$

In comparing the sensitivities of open and closed loop systems it is necessary to establish a terminal equivalence between the two system types. In Fig. 2 an open loop system which is comparable to the closed loop system of Fig. 1 is depicted. Since ξ and η are disturbance elements, the systems of Figs. 1 and 2 are called *nominally equivalent* if the $(\xi, \eta = 0)$ terminal mapping $u \rightarrow x$ is the same in both cases. From Eq. 2 and Fig. 2 it follows that the two linear systems are nominally equivalent whenever $(I + FGM)^{-1}FG = FQ$ which, assuming that $I - MFQ$ is invertible, is satisfied whenever

$$G = Q(I - MFQ)^{-1}. \tag{3}$$

In Eq. 3, F and Q are fixed transformations. This equality then relates G explicitly to the transformation M which remains as the independent compensation transformation of the closed loop system. In addition to Eq. 3, several other equalities summarized by the following lemma are useful in the following discussion.

Lemma 1. Suppose that all transformations are linear. Then, when the indicated inverses exist, the equality

(i) $G = Q(I - MFQ)^{-1}$ implies the identities

(ii) $(I + FGM)^{-1} = I - FQM$ (iv) $(I - FGM)^{-1}FG = FQ$

(iii) $(I + MFG)^{-1} = I - MFQ$

(v) $[I - M(I + FGM)]^{-1}FG = I - MFQ.$

Proof: Identity (v) follows immediately from (iv) which follows from (i). To prove (ii) and (iii) it is necessary only to clear fractions. For example, if (i) holds then

$$\begin{aligned}(I + MFG)(I - MFQ) &= (I - MFQ) + MF[G(I - MFQ)] \\ &= (I - MFQ) + MF[Q] = I.\end{aligned}$$

It is emphasized that Lemma 1 assumes the invertibility of the operators in question. It is not difficult to formulate sufficient conditions which imply this assumption. For example, one rather severe condition [see (**10**), p. 164] states that if $A: B \to B$ and $\| A \| < 1$ then $(I - A)^{-1}$ exists and is bounded on B. For stationary systems on the interval $(-\infty, \infty)$ this condition can be interpreted as requiring the system frequency response plot to lie inside the circle of unit radius about the origin. A second sufficient condition is given in Appendix A.

III. Perturbation Equations

We now return to the investigation of the effects of feedback on system sensitivity. Consider the systems of Figs. 1 and 2 with all transformations being linear and the five identities in Lemma 1 holding. The case $\xi = 0$, $\eta = 0$ is taken as the nominal for both systems. We consider two types of perturbations from this nominal: First, the disturbance elements $\{\xi, \eta\}$ which are already included; Second, a bounded additive variation δF in the transformation F which is *not* necessarily linear. It is clear from Fig. 2 that $\bar{e}_2 = Qu$, $\bar{e}_3 = FQu$ and $\bar{x} = KFQu$ are the respective open loop system nominal responses for the input u. The deviations δe_3 and δx from nominal are also immediate, being given by

$$\delta e_3 = FQ\xi + \eta + \delta FQ(u + \xi) \tag{4}$$

$$\delta x = K\delta e_3. \tag{5}$$

Turning now to the closed loop system of Fig. 1 and specifically to Eq. set 1 we have the perturbation relationships

$$\begin{aligned}\bar{e}_3 + \delta e_3 &= \eta + (F + \delta F)G[u + \xi - M(\bar{e}_3 + \delta e_3)] \\ \bar{e}_1 + \delta e_1 &= u + \xi - M[\eta + (F + \delta F)G(\bar{e}_1 + \delta e_1)]\end{aligned}$$

where $\bar{e}_3$, $\bar{e}_1$ satisfy Eq. set 1 (with $\xi = 0, \eta = 0$). Using the form of $\bar{e}_3$ and $\bar{e}_1$

it follows easily that

$$(I + FGM)\delta e_3 = \eta + FG\xi + \delta FG[u + \xi - M(\bar{e}_3 + \delta e_3)]$$
$$(I + MFG)\delta e_1 = \xi - M\eta - M\delta FG(\bar{e}_1 + \delta e_1),$$

which further simplify to the forms

$$(I + FGM)\delta e_3 = \eta + FG\xi + \delta F[Qu + G\xi - GM\delta e_3] \tag{6}$$

$$(I + MFG)\delta e_1 = \xi - M\eta - M\delta F[Qu + G\delta e_1] \tag{7}$$

which implicitly[1] define δe_3 and δe_1.

Several results of current interest may be obtained from Eqs. 6 and 7. The most interesting of these deal with design procedures for minimizing the effects of the system disturbances. Since the nominal closed loop transfer function is constrained at the fixed valve, FQ, the effects of feedback on the input disturbance, ξ, are second order at best. Thus we consider the case $\xi = 0$.

Remark 1. Suppose that δF is linear (and $\xi = 0$). For convenience δe_o and δe_c denote the perturbation δe_3 in the open loop and closed loop cases, respectively, also the notation $F_a = F + \delta F$ is useful. From Eq. 4 it follows that $\delta e_o = \eta + \delta FQu$ which, in conjunction with Eq. 6, results in $(I + F)\delta e_c = \eta + \delta FQu = \delta e_o$.

Assume now that $I + F_aGM$ has bounded inverse and define $N_a = (I + F_aGM)^{-1}$. Then, from the expression $\delta e_c = N_a\delta e_o$ a generalization of a result due to Perkins and Cruz (**3**) is easily obtained. For Hilbert spaces clearly

$$\| \delta e_c \|^2 = \langle N_a\delta e_o, N_a\delta e_o \rangle = \| \delta e_o \|^2 - \langle \delta e_o, (I - N_a{}^*N_a)\delta e_o \rangle$$

and consequently a sufficient condition for reduced (strictly reduced) sensitivity in the feedback case is that $I - N_a{}^*N_a$ be a positive (strictly positive) operator.

Cruz and Perkins (**1**), also (**17**), consider stationary multivariate systems. To obtain their result let the transformations in question act between finite cartesian products of $L_2(-\infty, \infty)$ equipped with the usual innerproduct. Now if F_a, Q, L, and J and hence N_a are stationary, bounded and linear, then, by Bochner's L_2 theorem [see (**11**)], these transformations must be representable by multiplicative frequency response matrices. Moreover using Plancharel's theorem [see (**12**)] it is easy to show that

$$I - N_a{}^*N_a \geq 0 \Leftrightarrow I - \hat{N}_a{}^*(\omega)\hat{N}_a(\omega) \geq 0, \qquad \text{all } \omega \in (-\infty, \infty)$$

where $\hat{N}_a$ is the frequency matrix representation of N_a. This condition is essentially the result of (**3**).

In comparing the sufficiency condition developed above with the result of

[1] Appendix A considers the question of existence and uniqueness for Eqs. 6 and 7 and derives an explicit relationship for δe_3.

(3) it should be realized that the abstract condition is *not* restricted to the stationary case. It also covers large classes of distributive, discrete and composite systems and moreover includes the output disturbance η in the analysis.

Remark 2. In Remark 1, δF is linear, however, the disturbances $\{\eta, \delta F\}$ are not necessarily small. Consider now η and δF small but δF not necessarily linear. Then a first order approximation[2] to Eqs. 6 and 7 is given by ($\xi = 0$)

$$\delta e_3 = (I - FQM)\{\eta + \delta FQu\} \tag{8}$$

$$\delta e_1 = -(I - MFQ)M\{\eta + \delta FQu\}. \tag{9}$$

In the next section consideration is given to the minimization of these expressions with respect to the compensator M.

In view of Eqs. 4 and 8 it is apparent that the results of Remark 1 have a convenient first order approximation. Indeed, setting $N = I - FQM$ it follows easily that the condition; $I - N^*N$ is positive (strictly positive) is sufficient to insure that the incremental (i.e., small disturbance) closed loop sensitivity is less (strictly less) than the incremental open loop sensitivity. From a design standpoint this incremental criterion has one big advantage over the earlier criterion in that N is independent of δF and hence, an *a priori* quantity.

Remark 3. Equations 8 and 9 treat the external disturbance η on an equal footing with the perturbation δF in the system response function. It is possible to strengthen this tie even more if the disturbance problem is viewed in the following manner. Let the tuplet $(e_1, e_3) \in B_1 \times B_2$ be taken as the relevant system response variables and let the tuplet $(u, \eta) \in B_1 \times B_2$ be thought of as the total system input. Define the transformation $(e_1, e_3) = V(u, \eta)$ by the equations

$$\begin{aligned} e_1 &= Iu + O\cdot\eta \\ e_2 &= Tu + O\cdot\eta \end{aligned} \qquad (u, \eta) \in B_1 \times B_2 \tag{10}$$

where $T = FQ$. Then, with the meaning of Eqs. 8 and 9, the deviation δV in V may be written as the operator matrix

$$\delta V = \begin{bmatrix} -(I - MT)M\delta T & -(I - MT)M \\ (I - TM)\delta T & (I - TM) \end{bmatrix}$$

where $\delta T = \delta FQ$. In other words, the disturbance η is thought of as being there all the time in a system with nominal response to these signals being multiplication by zero.

As a consequence of these observations it is also possible to measure the total disturbance δV as a parameter variation problem. For example, if B_1, B_2 are the Hilbert spaces H_1, H_2, and if δF is linear the sensitivity measures proposed

[2] Appendix A considers sufficient conditions for the validity of this approximation.

in (6) can be used. Here it is necessary to compute the operator $I + V^*V$ on $H_1 \times H_2$ which is used to determine a normalized disturbance operator, namely $\delta V(I + V^*V)^{-1/2}$. In the present setting it is not difficult to show that

$$\delta V(I + V^*V)^{-1/2} = \begin{bmatrix} -(I - MT)M\delta T(2I + T^*T)^{-1/2} & -(I - MT)M \\ (I - TM)\delta T(2I + T^*T)^{-1/2} & (I - TM) \end{bmatrix}.$$

In (6) it is shown that by using a suitable norm on this operator (the usual supremum, the Hilbert–Schmidt, or a partitioned Hilbert–Schmidt results in a sensitivity measure with several desirable properties. The reader is referred to the reference cited for the physical reasoning underlying this choice of sensitivity measure.

IV. A Sensitivity Minimization Problem

The study of the interrelationships between feedback and parameter changes in pure gain amplifiers is the historical origin of contemporary system sensitivity analysis. One early result is that a feedback amplifier with a high forward loop gain is less affected by gain changes than an equivalent open loop amplifier [see (1)]. This elementary principle carries over to frequency dependent systems [see (2)] and elsewhere and has attained the status of a "folk theorem" among system analysts. In this section we establish the validity of the principle in a function space setting.

Consider now the incremental sensitivity problem of Remark 2. Specifically the equation $\delta e_c = (I - TM)\delta e_o$, where $T = FQ$, which relates the open loop and closed loop disturbances. An apparent question is: *How can M be chosen to make $I - TM$ small?* (This is equivalent to maximizing the positiveness of the operator $I - N^*N$ discussed in Remark 2.) We should also be interested in the form of G, which in the present notation must satisfy the defining equation $G = Q(I - MT)^{-1}$. To make $I - TM$ small it is evidently both necessary and sufficient to make TM approach the identity operator as closely as possible. To phrase this objective another way, M should be the transformation with largest domain such that I is an extension of TM, that is $TMx = x, x \in D(TM)$.

Now suppose that $\langle R(T) \rangle$, the closure of the range of T, is a proper closed subspace of B_2. If I_1 and I denote the identities on $\langle R(T) \rangle$ and B_2, respectively, then I extends TM if and only if I_1 extends TM. Hence, without loss of generality, the assumption $\langle R(T) \rangle = B_2$ can be made.

Theorem I. Let $T: B_1 \to B_2$ be a bounded one-to-one transformation with dense range. Then T^{-1} exists as a closed densely defined linear transformation. Furthermore: (1) I extends TM for some linear M if and only if T^{-1} extends M; (2) No transformation M exists such that; I extends TM and $I - MT$ has a bounded inverse.

Proof: Let M be an operator with domain $D(M)$ in B_2 such that I extends TM. Then if $y \in D(M)$ we have $y = T(My)$ so that y belongs to the range on T and hence $D(T^{-1}) \supset D(M)$. Moreover, $T^{-1}y = My$ and hence T^{-1} extends M. Conversely, if T^{-1} extends M then $T^{-1}x = Mx$ for $x \in D(M)$ and hence $TMx = TT^{-1}x = x$ holds for $x \in D(M)$ which proves that I extends TM. In other words, T^{-1} is the maximal transformation in the class satisfying 1.

Next we show that if T^{-1} extends M, then $I - MT$ is not invertible. Indeed if $y = x - MTx$ belongs to the range of $I - MT$, then Tx necessarily belongs to $D(M) \subset D(T^{-1})$ hence $MTx = T^{-1}Tx = x$. Therefore $R(I - MT) = \{0\}$ and $I - MT$ is not invertible.

While part of this theorem rules out the ideal case it is important to note, however, that if $\lambda \neq 0, 1$, then $M = \lambda T^{-1}$ satisfies (a) $TMy = \lambda y$ all $y \in R(T)$; (b) $(I - MT)^{-1} = (1 - \lambda)^{-1}I$. Indeed if $M = \lambda T^{-1}$ then $I - MT = (1 - \lambda)I$ has a bounded inverse and if $y = Tx$ belongs to the range of T, $TMy = T\lambda T^{-1}y = \lambda y$. Thus assuming that $R(\delta T) \subset R(T)$ and choosing $M = \lambda T^{-1}$ then sensitivity reduction $\delta e_3 = (I - TM)\delta Tu = (1 - \lambda)\delta Tu$ may be achieved. Since δT is assumed to be bounded, the ratio $(\| \delta e_3 \| / \| u \|) \leq | 1 - \lambda | \cdot \| \delta T \|$ can therefore be reduced to any small but finite *a priori* value by letting $\lambda \rightarrow 1$. This reduction in sensitivity is bought at the expense of increasing gain in the forward compensator: $G = (I - MT)^{-1} = (1 - \lambda)^{-1}I$.

Assume now that T satisfies the conditions of Theorem I with the exception that T is not one-to-one. The null space $N(T)$ of T is closed. Suppose that there exists[3] a complementary closed subspace A such that B_1 can be decomposed into the direct sum $B_1 = N(T) \oplus A$. Then the projection P of B_1 onto A is continuous and T_A, the restriction of T to A, is one-to-one with $R(T_A) = R(T)$. A densely defined pseudo inverse,[4] $T^{\dagger}$, for T may be obtained by choosing $T^{\dagger} = T_A^{-1}$. It has the properties: $T^{\dagger}T = P$ and $TT^{\dagger}y = y, y \in R(T)$.

If $M: B_2 \rightarrow B_1$ is any linear operator, then $x = TMx$, $x \in D(M)$ if and only if $x = T_APMx$. Hence I extends TM if and only if I extends T_APM which (by Theorem I) happens if and only if $T^{\dagger}$ extends PM. If $T^{\dagger}$ extends PM, however, then the equality chain $MTx = MT_APx = (I - P)MT_APx + PMT_APx = (I - P)MT_APx + Px$ holds, which shows that $PMTx = Px$. Thus if $y = x - MTx$ is in the range of $I - MT$ then $Py = Px - PMTx = 0$ and hence $R(I - MT) \subset N(T) \neq B_1$. These observations lead to the following corollary.

Corollary. The condition "that T be one-to-one" of Theorem I can be deleted provided that a closed direct sum complement exists for $N(T)$. Condition 1 should then read: I extends TM for some linear transformation M if and only if T_A^{-1} extends PM. Condition 2 remains intact.

[3] In Hilbert spaces A may be taken as the orthogonal complement of $N(T)$. In Banach spaces, however, this assumption is not to be taken lightly. Refer to the discussion in (10), Sec. 4.8, of this point.

[4] Obviously $T_A^{\dagger}$ changes with the choice of the subspace A. In Hilbert spaces by choosing $A = N(T)$ one obtains a $T^{\dagger}$ which carries every $y \in R(T)$ into its unique preimage of minimum norm. In some Banach spaces (for example rotund reflexive space) this latter property can be taken as the criterion for defining $T^{\dagger}$ [see (9), Chap. 4]. Such considerations, however, lead to a nonlinear pseudo inverse which conflicts with the linearity assumed in Section II.

The case where the compensator M is constrained to be of the form $M = LJ$ leads to analogous results. For instance, with Hilbert spaces and T and J both onto, the choice $L = \lambda T^{\dagger}J^{\dagger}$ produces the results

$$I - TLJ = I - \lambda P_{\Omega}$$
$$L - LJT = I - \lambda P_{\Delta}$$

where P_{Ω}, P_{Δ} are the orthogonal projections on the subspaces; $\Omega = N(J)$ and $\Delta = \{N(T) \cup T^{-1}[N(J)]\}^{\perp}$, respectively.

Example 1. To illustrate the results of Sections III and IV consider a linear dynamic plant satisfying the vectoral differential equation

$$\dot{e}_3(t) = A(t)e_3(t) + B(t)u(t); \qquad e_3(t_0) = 0, \quad t \in [t_0, t_f].$$

Here e_2 and e_3 denote m and n tuples of functions respectively while the matrices A and B have compatible dimensions. The transformation $e_3 = Fe_2$ is perhaps better identified through the equation

$$e_3(t) = (Fe_2)(t) = \int_{t_0}^{t} \Phi(t, s)B(s)e_2(s)\, ds, \qquad t \in [t_0, t_f] \qquad (11)$$

where Φ denotes the usual transition matrix for the system. If F acts between the Hilbert spaces $H_1 = [L_2(t_0, t\ )]^m$ and $H_2 = [L_2(t_0, t_f)]^n$ (equipped with the usual norm) then F^* is computed by the equation

$$(F^*y)(s) = \int_{s}^{t_f} B^*(s)\Phi^*(t, s)y(t)\, dt, \qquad t \in [t_0, t_f].$$

A reasonable question to ask is whether a time varying matrix M can be found, which when used in the feedback loop of Fig. 1, results in a reduction of incremental system sensitivity. Assuming for simplicity that $Q = I$ the function N of Remark 2 is given by

$$(Nz)(t) = z(t) - \int_{t_0}^{t} \Phi(t, s)B(s)M(s)z(s)\, ds, \qquad t \in [t_0, t_f].$$

Consequently the operator $I - N^*N$ may be explicitly computed by the formula

$$(z - N^*Nz)(t) = \int_{t}^{t_f} M^*(t)B^*(t)\Phi^*(\beta, t)z(\beta)\, d\beta$$
$$+ \int_{t_0}^{t} \Phi(t, \beta)B(\beta)M(\beta)z(\beta)\, d\beta - \int_{t}^{t_f} M^*(t)B^*(t)\Phi^*(s, t)$$
$$\times \int_{t_0}^{s} \Phi(s, \beta)B(\beta)M(\beta)z(\beta)\, d\beta\, ds.$$

In order for M to have the desired properties this integral operator must be positive definite. The feasibility of using this criteria is examined in detail in (**20**).

If the matrices A, B and M are all stationary and if the Hilbert spaces are finite products of $L_2(-\infty, \infty)$ then the sufficiency condition simplifies to the requirement that the matrix

$$M^*B^*\Phi^*(\omega) + \Phi(\omega)BM + M^*B^*\Phi^*(\omega)\Phi(\omega)BM, \qquad \omega \in (-\infty, \infty)$$

be positive definite at all frequencies. [Here $\Phi(\omega) = (j\omega I - A)^{-1}$.] In (**16**) the use of this criteria as a design tool is considered.

Example 2. This example deals also with the linear plant of Example 1. Without loss of generality, assume that $n > m$ and that the columns of $B(t)$ are linearly independent vectors in E^n for (almost) all $t \in [t_0, t_f]$. Then a matrix $B^\dagger$ exists such that $B^\dagger(t)B(t)$, $t \in [t_0, t_f]$, is the identity on E^m and $B(t)B^\dagger(t)$, $t \in [t_0, t_f]$, is the orthogonal projection, P_t, on the instantaneous range space (column space) of $B(t)$ in E^n. The transformation F defined in Eq. 11 is one-to-one and has dense range. Its inverse may be computed by the rule

$$(F^{-1}e_3)(t) = B^\dagger(t)[\dot{e}_3(t) - A(t)e_3(t)]; \qquad t \in \tau, e_3 \in R(F). \qquad (12)$$

If $L = \lambda F^{-1}$, $0 < \lambda < 1$, it may be verified directly that $e_2 = Ge_1 = (1 - \lambda)^{-1}e_1$ and hence the system equations take the form,

$$\begin{aligned} e_2(t) &= (1 - \lambda)^{-1}u(t) - \lambda(1 - \lambda)^{-1}B^\dagger(t)[\dot{e}_3(t) - A(t)e_3(t)], && t \in \tau \\ \dot{e}_3(t) &= [A(t) + \delta A(t)]e_3(t) + B(t)e_2(t), && t \in \tau. \end{aligned}$$

Eliminating e_2 from this set produces the result

$$\begin{aligned}[][I + \lambda(1 - \lambda)^{-1}P_t]\dot{e}_3(t) = {} & [(I + \lambda(1 - \lambda)^{-1}P_t)A(t) + \delta A(t)]e_3(t) \\ & + (1 - \lambda)^{-1}B(t)u(t), \qquad t \in \tau. \end{aligned}$$

Multiplying through by $(1 - \lambda)$ and letting $S_t = I - P_t$ denote the instantaneous projection on the orthogonal complement of the column space of $B(t)$ in E^n this expression becomes

$$[I - \lambda S_t]\dot{e}_3(t) = [I - \lambda S_t]A(t)e_3(t) + (1 - \lambda)\delta A(t)e_3(t) + B(t)u(t) \qquad t \in \tau.$$

Using the fact that $S_tB(t) = 0$, it easily follows that

$$\dot{e}_3(t) = A(t)e_3(t) + B(t)u(t) + (1 - \lambda)[I - \lambda S_t]^{-1}\delta A(t)e_3(t), \qquad t \in \tau \qquad (13)$$

describes the behavior of the compensated system.

In the case where $m = n$ and $B(t)$ is nonsingular for all $t \in [t_0, t_f]$ (and hence $S_t = 0$) Eq. 13 clearly shows that the nominal system function is preserved and that the perturbation is reduced by the factor $1 - \lambda$. In the more general case the identity $(1-\lambda)[I-\lambda S_t]^{-1} = S_t + (1-\lambda)P_t$ when used in Eq. 13 shows that the portion of $\delta A(t)$ in the column space of $B(t)$ is reduced by the $(1-\lambda)$ factor while the orthogonal part of this disturbance remains unaffected. Since the feedback signal must pass through the matrix B before reaching the point of disturbance this latter result is intuitive.

IV. A Second Minimization Problem (18, 19)

An important aspect of the sensitivity minimization problem posed in Section II is that the ideal feedback function M may well be an unbounded operator. This is the case in Example 2 as testified to by the presence of the derivative in Eq. 12. Thus, it is entirely feasible that in keeping δe_3 small the spurious signals δe_1 and δe_2 may be quite large in both a Hilbert space and a point wise sense. From an engineering standpoint this can be a serious defect.

In the system of Fig. 1 the variable e_2 represents the actual plant input. Consequently, the actual fuel, energy, etc. expended by the system is likely to be related to this variable. Continuing the assumption $\xi = 0$, it is easily shown that $\delta e_2 = -GM\delta e_3 = -QM\delta e_o$ describes the disturbance in e_2. Thus it is natural to attempt to minimize some appropriate combination of the errors; $\delta e_3 = (I - TM)\delta e_o$ and $\delta e_2 = -QM\delta e_o$. Since these two errors are in different function spaces, algebraic combinations are *not* appropriate. Norm combinations, however, such as $\|\delta e_3\|^2 + \|\delta e_2\|^2$ may obviously be considered.

We now restrict attention to the case $Q = I$ and Hilbert spaces and consider the function $\epsilon(\delta e_o) = \|R(I - TM)\delta e_o\|^2 + \|KM\delta e_o\|^2$ on δe_o where R is a bounded invertible operator on H_2 and K is a bounded invertible transformation from H_1 onto H_2. A simple computation shows that $\epsilon(\delta e_o) = \langle \delta e_o, W(M)\delta e_o\rangle$ holds where $W(M)$ is the positive self-adjoint operator on H_2 defined by $W(M) = (I - M^*T^*)R^*R(I - TM) + M^*K^*KM$. Furthermore, to a first order approximation in δM, it may be shown that

$$W(M + \delta M) - W(M) = \delta M^*\{[T^*R^*RT + K^*K]M - T^*R^*R\} + \{\quad\}^*\delta M.$$

where the coefficient preceding δM is the adjoint of the coefficient succeeding δM^*. Hence if a transformation M_o exists which minimizes $\epsilon(\delta e_o)$ independent of δe_o it must of necessity satisfy

$$[T^*R^*RT + K^*K]M_o = T^*R^*R.$$

Since K is by assumption invertible it follows that

$$M_o = (K^*K)^{-1}[I + T^*R^*RT(K^*K)^{-1}]^{-1}T^*R^*R \qquad (14)$$

defines the optimal choice of the feedback compensator.

This optimization does not *a priori* include a physical realizability constraint. In Example 4 we return to consider this problem in a concrete setting. It is fruitful however, to complete the present development first. To do so the following equalities will be helpful.

(i) $I - (K^*K)^{-1}[I + T^*R^*RT(K^*K)^{-1}]^{-1}T^*R^*RT = [I + (K^*K)^{-1}T^*R^*RT]^{-1}.$

(ii) $I - T(K^*K)^{-1}[I + T^*R^*RT(K^*K)^{-1}]^{-1}T^*R^*R = [I + T(K^*K)^{-1}T^*R^*R]^{-1}.$

(iii) $[I + R^*RT(K^*K)^{-1}T^*]^{-1}R^*R = R^*R[I + T(K^*K)^{-1}T^*R^*R]^{-1}.$

(iv) $[I + (K^*K)^{-1}T^*R^*RT]^{-1}(K^*K)^{-1}T^*R^*R = (K^*K)^{-1}T^*R^*R[I + T(K^*K)^{-1}T^*R^*R]^{-1}.$

The inverses of the form $(I + A)^{-1}$ in these equalities exist because the operator A is self-adjoint and positive definite. Each of these equalities may be verified directly by clearing fractions (as suggested in the proof of Lemma 1).

Now that the form of M_o is known the companion compensator G_o is defined by the formula

$$G_o^{-1} = I - M_oT = I - (K^*K)^{-1}[I + T^*R^*RT(K^*K)^{-1}]^{-1}T^*R^*RT.$$

Thus as a consequence of equality (i) it follows that

$$G_o = I + (K^*K)^{-1}T^*R^*RT. \tag{15}$$

The form of $W(M_o)$ is also of interest. Equality (ii) is an intermediate form of the identity $I - TM_o = [I + T(K^*K)^{-1}T^*R^*R]^{-1}$. Using the fact that $(I - TM_o)^* = [I + R^*RT(K^*K)^{-1}T^*]^{-1}$ and identify (iii) it follows that

$$(I - TM_o)^*R^*R(I - TM_o) = R^*R[I + T(K^*K)^{-1}T^*R^*R]^{-2}. \tag{16}$$

Using equality (iv) it can likewise be shown that

$$M_o^*K^*KM_o = R^*RT(K^*K)^{-1}T^*R^*R[I + T(K^*K)^{-1}T^*R^*R]^{-2}. \tag{17}$$

Thus adding Eqs. 16 and 17 the operator $W(M_o)$ may be identified as

$$W(M_o) = R^*R[I + T(K^*K)^{-1}T^*R^*R]^{-1}. \tag{18}$$

Remark 4. It is helpful to examine a special case of these results. In particular the case where $R = kI$ and K is of the form $K = (I - R)V$ where $0 < k < 1$ and V is an unitary operator from H_1 onto H_2. Then $K^*K = (1 - k)^2I$ and $R^*R = k^2I$. Consequently the optimal form of M_o becomes

$$M_o = k^2/(1 - k)^2\{I + [k^2/(1 - k)^2]T^*T\}^{-1}T^*, \tag{19}$$

the companion compensator G_o takes the form

$$G_o = I + [k^2/(1-k)^2]T^*T \tag{20}$$

and the operator $W(M_o)$ becomes $W(M_o) = k^2\{I + [k^2/(1-k)^2]TT^*\}^{-1}$.

If the transformation M_o is written in the equivalent form

$$M_o = T^*\{[(1-k)^2/k^2]I + TT^*\}^{-1}$$

the limiting case as $k \to 0$, namely $M_o = 0$, is also apparent from the equivalent limit $\epsilon(\delta e_o) \to \| M\delta e_o \|^2$.

Example 3. In this example the solution of the latter sensitivity problem is shown to have implications which are not immediately evident from the abstract form. To do this in a fairly simple setting the system of Example 1 is considered. With the assumptions: $B(t) = I$, $t \in [t_0, t_f]$ and $H_1 = H_2 = [L_2(t_0, t_f)]^n$ transformation $e_3 = Te_2$ is defined by Eq. 11. The operators R and K are taken as kI and $(1-k)I$, respectively. Thus, Eqs. 19 and 20 designate the compensators of interest.

To synthesize M_o or G_o we must first determine T^*. In the present case it is easily shown that T^* is defined by

$$(T^*x)(t) = \int_t^{t_f} \Phi^*(t_f, s)x(s)\, ds \qquad t \in [t_0, t_f] \tag{21}$$

which may be also written as

$$(T^*x)(t) = \Psi(t, t_0) \int_{t_0}^{t_f} \Psi(t_0, s)x(s)\, ds - \Psi(t, t_0) \int_{t_0}^{t} \Psi(t_0, s)x(s)\, ds \qquad t \in [t_0, t_f] \tag{22}$$

where $\Psi(t, s) = \Phi^*(s, t)$. From Eq. 21 it is clear that T^* is a *pure predictor* and hence in the general context of this example completely nonrealizable. Equations 19 and 20 show that this nonrealizability is passed along to the compensators G_o and M_o. Thus, although the construction of G_o, M_o and $W(M_o)$ is straightforward, its principal value is as a benchmark for comparison with realizable compensators.

Remark 5. Equation 22 shows that if the quantity

$$\xi_0 = \int_{t_0}^{t_f} \psi(t_0, s)x(s)\, ds$$

could be computed then T^* could be realized by the system

$$\dot{z}(t) = -A^*(t)z(t) - x(t); \qquad z(t_0) = \xi_0, \qquad t \in [t_0, t_f]$$

where Ψ is the transition matrix of this latter system and $z = T^*x$. Note also that since $e_3 = \bar{e}_3 + \delta e_3$ and $\bar{e}_3 = Tu$, it is only the variation δe_3 that must be predicted. In specific cases where $\bar{u}$ is fixed and sufficient *a priori* information about this disturbance exists it follows that the predictive nature of T^* and (and hence M_o, G_o) can be (perhaps approximately) realized.

A classic case in point is that of the ballistic missile guidance system. In this case the elements x, u denote deviations from nominal trajectory and nominal thrust profile respectively. The times t_0, t_f represent the beginning of free fall and target impact. The system disturbance arises from spurious atmospheric effects prior to t_0 and takes the form $\delta e_3(t) = \Phi(t, t_0)\delta x^0$, $t \in [t_0, t_f]$ during the true period of interest. For signals of this form Eq. 21 can obviously be realized by sampling x at time t_0 and precomputing the matrix

$$\int_t^{t_f} \Phi^*(t_f, s)\Phi(s, t_0)\, ds.$$

Example 4. In this example all operators are assumed to be defined on L_2. For convenience set $A = T^*R^*RT + K^*K$ and $B = T^*R^*R$. The frequency response of A and B is denoted by $\hat{A}$ and $\hat{B}$, respectively. Using Plancherel's theorem and picking up the analysis just prior to Eq. 14 it follows that, in the present example,

$$\langle \delta e_o, (M + \delta M)\delta e_o \rangle - \langle \delta e_o, M\delta e_o \rangle$$
$$= 2\int_{-\infty}^{\infty} \delta M^*(\omega)[\hat{A}(\omega)\hat{M}(\omega) - \hat{B}(\omega)] \mid \delta e_o(\omega) \mid^2 d\omega.$$

At this point it is possible to impose a realizability constraint on M. The procedure is exactly the same as the well known Weiner–Hopf technique for determining optimal filters [see (**13**)]. If M is to be nonanticipatory and δM also of this class, then $\hat{\delta M}^*$ is analytic in the lower half plane and $\hat{M}$ is analytic in the upper half plane. From the definition of $\hat{A}$ it follows that this function is symmetric about the real axis. A factorization $\hat{A}(z) = a(z)\bar{a}(z)$ of $\hat{A}$ is assumed to exist and the previous equality reorganized in the form

$$\int_{-\infty}^{\infty} \delta M^*(\omega)\hat{\delta e}_o(\omega)\bar{a}(\omega)\left[a(\omega)\hat{\delta e}_o(\omega)\hat{M}(\omega) - \frac{\hat{B}(\omega)\hat{\delta e}_o(\omega)}{\bar{a}(\omega)}\right] d\omega = 0$$

and hence if M is chosen to be

$$M(\omega) = [a(\omega)\hat{\delta e}_o(\omega)]^{-1}[\hat{B}(\omega)\hat{\delta e}_o(\omega)/\bar{a}(\omega)]_{\text{u.h.p.}}$$

then the above integrand is analytic in the lower half plane and hence by a theorem of Cauchy, the integral is zero.

As a specific example consider the case where $K(\omega) = k$, $R(\omega) = (1 - k^2)^{1/2}$ for $0 \leq k \leq 1$ and $T(\omega) = 1/(1 + j\omega)$. Then a simple computation reveals that

$$a(\omega)\bar{a}(\omega) = (1 + jk\omega)/(1 + j\omega) \cdot (1 - jk\omega)/(1 - j\omega)$$
$$B(\omega) = (1 - k^2)/(1 - jk\omega)$$

and consequently the optimal feedback compensator takes the form

$$M_o(\omega) = [(1 + j\omega)/(1 + jk\omega)\hat{\delta e}_o(\omega)][(1 - k^2)\hat{\delta e}_o(\omega)/(1 - jk\omega)]_{u.h.p.}$$

Notice that the realizable solution depends on δT while the earlier solution did not.

Conclusions

In this paper the problem of reducing system sensitivity by means of feedback is considered. A function space formulation is used as the vehicle for the analysis. The principal results of the investigation are the following.

In Section III a perturbation analysis is used to establish the system first order sensitivity equations (see Eqs. 6 and 7). It is assumed that the compensators and the nominal plant characteristic are linear while the plant perturbation must be additive but not necessarily linear. Remark 1 indicates how the sufficiency condition for sensitivity reduction established by Cruz and Perkins (3) is a concrete manifestation of a simple abstract inequality which itself applies to many nonstationary, distributive, discrete or composite systems. Remark 2 shows that these results hold also for small nonlinear plant disturbances. Section IV then proceeds to the minimization of the abstract sensitivity operator. Theorem I and related remarks establish a fundamental limitation to sensitivity reduction.

In Section V a second sensitivity minimization problem is formulated. The problem is first solved in abstract and the solution illustrated in Example 3. A physical realizability constraint is then imposed on the problem and the solution obtained once more by methodology similar to the Weiner–Hopf technique.

Appendix A.

We denote H as a Hilbert space while $f: H \rightarrow H$ is a continuous function on H. With Eqs. 1 through 7 as motivation, we consider the functional equation

$$\lambda z = f(z) + y \tag{A-1}$$

where λ is a scalar and $y, z \in H$.

The *Lipschitz norm* of the function f is the number ($+\infty$ being allowed)

$$\|f\| = \sup \{\|f(z_1) - f(z_2)\|/\|z_1 - z_2\|\}$$

where the sup is taken over all pairs of distinct points in H. When f is linear the Lipschitz norm agrees with the usual norm for linear operators. When f represents a time and frequency invariant amplifier $\|f\|$ is the supremum of all incremental amplification gains. The function f is said to be Lipschitzian whenever $\|f\| < \infty$.

The *numerical range*, $\eta(f)$, of f is the set of scalars

$$\eta(f) = \{\langle f(z_1) - f(z_2), z_1 - z_2\rangle/\|z_1 - z_2\|^2 : z_1 \neq z_2\}.$$

For any scalar λ the symbol $d(\lambda, f)$ denotes the distance from λ to $\eta(f)$, that is, $d(\lambda,f) = \inf\{|\lambda - n| : n \epsilon \eta(f)\}$. The major result of this Appendix is the theorem.

Theorem II. For any λ such that $d(\lambda, f) > 0$ Eq. A-1 has a unique solution z for every $y \in H$. Moreover, the function $(\lambda I - f)^{-1}$ is Lipschitzian with $\|(\lambda I - f)^{-1}\| \leq 1/d(\lambda, f)$.

This theorem in a somewhat more general form is due to Zarantonello (14) who also discussed the computation of the inverse in question. The continuity assumption on f can be loosened somewhat and related results are also known for Banach spaces. This particular statement of Zarantonello's theorem, however, is sufficient for the present objectives. From the elementary inequality

$$|\langle f(z_1) - f(z_2), z_1 - z_2 \rangle| \leq \|f(z_1) - f(z_2)\| \cdot \|z_1 - z_2\|$$

and the definitions of $\eta(f)$ and $\|f\|$ it follows easily that if $\|f\| < 1$ then $\eta(f)$ lies strictly inside the unit circle of the scalar plane. Hence, $\|f\| < 1$ implies $d(1, \pm f) \geq 1 - \|f\| > 0$ and consequently $(I \pm f)$ has a bounded inverse satisfying $\|(I \pm f)^{-1}\| \leq 1/(1 - \|f\|)$. Thus the condition $\|f\| < 1$ noted after Lemma 1 in Section 2 is more severe than the condition imposed by Zarantonello's theorem.

In Remark 2 of Section 3 the approximation $\eta + \delta F(Qu - GM\delta e_3) \simeq \eta + \delta FQu$ is used to advantage. Suppose that δF is Lipschitz and that $\|\delta F\|$ and $\|GM\delta e_3\|$ are both comparable with $\|\eta\|$ which is small. The approximation error then satisfies $\|\delta F(Qu - GM\delta e_3) - \delta FQu\| \leq \|\delta F\| \cdot \|GM\delta e_3\|$ which shows it to be a second order effect.

To illustrate the use of Zarantonello's theorem consider Eq. 6 (with $\xi \equiv 0$). For convenience define the variable; $z = Qu - GM\delta e_3$, in which case Eq. 6 may be rewritten

$$\delta e_3 = (I - FQM)\{\eta + \delta Fz\}. \tag{A-2}$$

Using this expression, in the definition of z, results in

$$\begin{aligned} z &= Qu - GM\delta e_3 \\ &= Qu - GM(I - FQM)\{\eta + \delta Fz\} \\ &= Q(u - M\eta) - QM\delta Fz \end{aligned}$$

where part (1) of Lemma 1 was used. Now, using Zarantonello's theorem we conclude that if $d(1, -QM\delta f) > 0$ then this last equation has a unique solution for every u and z which may be written in the form

$$z = (I + QM\delta F)^{-1}Q(u - M\eta)$$

where $\|(I + QM\delta F)^{-1}\| \leq 1/d(1 - QM\delta F)$. It then follows from Eq. A-2 that for every u and z which may be written in the form

$$z = (I + QM\delta F)^{-1}Q(u - M\eta)$$

where $||(I + QM\delta F)^{-1}|| \leq 1/d(1 - QM\delta F)$. It then follows from Eq. A-2 that

$$\delta e_3 = (I - FQM)\{\eta + \delta F(I + QM\delta F)^{-1}Q(u - M\eta)\} \tag{A-3}$$

is a valid explicit relationship for δe_3. From Eq. *A*-3 it is clear that if δF is Lipschitz then δe_3 is bounded. Indeed for the case $\eta = 0$ it follows that

$$|| \delta e_3 ||/|| u || \leq || (I - FQM) || \cdot || \delta F || \cdot || Q ||/d(\lambda, -QM\delta F).$$

Since δe_3 represents the net effects of the nonlinearity, the righthand side of this inequality can serve as a conservative linear equivalent gain to the system nonlinearity. In an earlier article Zames (15), who did not have the advantage of Zarantonello's theorem, established a somewhat weaker result in a more concrete setting.

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