AFCRL-67-0360

LOW FREQUENCY SCATTERING BY SPHEROIDS AND DISCS

J.S. Asvestas and R.E. Kleinman

The University of Michigan
Radiation Laboratory
201 Catherine Street
Ann Arbor, Michigan 48108

Contract No. AF 19(628)-4328
Project 5635
Task 563502
Work Unit No. 56350201

Scientific Report No. 4

June 1967

Contract Monitor
Robert A. Shore
Microwave Physics Laboratory

Prepared for

Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
Bedford, Massachusetts 01730

ABSTRACT

The problem of scattering of a scalar plane wave by a spheroid of revolution is solved for either Dirichlet or Neumann boundary conditions, arbitrary major to minor axis ratio, and arbitrary incident direction. The solution is obtained by using an iterative method applied to solutions of the corresponding potential problem and is expressed as a series of products of Legendre and trigonometric functions, and ascending powers of wave number. A recursion relation for the coefficients in this series is derived. These results are employed to calculate the scattering cross sections for 2:1, 5:1, and 10:1 prolate spheroids.

TABLE OF CONTENTS

I
INTRODUCTION

This report presents the complete low frequency expansion of the field scattered when a scalar plane wave is incident from an arbitrary direction on a spheroid of revolution (prolate, oblate, or disc), on which either Dirichlet or Neumann boundary conditions are imposed. The expressions for the field are valid everywhere in space and for all values of the ratio of spheroid dimension to wavelength within the radius of convergence of the low frequency expansion.

The work began as a demonstration of the efficacy of a recently derived technique for solving boundary value problems for the Helmholtz equation by iterating the Green's function for Laplace's equation. This new method had been applied to the problem of scattering by a sphere both for a Dirichlet boundary condition (Kleinman, 1965) and a Neumann boundary condition (Ar and Kleinman, 1966). The prolate spheroid was selected to provide a more substantial test of these methods, which proved to work even better than anticipated.

The problem of scalar scattering by a prolate spheroid for both Dirichlet and Neumann boundary conditions has been extensively treated. F.B. Sleator (1964) presents an exhaustive bibliography. Exact solutions are known in terms of spheroidal wave functions and both low and high frequency approximations have been found. The standard methods for obtaining low frequency approximations, either by direct expansion of the terms of the spheroidal function series in powers of wave number or by determining each term in the expansion as the solution of a potential problem (cf. Noble, 1962), are somewhat cumbersome. One may question the purpose of finding low frequency expansions if the exact solution is known. The answer lies in the complexity of the spheroidal functions which make analysis and computation difficult.

The present approach, although certainly not a trivial calculation, avoids entirely the use of spheroidal functions on the one hand and, on the other, obviates the need for solving more than one potential problem. The solution is found in the form of a series of products of spheroidal potential functions, i.e. Legendre functions, whose coefficients are determinated iteratively. While this in itself might be ample justification for presenting the results, their value is considerably enhanced by the fact that a recurrence relation for the coefficients is found. This means, in effect, that the iteration process may be carried out completely and the complete low frequency expansion obtained.

This is carried out explicitly for a plane wave incident from an arbitrary direction on a prolate spheroid for both Dirichlet and Neumann boundary conditions. In addition to expressions for the field valid everywhere in space, the simplifications occurring in the limiting cases of far zone and nose-on incidence are explicitly given as is the expression for scattering cross section. The corresponding results for an oblate spheroid and the important limiting case, the disc, may be obtained by a simple transformation and these results are also presented explicitly. Some numerical calculations of scattering cross sections of prolate spheroids have been carried out. These results are presented and compared, where possible, with existing data.

In Section II, the iteration method is adapted to take advantage of the symmetry of prolate spheroid geometry. The method is applied to the Dirichlet problem for the prolate spheroid in Section III and the Neumann problem in Section IV. Section V contains the detailed analytic results for oblate spheroids and discs. The numerical calculations for prolate spheroids are presented in Section VI. Much of the detailed mathematical analysis has been relegated to a series of appendices in the hope of making the method and the results more accessible.

This work was supported by the Air Force Cambridge Research Laboratories under contract AF 19(628)-4328 and by the National Science Foundation under Grant No. GP 6140.

II
GENERAL CONSIDERATIONS

In this section we present the problem, the method of attack, and some definitions essential to a clear understanding of the procedures followed.

The problem we are concerned with is the determination of the scattered field which results when a plane wave of arbitrary incidence impinges upon a prolate spheroid. With respect to a rectangular system of coordinates (x, y, z), the prolate spheroid is oriented with its axis of revolution (major axis 2a) coinciding with the z-axis, and its geometrical center at the origin. The minor axis is 2b. Then the relations between prolate spheroidal coordinates (ξ, η, ϕ) and rectangular coordinates (x, y, z) are[+]

$$x = c\sqrt{(\xi^2 - 1)(1 - \eta^2)}\ \cos\phi \tag{2.1}$$

$$y = c\sqrt{(\xi^2 - 1)(1 - \eta^2)}\ \sin\phi \tag{2.2}$$

$$z = c\xi\eta \tag{2.3}$$

where c is half the interfocal distance of the spheroid, and $1 \leqslant \xi < \infty$, $-1 \leqslant \eta \leqslant +1$, $0 \leqslant \phi \leqslant 2\pi$. The surfaces ξ = constant represent confocal prolate spheroids. The metric coefficients of the spheroidal variables are given by by

$$h_\xi = c\sqrt{\frac{\xi^2 - \eta^2}{\xi^2 - 1}}\ ; \quad h_\eta = c\sqrt{\frac{\xi^2 - \eta^2}{1 - \eta^2}}\ ; \quad h_\phi = c\sqrt{(\xi^2 - 1)(1 - \eta^2)}\ . \tag{2.4}$$

Having defined the prolate spheroid, we now turn to the definition of the incident plane wave. Without loss of generality, we take the x-z plane as the plane of incidence. The direction of propagation forms an angle θ_o with the positive z-axis,

---

+ For a detailed discussion of the geometry of the prolate spheroid see Sleator (1964).

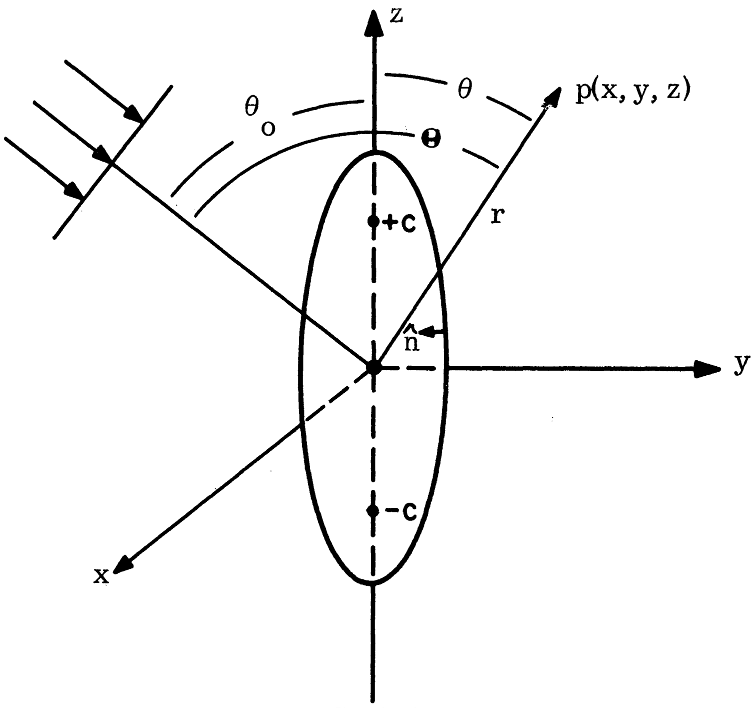

FIG. 1-1

(see Fig. 1-1), and, if p is the observation point with coordinates (r, θ, ϕ), we write

$$u^i(p) = e^{-ikr\cos\Theta} , \tag{2.5}$$

where $u^i(p)$ denotes the plane wave as observed at p and

$$\cos\Theta = \cos\theta\cos\theta_o + \sin\theta\sin\theta_o\cos\phi . \tag{2.6}$$

When θ_o is reduced to zero, the plane wave is seen to propagate along the negative z-axis. The time dependence is harmonic $(e^{-i\omega t})$.

We now state the problem:

Let S designate the surface of a prolate spheroid with surface coordinate ξ_s, and let V be the volume exterior to it. Designate by $\overline{V}$ the union of S and V: $\overline{V} = S \cup V$. Finally, let $u^s(p)$ be the resulting scattered field due to the presence of the prolate spheroid. We wish to determine a function u(p) such that

(i) $$u(p) = u^i(p) + u^s(p)\ , \qquad p \in \overline{V} \tag{2.7}$$

(ii) $$(\nabla^2 + k^2)\, u^s(p) = 0\ , \qquad p \in V \tag{2.8}$$

(iii) $$\lim_{r \to \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \tag{2.9}$$

(iv) Either

(a) $$u(p_s) = 0\ , \qquad p_s \in S \tag{2.10a}$$

or

(b) $$\frac{\partial u(p_s)}{\partial n} = 0\ , \qquad p_s \in S \tag{2.10b}$$

Equation (2.9) implies a suppressed time harmonic dependence $e^{-i\omega t}$. Moreover, boundary condition (2.10a) refers to the Dirichlet problem and (2.10b) to the Neumann problem and the two problems will be treated separately.

The approach employed in solving the problem is based on a new method of finding itérative solutions of the Helmholtz equation (Kleinman, 1965; Ar and Kleinman, 1966). Inherent to this method is the assumption of long wavelength compared to the dimensions of the scatterer. The original iteration scheme was phrased in spherical coordinates and much of the analysis depended upon expansions in these variables. Here we essentially rederive these results in prolate spheroidal coordinates in which form the iteration becomes more tractable.

We start with a representation theorem (Kleinman, 1965; Ar and Kleinman, 1966):

Theorem: Any function $\omega(p)$, defined for all $p \in \overline{V}$, which is twice differentiable in V, and regular at infinity satisfies the integral equation

$$\omega(p_1) = \int_V G_o^D(p_1, p_v)\nabla^2\omega(p_v)dV + \int_S \omega(p_s)\frac{\partial}{\partial n}G_o^D(p_1, p_s)dS \ , \tag{2.11a}$$

where G_o^D is the normalized static Green's function of the first kind, and the integral equation

$$\omega(p_1) = \int_V G_o^N(p_1, p_v)\nabla^2\omega(p_v)dV - \int_S G_o^N(p_1, p_s)\frac{\partial}{\partial n}\omega(p_s)dS \ , \tag{2.11b}$$

where G_o^N is the normalized static Green's function of the second kind. The normalized static Green's function $G_o(p_1, p)$ of either kind is defined as follows:

$$\begin{array}{lll} \text{(i)} & \nabla^2 G_o(p_1, p) = \delta(p_1 | p) \ , & p_1, p \in V \\ \text{(ii)} & G_o(p_1, p) \ \text{regular at infinity} & \\ \text{(iii)} & \text{(a)} \ G_o^D(p_1, p_s) = 0 & \text{(first kind)} \\ & \text{(b)} \ \frac{\partial}{\partial n} G_o^N(p_1, p_s) = 0 & \text{(second kind)} \ . \end{array} \tag{2.12}$$

The normal is directed out of the volume V. Moreover, we define a function f(p) to be regular at infinity if it satisfies the Kellog (1929) conditions

$$\lim_{r\to\infty} \left| rf(p) \right| < \infty \quad \text{and} \quad \lim_{r\to\infty} \left| r^2 \frac{\partial f(p)}{\partial r} \right| < \infty \ , \quad \begin{array}{l} 0 \leqslant \theta \leqslant \pi \\ 0 \leqslant \phi \leqslant 2\pi \ . \end{array} \tag{2.13}$$

Using expressions (2.4), it can be readily shown that in prolate spheroidal coordinates

$$\nabla^2 = \frac{1}{c^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial}{\partial\xi}\right]+\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta}\right]+\frac{\partial}{\partial\phi}\left[\frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial}{\partial\phi}\right]\right\} \tag{2.14}$$

$$dV = c^3(\xi^2-\eta^2)d\xi d\eta d\phi \tag{2.15}$$

$$dS = c^2\sqrt{(\xi_s^2-\eta^2)(\xi_s^2-1)}\; d\eta d\phi \tag{2.16}$$

$$\frac{\partial}{\partial n}\bigg|_{\xi=\xi_s} = -\frac{1}{c}\sqrt{\frac{\xi_s^2-1}{\xi_s^2-\eta^2}}\;\frac{\partial}{\partial\xi_s}\,. \tag{2.17}$$

The function we wish to substitute in the representation theorem equations is the scattered field $u^s(p)$. This function, however, is not regular at infinity but, as we have shown in Appendix A, the function $e^{-ikc(\xi\pm\eta)}u^s(p)$ is. For this reason we let

$$\omega(p) = e^{-ikc(\xi\pm\eta)}u^s(p) \tag{2.18}$$

in equations (2.11a) and (2.11b). From (2.14) and the Helmholtz equation (2.9) we have

$$\nabla^2\omega(p) = -\frac{2ik}{c(\xi^2-\eta^2)}\left[(\xi^2-1)\frac{\partial\omega(p)}{\partial\xi}\mp(\eta^2-1)\frac{\partial\omega(p)}{\partial\eta}+(\xi\mp\eta)\omega(p)\right]. \tag{2.19}$$

Substitution of equations (2.15), (2.16), (2.17) and (2.19) in (2.11a) and (2.11b) gives,

$$\omega(p_1) = -2ikc^2 \int_{\xi_s}^{\infty} d\xi \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi \, G_o^D(p_1, p) \left[(\xi^2 - 1) \frac{\partial \omega(p)}{\partial \xi} \mp (\eta^2 - 1) \frac{\partial \omega(p)}{\partial \eta} + (\xi \mp \eta) \omega(p) \right]$$

$$- c(\xi_s^2 - 1) \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi \, \omega(p_s) \frac{\partial}{\partial \xi_s} G_o^D(p_1, p_s), \qquad (2.20a)$$

for the Dirichlet case, and

$$\omega(p_1) = -2ikc^2 \int_{\xi_s}^{\infty} d\xi \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi \, G_o^N(p_1, p) \left[(\xi^2 - 1) \frac{\partial \omega(p)}{\partial \xi} \mp (\eta^2 - 1) \frac{\partial \omega(p)}{\partial \eta} + (\xi \mp \eta) \omega(p) \right]$$

$$+ c(\xi_s^2 - 1) \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi \, G_o^N(p_1, p_s) \frac{\partial}{\partial \xi_s} \omega(p_s) , \qquad (2.20b)$$

for the Neumann case.

These are the integrodifferential equations that we have to solve. The first one involves the normalized static Green's function of the first kind (Dirichlet boundary condition) defined by (2.12) and given by

$$G_o^D(p_1, p) = -\frac{1}{4\pi c} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^m \epsilon_m (2n+1) \left[\frac{(n-m)!}{(n+m)!} \right]^2 \cos m(\phi_1 - \phi)$$

$$\cdot P_n^m(\eta_1) P_n^m(\eta) \left[\begin{Bmatrix} P_n^m(\xi_1) Q_n^m(\xi) \\ \\ P_n^m(\xi) Q_n^m(\xi_1) \end{Bmatrix} - \frac{P_n^m(\xi_s)}{Q_n^m(\xi_s)} Q_n^m(\xi_1) Q_n^m(\xi) \right]_{\xi < \xi_1}^{\xi > \xi_1} \qquad (2.21)$$

[See for example, Morse and Feshbach (1953, p.1291). The existing differences are due to a different normalization and a different definition of the Legendre functions.

The corresponding Green's function of the second kind is of similar form except for involving the ratio of the derivatives of Legendre functions so that the boundary condition (2.12.iiib) is satisfied.] Equation (2.20b) involves the normalized static Green's function of the second kind (Neumann boundary condition) defined by (2.12) and given by

$$G_o^N(p_1, p) = -\frac{1}{4\pi c}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^m \epsilon_m (2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2 \cos m(\phi_1 - \phi)$$

$$\cdot P_n^m(\eta_1)P_n^m(\eta)\left[\left\{\begin{matrix} P_n^m(\xi_1)Q_n^m(\xi) \\ \\ P_n^m(\xi)Q_n^m(\xi_1)\end{matrix}\right\} - \frac{P_n^m(\xi_s)'}{Q_n^m(\xi_s)'}\,Q_n^m(\xi_1)Q_n^m(\xi)\right]\begin{matrix}\xi > \xi_1 \\ \\ \xi < \xi_1\end{matrix}\quad, \qquad (2.22)$$

where a prime on a function denotes differentiation with respect to ξ_s. The symbol ϵ_m is the Neumann factor defined by

$$\epsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m = 1, 2, 3, \ldots \end{cases} \qquad (2.23)$$

The associated Legendre functions are defined as follows:

$$P_n^m(\mu) = \frac{1}{2^m}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)\Gamma(m+1)}(1-\mu^2)^{m/2}\,{}_2F_1(m-n,\, n+m+1; m+1; \frac{1-\mu}{2}),$$

$$|\mu - 1| < 2 \,. \qquad (2.24)$$

$$P_n^m(\mu) = 2^n\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-m+1)\Gamma(1/2)}(\mu^2-1)^{m/2}\mu^{n-m}\,{}_2F_1\left(\frac{m-n+1}{2}, \frac{m-n}{2}; \frac{1}{2}-n; \frac{1}{\mu^2}\right),$$

$$|\mu| > 1;\ |\arg(\mu \pm 1)| < \pi \,. \qquad (2.25)$$

$$Q_n^m(\mu) = \frac{(-1)^m}{2^{n+1}} \frac{\Gamma(n+m+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} \frac{(\mu^2-1)^{m/2}}{\mu^{n+m+1}} \; {}_2F_1\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}; n+\frac{3}{2}; \frac{1}{\mu^2}\right),$$

$$|\mu| > 1; \; |\arg(\mu - 1)| < \pi \; . \qquad (2.26)$$

Definitions (2.25) and (2.26) agree with those given by Magnus and Oberhettinger (1949, pp 64 and 60, respectively), while (2.24) differs by a factor of $(-1)^m$.

To solve the integrodifferential equations (2.20) for $\omega(p)$ we proceed as follows:

We write $\omega(p)$ as a power series in k of the form

$$\omega(p) = \sum_{M=0}^{\infty} (-ikc)^M \omega_M(p)$$

and we substitute in equations (2.20) to obtain an iteration scheme for $\omega_M(p)$. We subsequently show that these coefficients of k are of a particular form and develop recurrence relations through which $\omega_M(p)$ can be found for arbitrary M.

III
THE DIRICHLET PROBLEM

3.1 The Iteration Scheme

The appropriate integrodifferential equation for the Dirichlet problem is (2.20a) which we repeat here for convenience

$$\omega(p_1) = -2ikc^2 \int_{\xi_s}^{\infty} d\xi \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi\, G_o^D(p_1, p) \left[(\xi^2 - 1) \frac{\partial \omega(p)}{\partial \xi} \mp (\eta^2 - 1) \frac{\partial \omega(p)}{\partial \eta} + (\xi \mp \eta)\omega(p) \right]$$

$$- c(\xi_s^2 - 1) \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi\, \omega(p_s) \frac{\partial}{\partial \xi_s} G_o^D(p_1, p_s) \,. \qquad (3.1)$$

The appropriate Green's function is given by (2.21) and the boundary condition satisfied by $\omega(p)$ is seen to be, from equations (2.7), (2.10a) and (2.14),

$$\omega(p_s) = -u^i(p_s) e^{-ikc(\xi_s \pm \eta)} \,. \qquad (3.2)$$

The incident plane wave $u^i(p)$ is given by (2.5) which can be written in prolate spheroidal coordinates as

$$u^i(p) = e^{-ikr\cos\Theta} = e^{-ikc\left[\cos\theta_o \xi\eta + \sin\theta_o \sqrt{\xi^2 - 1}\sqrt{1 - \eta^2}\cos\phi\right]} \,. \qquad (3.3)$$

Denote the surface integral of (3.1) by $I^s(p_1)$:

$$I^s(p_1) = -c(\xi_s^2 - 1) \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi\, \omega(p_s) \frac{\partial}{\partial \xi_s} G_o^D(p_1, p_s) \,. \qquad (3.4)$$

In Appendix B we show that $I^s(p_1)$ may be written in the following form

$$I^s(p_1) = e^{-ikc\xi_s} \sum_{M=0}^{\infty} (-ikc)^M I_M^s(p_1) \tag{3.5}$$

where

$$I_M^s(p_1) = \sum_{\ell=0}^{M} \sum_{m=0}^{\ell} A_\ell^{M,m}(\xi_s) P_\ell^m(\eta_1) Q_\ell^m(\xi_1) \cos m\phi_1 \tag{3.6}$$

with

$$A_\ell^{M,m}(\xi_s) = \begin{cases} -\epsilon_m \sqrt{\pi} \dfrac{(\xi_s \pm \cos\theta_o)^M}{2^{M+1}} (2\ell+1) \dfrac{(\ell-m)!}{(\ell+m)!} \dfrac{P_\ell^m\left(\dfrac{\xi_s \cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o}\right)}{\left(\dfrac{M-\ell}{2}\right)! \, \Gamma\left(\dfrac{M+\ell}{2} + \dfrac{3}{2}\right) Q_\ell^m(\xi_s)} , & M+\ell \text{ even} \\ 0 , & M+\ell \text{ odd} . \end{cases} \tag{3.7}$$

Note that I_M^s is independent of k. Moreover, let

$$\psi(p_1) = e^{+ick\xi_s} \omega(p_1) \tag{3.8}$$

where $\psi(p_1)$ is assumed to have a power series expansion in k of the form

$$\psi(p) = \sum_{M=0}^{\infty} (-ikc)^M \psi_M(p) . \tag{3.9}$$

Substitution of (3.9) in (3.8) and the resulting equation together with (3.5) in (3.1) gives

$$\sum_{M=0}^{\infty}(-ikc)^M\psi_M(p_1) = -2ikc^2\sum_{M=0}^{\infty}(-ikc)^M\int_{\xi_s}^{\infty}d\xi\int_{-1}^{+1}d\eta\int_0^{2\pi}d\phi\, G_o^D(p_1,p)$$

$$\cdot\left[(\xi^2-1)\frac{\partial\psi_M(p)}{\partial\xi}\mp(\eta^2-1)\frac{\partial\psi_M(p)}{\partial\eta}+(\xi\mp\eta)\psi_M(p)\right]+\sum_{M=0}^{\infty}(-ikc)^M I_M^s(p_1)\ .$$

The interchange in differentiation and summation, and summation and integration was made by assuming (3.9) to converge absolutely and uniformly and to be term by term differentiable with respect to each of the variables and the resulting series to be uniformly and absolutely convergent. Collecting the coefficients of equal powers of k in the above equation, we arrive at the following iteration scheme:

$$\psi_o(p_1) = I_o^s(p_1) \tag{3.10a}$$

$$\psi_{M+1}(p_1) = 2c\int_{\xi_s}^{\infty}d\xi\int_{-1}^{+1}d\eta\int_0^{2\pi}d\phi\, G_o^D(p_1,p)\left[(\xi^2-1)\frac{\partial\psi_M(p)}{\partial\xi}\mp(\eta^2-1)\frac{\partial\psi_M(p)}{\partial\eta}\right.$$
$$\left.+(\xi\mp\eta)\psi_M(p)\right]+I_{M+1}^s(p_1)\ ,\quad M=0,1,2,\ldots \tag{3.10b}$$

3.2 The Recurrence Relations

We shall now solve for the M+1st iterate in (3.10b). In order to do this we need to establish the fact that ψ_M may be written as

$$\psi_M(p) = \sum_{t=0}^{M}\sum_{r=0}^{M}\sum_{\ell=0}^{t} D_{r,t}^{M,\ell}(\xi_s)Q_r^{\ell}(\xi)P_t^{\ell}(\eta)\cos\ell\phi \tag{3.11}$$

for all M (M = 0, 1, 2, ...). This is accomplished using the principle of mathematical induction, that is, first we show that (3.11) holds for M = 0 and secondly we show that if it holds for M, it also holds for M+1.

That the representation holds for $M=0$ is obvious since, with equations (3.10a) and (3.6),

$$\psi_0(p) = A_0^{0,0}(\xi_s)Q_0(\xi) \tag{3.12}$$

which clearly is of the form (3.11).

Next assume that (3.11) holds for M. We wish to show that $\psi_{M+1}(p)$ may then be written as

$$\psi_{M+1}(p) = \sum_{t=0}^{M+1} \sum_{r=0}^{M+1} \sum_{\ell=0}^{t} D_{r,t}^{M+1,\ell}(\xi_s)Q_r^\ell(\xi)P_t^\ell(\eta)\cos\ell\phi$$

The analysis which establishes this is somewhat tedious; however, in the process we actually arrive at an expression for $D_{r,t}^{M+1,\ell}$ in terms of $D_{r,t}^{M,\ell}$ which in fact is the major goal of this section.

First note that the second term in (3.10b) has already been shown to be of the form (3.11) [see (3.6), (3.7)]. Next denote the volume integral of (3.10b) by $I_{M+1}^V(p_1)$ and substitute in it the Green's function of (2.21). Then,

$$I_{M+1}^V(p_1) = -\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^m\epsilon_m(2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2 P_n^m(\eta_1)$$

$$\cdot\int_{\xi_s}^{\infty} d\xi\, C_n^m(\xi,\xi_1,\xi_s)\int_{-1}^{+1} d\eta\, P_n^m(\eta)\int_0^{2\pi} d\phi\cos m(\phi-\phi_1)\Bigg[(\xi^2-1)\frac{\partial\psi_M(p)}{\partial\xi}$$

$$\mp(\eta^2-1)\frac{\partial\psi_M(p)}{\partial\eta}+(\xi\mp\eta)\psi_M(p)\Bigg] \quad , \tag{3.13}$$

where

$$C_n^m(\xi,\xi_1,\xi_s) = \begin{Bmatrix} P_n^m(\xi_1)Q_n^m(\xi) \\ \\ P_n^m(\xi)Q_n^m(\xi_1) \end{Bmatrix} - \frac{P_n^m(\xi_s)}{Q_n^m(\xi_s)} Q_n^m(\xi_1)Q_n^m(\xi) \qquad \begin{matrix} \xi > \xi_1 \\ \\ \xi < \xi_1 \end{matrix} \quad . \tag{3.14}$$

Substitution of (3.11) in (3.13) leads to

$$I_{M+1}^{v}(p_1) = -\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{t=0}^{M}\sum_{r=0}^{M}\sum_{\ell=0}^{t}(-1)^m\epsilon_m(2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2$$

$$\cdot D_{r,t}^{M,\ell}(\xi_s)P_n^m(\eta_1)\int_{\xi_s}^{\infty} d\xi\, C_n^m(\xi,\xi_1,\xi_s)\int_{-1}^{+1} d\eta\, P_n^m(\eta)$$

$$\cdot \int_0^{2\pi} d\phi \cos m(\phi-\phi_1)\cos\ell\phi \left[(\xi^2-1)P_t^\ell(\eta)\frac{dQ_r^\ell(\xi)}{d\xi} \mp (\eta^2-1)Q_r^\ell(\xi)\frac{dP_t^\ell(\eta)}{d\eta}\right.$$

$$\left. + (\xi \mp \eta)Q_r^\ell(\xi)P_t^\ell(\eta)\right] \quad .$$

Performing the angular integration and rearranging terms we get

$$I_{M+1}^{v}(p_1) = -\sum_{n=0}^{\infty}\sum_{t=0}^{M}\sum_{r=0}^{M}\sum_{\ell=0}^{t}(-1)^\ell(2n+1)\left[\frac{(n-\ell)!}{(n+\ell)!}\right]^2 D_{r,t}^{M,\ell}(\xi_s)$$

$$\cdot P_n^\ell(\eta_1)\cos\ell\phi_1\int_{\xi_s}^{\infty} d\xi\, C_n^\ell(\xi,\xi_1,\xi_s)\int_{-1}^{+1} d\eta\, P_n^\ell(\eta)\left\{\left[(\xi^2-1)\frac{d}{d\xi}+\xi\right]P_t^\ell(\eta)\right.$$

$$\left. \cdot Q_r^\ell(\xi) \mp \left[(\eta^2-1)\frac{d}{d\eta}+\eta\right]P_t^\ell(\eta)Q_r^\ell(\xi)\right\} \quad . \tag{3.15}$$

To perform the η integration we use the relation (Magnus and Oberhettinger, 1949, pp 61-62)

$$\left[(z^2-1)\frac{d}{dz}+z\right]\begin{Bmatrix} P_n^\ell(z) \\ Q_n^\ell(z)\end{Bmatrix} = \frac{(n+1)(n-\ell+1)}{2n+1}\begin{Bmatrix} P_{n+1}^\ell(z) \\ Q_{n+1}^\ell(z)\end{Bmatrix} - \frac{n(n+\ell)}{2n+1}\begin{Bmatrix} P_{n-1}^\ell(z) \\ Q_{n-1}^\ell(z)\end{Bmatrix} \; ; \quad n,\ell = 0,1,2,\ldots \tag{3.16}$$

Substituting this relation in (3.15), we get

$$I_{M+1}^{V}(p_1) = -\sum_{n=0}^{\infty}\sum_{t=0}^{M}\sum_{r=0}^{M}\sum_{\ell=0}^{t}(-1)^\ell(2n+1)\left[\frac{(n-\ell)!}{(n+\ell)!}\right]^2 D_{r,t}^{M,\ell}(\xi_s)P_n^\ell(\eta_1)\cos\ell\phi_1$$

$$\cdot\int_{\xi_s}^{\infty} d\xi\, C_n^\ell(\xi,\xi_1,\xi_s)\int_{-1}^{+1} d\eta\, P_n^\ell(\eta)\left\{\mp\frac{(t+1)(t-\ell+1)}{2t+1}P_{t+1}^\ell(\eta)Q_r^\ell(\xi)\right.$$

$$\left.+\left[(\xi^2-1)\frac{d}{d\xi}+\xi\right]Q_r^\ell(\xi)P_t^\ell(\eta)\pm\frac{t(t+\ell)}{2t+1}P_{t-1}^\ell(\eta)Q_r^\ell(\xi)\right\}$$

$$= -\sum_{n=0}^{\infty}\sum_{t=0}^{M+1}\sum_{r=0}^{M}\sum_{\ell=0}^{t}(-1)^\ell(2n+1)\left[\frac{(n-\ell)!}{(n+\ell)!}\right]^2 P_n^\ell(\eta_1)\cos\ell\phi_1$$

$$\cdot\int_{\xi_s}^{\infty} d\xi\, C_n^\ell(\xi,\xi_1,\xi_s)\int_{-1}^{+1} d\eta\, P_n^\ell(\eta)P_t^\ell(\eta)\left\{\mp\frac{t(t-\ell)}{2t-1}D_{r,t-1}^{M,\ell}(\xi_s)Q_r^\ell(\xi)\right.$$

$$\left.+D_{r,t}^{M,\ell}(\xi_s)\left[(\xi^2-1)\frac{d}{d\xi}+\xi\right]Q_r^\ell(\xi)\pm\frac{(t+1)(t+\ell+1)}{2t+3}D_{r,t+1}^{M,\ell}(\xi_s)Q_r^\ell(\xi)\right\}, \tag{3.17}$$

where in this last expression we have adopted the convention that $D^{M,\ell}_{r,t}(\xi_s)$ is identically zero whenever any of the subscripts or the other superscript is greater than M. We now employ the following orthogonality property for the Legendre functions of the first kind (Magnus and Oberhettinger, 1949, p. 54),

$$\int_{-1}^{+1} dx\, P^{\ell}_{n}(x) P^{\ell}_{m}(x) = \frac{2}{2n+1} \frac{(n+\ell)!}{(n-\ell)!} \delta_{nm}$$

to obtain

$$I^{V}_{M+1}(p_1) = -2 \sum_{t=0}^{M+1} \sum_{r=0}^{M} \sum_{\ell=0}^{t} (-1)^{\ell} \frac{(t-\ell)!}{(t+\ell)!} P^{\ell}_{t}(\eta_1) \cos \ell\phi_1$$

$$\cdot \int_{\xi_s}^{\infty} d\xi\, C^{\ell}_{t}(\xi, \xi_1, \xi_s) \left\{ \left[\mp \frac{t(t-\ell)}{2t-1} D^{M,\ell}_{r,t-1}(\xi_s) \pm \frac{(t+1)(t+\ell+1)}{2t+3} D^{M,\ell}_{r,t+1}(\xi_s) \right] Q^{\ell}_{r}(\xi) \right.$$

$$\left. + D^{M,\ell}_{r,t}(\xi_s) \left[(\xi^2 - 1) \frac{d}{d\xi} + \xi \right] Q^{\ell}_{r}(\xi) \right\} .$$

Employing once more the relation (3.16) in the equation above, we write

$$I^{V}_{M+1}(p_1) = -2 \sum_{t=0}^{M+1} \sum_{r=0}^{M} \sum_{\ell=0}^{t} (-1)^{\ell} \frac{(t-\ell)!}{(t+\ell)!} P^{\ell}_{t}(\eta_1) \cos \ell\phi_1 \int_{\xi_s}^{\infty} d\xi\, C^{\ell}_{t}(\xi, \xi_1, \xi_s)$$

$$\cdot \left\{ \frac{(r+1)(r-\ell+1)}{2r+1} D^{M,\ell}_{r,t}(\xi_s) Q^{\ell}_{r+1}(\xi) + \left[\mp \frac{t(t-\ell)}{2t-1} D^{M,\ell}_{r,t-1}(\xi_s) \right. \right.$$

$$\left. \left. \pm \frac{(t+1)(t+\ell+1)}{2t+3} D^{M,\ell}_{r,t+1}(\xi_s) \right] Q^{\ell}_{r}(\xi) - \frac{r(r+\ell)}{2r+1} D^{M,\ell}_{r,t}(\xi_s) Q^{\ell}_{r-1}(\xi) \right\}$$

(cont'd)

$$= -2 \sum_{t=0}^{M+1} \sum_{r=0}^{M+1} \sum_{\ell=0}^{t} (-1)^{\ell} \frac{(t-\ell)!}{(t+\ell)!} P_t^{\ell}(\eta_1) \cos \ell \phi_1 \left\{ \frac{r(r-\ell)}{2r-1} D_{r-1,\,t}^{M,\ell}(\xi_s) \right.$$

$$\mp \frac{t(t-\ell)}{2t-1} D_{r,\,t-1}^{M,\ell}(\xi_s) \pm \frac{(t+1)(t+\ell+1)}{2t+3} D_{r,\,t+1}^{M,\ell}(\xi_s)$$

$$\left. - \frac{(r+1)(r+\ell+1)}{2r+3} D_{r+1,\,t}^{M,\ell}(\xi_s) \right\} \int_{\xi_s}^{\infty} d\xi \, C_t^{\ell}(\xi, \xi_1, \xi_s) Q_r^{\ell}(\xi) \; . \tag{3.19}$$

In arriving at this last expression, one must bear in mind that $D_{r,t}^{M,\ell}(\xi_s)$ is identically zero whenever r, t or ℓ is greater than M. As shown in Appendix C,

$$\int_{\xi_s}^{\infty} d\xi \, C_t^{\ell}(\xi, \xi_1, \xi_s) Q_r^{\ell}(\xi) = (-1)^{\ell} \frac{1}{r(r+1)-t(t+1)} \frac{(t+\ell)!}{(t-\ell)!} \left[\frac{Q_r^{\ell}(\xi_s)}{Q_t^{\ell}(\xi_s)} Q_t^{\ell}(\xi_1) - Q_r^{\ell}(\xi_1) \right],$$

$$r \neq t \; . \tag{3.20}$$

Furthermore, whenever $r = t$ in (3.19), the bracketed coefficient is equal to zero. This follows from the fact, established in Appendix D, that the relationship

$$D_{r,t}^{M,\ell}(\xi_s) = (\pm 1)^{r+t} D_{t,r}^{M,\ell}(\xi_s) \tag{3.21}$$

holds among these coefficients. Thus we need not evaluate terms in (3.19) when $r = t$. Substituting, then, (3.20) in (3.19) we get

$$I^{v}_{M+1}(p_1) = -2 \sum_{t=0}^{M+1} \sum_{r=0}^{M+1}{}' \sum_{\ell=0}^{t} \frac{1}{r(r+1)-t(t+1)} \left[\frac{r(r-\ell)}{2r-1} D^{M,\ell}_{r-1,t}(\xi_s) \right.$$

$$\mp \frac{t(t-\ell)}{2t-1} D^{M,\ell}_{r,t-1}(\xi_s) \pm \frac{(t+1)(t+\ell+1)}{2t+3} D^{M,\ell}_{r,t+1}(\xi_s)$$

$$\left. - \frac{(r+1)(r+\ell+1)}{2r+3} D^{M,\ell}_{r+1,t}(\xi_s) \right] \cdot \left[\frac{Q^{\ell}_{r}(\xi_s)}{Q^{\ell}_{t}(\xi_s)} Q^{\ell}_{t}(\xi_1) - Q^{\ell}_{r}(\xi_1) \right] P^{\ell}_{t}(\eta_1) \cos \ell\phi_1 \; , \tag{3.22}$$

where the prime on the summation for r indicates that the term $t = r$ must be deleted. This may be rewritten as

$$I^{v}_{M+1}(p_1) = \sum_{t=0}^{M+1} \sum_{r=0}^{M+1} \sum_{\ell=0}^{t} E^{M+1,\ell}_{r,t}(\xi_s) Q^{\ell}_{r}(\xi_1) P^{\ell}_{t}(\eta_1) \cos \ell\phi_1 \; , \tag{3.23}$$

where

$$E^{M+1,\ell}_{r,t}(\xi_s) = \frac{2}{r(r+1)-t(t+1)} \left[\frac{r(r-\ell)}{2r-1} D^{M,\ell}_{r-1,t}(\xi_s) \mp \frac{t(t-\ell)}{2t-1} D^{M,\ell}_{r,t-1}(\xi_s) \right.$$

$$\left. \pm \frac{(t+1)(t+\ell+1)}{2t+3} D^{M,\ell}_{r,t+1}(\xi_s) - \frac{(r+1)(r+\ell+1)}{2r+3} D^{M,\ell}_{r+1,t}(\xi_s) \right] \qquad r \neq t \; , \tag{3.24a}$$

$$E^{M+1,\ell}_{t,t}(\xi_s) = - \sum_{r=0}^{M+1}{}' \frac{Q^{\ell}_{r}(\xi_s)}{Q^{\ell}_{t}(\xi_s)} E^{M+1,\ell}_{r,t}(\xi_s) \; , \tag{3.24b}$$

and the prime on the summation indicates that the term $r = t$ must be deleted (see Appendix E). Thus (3.10b) can be written as

$$\psi_{M+1}(p_1) = I^V_{M+1}(p_1) + I^S_{M+1}(p_1) = \sum_{t=0}^{M+1} \sum_{r=0}^{M+1} \sum_{\ell=0}^{t} D^{M+1,\ell}_{r,t}(\xi_s) Q^\ell_r(\xi) P^\ell_t(\eta) \cos \ell\phi$$

(3.25)

where

$$D^{M+1,\ell}_{r,t} = E^{M+1,\ell}_{r,t}(\xi_s) \,, \qquad r \neq t$$

$$= E^{M+1,\ell}_{t,t}(\xi_s) + A^{M+1,\ell}_t \,, \qquad r = t \quad .$$

Equation (3.25) is clearly of the form (3.11) which is what we wished to establish. Not only have we completed this inductive proof but, in the process, we have derived recurrence relations for the coefficients $D^{M,\ell}_{r,t}(\xi_s)$:

$$D^{M+1,\ell}_{r,t}(\xi_s) = \frac{2}{r(r+1)-t(t+1)} \left[\frac{r(r-\ell)}{2r-1} D^{M,\ell}_{r-1,t}(\xi_s) \mp \frac{t(t-\ell)}{2t-1} D^{M,\ell}_{r,t-1}(\xi_s) \right.$$

$$\left. \pm \frac{(t+1)(t+\ell+1)}{2t+3} D^{M,\ell}_{r,t+1}(\xi_s) - \frac{(r+1)(r+\ell+1)}{2r+3} D^{M,\ell}_{r+1,t}(\xi_s) \right] ; \qquad \begin{array}{l} r \neq t \\ M = 0, 1, 2, \ldots \end{array}$$

(3.26a)

$$D^{M+1,\ell}_{t,t}(\xi_s) = - \sum_{r=0}^{M+1}{}' \frac{Q^\ell_r(\xi_s)}{Q^\ell_t(\xi_s)} D^{M+1,\ell}_{r,t}(\xi_s) + A^{M+1,\ell}_t(\xi_s) \, ; \quad M = 0, 1, 2, \ldots$$

(3.26b)

with

$$D^{o,o}_{o,o}(\xi_s) = A^{o,o}_{o}(\xi_s) , \qquad (3.26c)$$

(see equation 3.12).

We are now ready to write the expression for the scattered field $u^s(p_1)$. By equations (2.18), (3.8), (3.9) and (3.11) we have

$$u^s(p_1) = e^{+ikc(\xi_1 \pm \eta_1)} \omega(p_1)$$

$$= e^{-ikc\xi_s} e^{+ikc(\xi_1 \pm \eta_1)} \psi(p_1)$$

$$= e^{-ikc\xi_s} e^{+ikc(\xi_1 \pm \eta_1)} \sum_{M=0}^{\infty} (-ikc)^M \psi_M(p_1)$$

$$= e^{ikc\xi_1} e^{-ikc(\xi_s \mp \eta_1)} \sum_{M=0}^{\infty} (-ikc)^M \sum_{t=0}^{M} \sum_{r=0}^{M} \sum_{\ell=0}^{t} D^{M,\ell}_{r,t}(\xi_s) \cdot Q^{\ell}_{r}(\xi_1) P^{\ell}_{t}(\eta_1) \cos \ell\phi_1 . \qquad (3.27)$$

If we expand $e^{-ikc(\xi_s \mp \eta_1)}$ in a power series of k and employ the Cauchy formula for the product of two infinite series, the above expression becomes

$$u^s(p_1) = e^{ikc\xi_1} \sum_{n=0}^{\infty} (-ikc)^n \sum_{M=0}^{n} \frac{(\xi_s \mp \eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} \sum_{r=0}^{M} \sum_{\ell=0}^{t} D^{M,\ell}_{r,t}(\xi_s) \cdot Q^{\ell}_{r}(\xi_1) P^{\ell}_{t}(\eta_1) \cos \ell\phi_1 , \qquad (3.28)$$

where, in both of the above equations, $D^{M,\ell}_{r,t}(\xi_s)$ is given by equations (3.26a, b, c) and, in turn, $A^{M,\ell}_{t}(\xi_s)$ is given by (3.7).

3.3 The Far Field and the Scattering Cross Section

From the definition of $Q^m_n(\xi)$ in (2.26) and (3.28) the far field is given by

$$u^{sf}(p_1) = \frac{e^{ikc\xi_1}}{\xi_1} \sum_{n=0}^{\infty} (-ikc)^n \sum_{M=0}^{n} \frac{(\xi_s \mp \eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} \sum_{\ell=0}^{t} (-1)^{\ell} \ell!\, D^{M,\ell}_{o,t}(\xi_s) \cdot P^{\ell}_{t}(\eta_1) \cos \ell \phi_1 \quad . \tag{3.29}$$

Since the incident wave is of unit amplitude and $r \sim c\xi$ in the far field, the scattering cross section is given by

$$\sigma = \lim_{\xi_1 \to \infty} 4\pi c^2 \xi_1^2 \left| u^{sf}(p_1) \right|^2 = 4\pi c^2 \left| \sum_{n=0}^{\infty} (-ikc)^n u^{sf}_n(p_1) \right|^2 , \tag{3.30}$$

where

$$u^{sf}_n(p_1) = \sum_{M=0}^{n} \frac{(\xi_s \mp \eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} \sum_{\ell=0}^{t} (-1)^{\ell} \ell!\, D^{M,\ell}_{o,t}(\xi_s) P^{\ell}_{t}(\eta_1) \cos \ell \phi_1 \quad . \tag{3.31}$$

Assuming k real, we can rewrite (3.30) as follows:

$$\sigma = 4\pi c^2 \left| \sum_{n=0}^{\infty} (-ikc)^n u_n^{sf}(p_1) \right|^2 = 4\pi c^2 \sum_{n=0}^{\infty} (-ikc)^n u_n^{sf}(p_1) \cdot \overline{\sum_{n=0}^{\infty} (-ikc)^n u_n^{sf}(p_1)}$$

$$= 4\pi c^2 \sum_{n=0}^{\infty} (-ikc)^n u_n^{sf}(p_1) \overline{\sum_{n=0}^{\infty} (-ikc)^n u_n^{sf}(p_1)}$$

$$= 4\pi c^2 \sum_{n=0}^{\infty} (-ikc)^n u_n^{sf}(p_1) \sum_{n=0}^{\infty} (ikc)^n u_n^{sf}(p_1)$$

$$= 4\pi c^2 \sum_{n=0}^{\infty} (-ikc)^n \sum_{m=0}^{n} (-1)^m u_{n-m}^{sf}(p_1) u_m^{sf}(p_1)$$

$$= 4\pi c^2 \sum_{n=0}^{\infty} (kc)^{2n} \sum_{m=0}^{2n} (-1)^{n+m} u_{2n-m}^{sf}(p_1) u_m^{sf}(p_1) \quad . \tag{3.32}$$

3.4 Nose-on Incidence

In the case of nose-on incidence ($\theta_o = 0$) quite a few simplifications occur.

If we set $\theta_o = 0$ in (3.7), it becomes obvious from the definition of the Legendre function $P_n^m(\mu)$, $|\mu - 1| < 2$, equation (2.24), that $A_\ell^{M,m}(\xi_s)$ becomes zero unless $m = 0$. We then conclude that in the case in which the incident plane wave propagates along the z-axis there is no dependence on the azimuthal angle ϕ. This simplifies the results as follows:

Equation (3.6) can be written

$$I_M^s(p_1) = \sum_{t=0}^{M} A_t^M(\xi_s) P_t(\eta_1) Q_t(\xi_1) \quad , \tag{3.33}$$

with

$$A_t^M(\xi_s) = \begin{cases} -\sqrt{\pi}\,\dfrac{(\xi_s \pm 1)^M (2t+1)}{2^{M+1}\left(\frac{M-t}{2}\right)!\,\Gamma\left(\frac{M+t}{2}+\frac{3}{2}\right) Q_t(\xi_s)}\ , & M+t \text{ even} \\ 0\ , \quad M+t \text{ odd} & . \end{cases} \tag{3.34}$$

Equations (3.26a, b, c) become

$$D_{r,t}^{M+1}(\xi_s) = \frac{2}{r(r+1)-t(t+1)}\left[\frac{r^2}{2r-1} D_{r-1,t}^M(\xi_s) \mp \frac{t^2}{2t-1} D_{r,t-1}^M(\xi_s) \right.$$
$$\left. \pm \frac{(t+1)^2}{2t+3} D_{r,t+1}^M(\xi_s) - \frac{(r+1)^2}{2r+3} D_{r+1,t}^M(\xi_s)\right] ; \quad \begin{matrix} r \neq t \\ M = 0, 1, 2, \ldots \end{matrix} \tag{3.35a}$$

$$D_{t,t}^{M+1}(\xi_s) = -\sum_{r=0}^{M+1}{}' \frac{Q_r(\xi_s)}{Q_t(\xi_s)} D_{r,t}^{M+1}(\xi_s) + A_t^{M+1}(\xi_s) \ ; \quad M = 0, 1, 2, \ldots \tag{3.35b}$$

with

$$D_{o,o}^{o}(\xi_s) = A_o^o(\xi_s) \ . \tag{3.35c}$$

The scattered field, $u^s(p_1)$, becomes

$$u^s(p_1) = e^{ikc\xi_1} \sum_{n=0}^{\infty} (-ikc)^n \sum_{M=0}^{n} \frac{(\xi_s \mp \eta_1)^{n-M}}{(n-M)!} \sum_{r=0}^{M} \sum_{t=0}^{M} D_{r,t}^M(\xi_s) Q_r(\xi_1) P_t(\eta_1) \ , \tag{3.36}$$

and the far field,

$$u^{sf}(p_1) = \frac{e^{ikc\xi_1}}{\xi_1} \sum_{n=0}^{\infty} (-ikc)^n \sum_{M=0}^{n} \frac{(\xi_s \mp \eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} D_{o,t}^M(\xi_s) P_t(\eta_1) \ . \tag{3.37}$$

The expression for the scattering cross section remains the same except for $u_n^{sf}(p_1)$:

$$u_n^{sf}(p_1) = \sum_{M=0}^{n} \frac{(\xi_s \mp \eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} D_{o,t}^{M}(\xi_s) P_t(\eta_1) \ . \tag{3.38}$$

IV
THE NEUMANN PROBLEM

4.1 The Iteration Scheme

The appropriate integrodifferential equation for the Neumann problem is (2.20b) which we repeat here,

$$\omega(p_1) = -2ikc^2 \int_{\xi_s}^{\infty} d\xi \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi\, G_o^N(p_1, p) \left[(\xi^2 - 1) \frac{\partial \omega(p)}{\partial \xi} \mp (\eta^2 - 1) \frac{\partial \omega(p)}{\partial \eta} \right.$$

$$\left. + (\xi \mp \eta)\, \omega(p) \right] + c(\xi_s^2 - 1) \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi\, G_o^N(p_1, p_s) \frac{\partial}{\partial \xi_s} \omega(p_s) \;, \tag{4.1}$$

with $G_o^N(p_1, p)$ given by (2.22). The appropriate boundary condition is given by (2.10b), which through equations (2.17) and (2.18), may be written,

$$\frac{1}{c} \sqrt{\frac{\xi_s^2 - 1}{\xi_s^2 - \eta^2}} \; \frac{\partial}{\partial \xi_s} \left[u^i(p_s) + e^{ikc(\xi_s \pm \eta)} \omega(p_s) \right] = 0. \tag{4.2}$$

Excluding the case in which the prolate spheroid degenerates to a wire of finite length ($\xi_s = 1$), we can write

$$\frac{\partial}{\partial \xi_s} \left[u^i(p_s) + e^{ikc(\xi_s \pm \eta)} \omega(p_s) \right] = 0 \tag{4.3}$$

from which

$$\frac{\partial \omega(p_s)}{\partial \xi_s} = -ikc\omega(p_s) - e^{-ikc(\xi_s \pm \eta)} \frac{\partial u^i(p_s)}{\partial \xi_s} \quad . \tag{4.4}$$

Substitution of (4.4) in (4.1) leads to

$$\omega(p_1) = -2ikc^2 \int_{\xi_s}^{\infty} d\xi \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi \, G_o^N(p_1, p) \Big[(\xi^2 - 1) \frac{\partial \omega(p)}{\partial \xi} \mp (\eta^2 - 1) \frac{\partial \omega(p)}{\partial \eta}$$

$$+ (\xi \mp \eta) \omega(p) \Big] - ikc^2(\xi_s^2 - 1) \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi \, \omega(p_s) G_o^N(p_1, p_s)$$

$$- c(\xi_s^2 - 1) e^{-ikc\xi_s} \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi \, e^{\mp ikc\eta} G_o^N(p_1, p_s) \frac{\partial u^i(p_s)}{\partial \xi_s} \, . \qquad (4.5)$$

Denote by $I^s(p_1)$ the second surface integral in (4.5)

$$I^s(p_1) = -c(\xi_s^2 - 1) e^{-ikc\xi_s} \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi \, e^{\mp ikc\eta} G_o^N(p_1, p_s) \frac{\partial u^i(p_s)}{\partial \xi_s} \, . \qquad (4.6)$$

In Appendix F we show that $I^s(p_1)$ may be written in the following form

$$I^s(p_1) = e^{-ikc\xi_s} \sum_{M=1}^{\infty} (-ikc)^M I_M^s(p_1) \, , \qquad (4.7)$$

where

$$I_M^s(p_1) = \sum_{\ell=0}^{M} \sum_{m=0}^{\ell} A_\ell^{M,m}(\xi_s) P_\ell^m(\eta_1) Q_\ell^m(\xi_1) \cos m\phi_1 \, , \qquad (4.8)$$

with

$$A_\ell^{M,m}(\xi_s) = -\epsilon_m \sqrt{\pi}\, \frac{(\xi_s \pm \cos\theta_o)^{M-1}}{2^{M+1}\left(\frac{M-\ell}{2}\right)!\, \Gamma\left(\frac{M+\ell}{2}+\frac{3}{2}\right) Q_\ell^m(\xi_s)'}\,(2\ell+1)\frac{(\ell-m)!}{(\ell+m)!}$$

$$\cdot \left\{ M P_\ell^m\left(\frac{\xi_s\cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o}\right) \pm \frac{1}{\xi_s^2 - 1}\left[\ell(\xi_s\cos\theta_o \pm 1)\, P_\ell^m\left(\frac{\xi_s\cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o}\right)\right.\right.$$

$$\left.\left. -(\ell+m)(\xi_s \pm \cos\theta_o) P_{\ell-1}^m\left(\frac{\xi_s\cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o}\right)\right]\right\}, \quad \text{for } M+\ell \text{ even,} \tag{4.9}$$

$$A_\ell^{M,m}(\xi_s) = 0\ , \qquad \text{for } M+\ell \text{ odd .} \tag{4.10}$$

From now on, the procedure for developing an iteration scheme parallels that of the Dirichlet problem. After writing

$$\psi(p_1) = e^{ikc\xi_s}\,\omega(p_1) \tag{4.11}$$

and assuming a low frequency expansion in powers of k for $\psi(p_1)$,

$$\psi(p_1) = \sum_{M=0}^{\infty} (-ikc)^M \psi_M(p_1)\ , \tag{4.12}$$

we substitute (4.11) in (4.12) and the resulting expression together with (4.7) in (4.5). Equating coefficients of equal powers of k, we obtain the following iteration scheme:

$$\psi_o(p_1) = 0 \tag{4.13a}$$

$$\psi_{M+1}(p_1) = 2c \int_{\xi_s}^{\infty} d\xi \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi\, G_o^N(p_1, p) \left[(\xi^2 - 1) \frac{\partial \psi_M(p)}{\partial \xi} \mp (\eta^2 - 1) \frac{\partial \psi_M(p)}{\partial \eta} \right.$$

$$\left. + (\xi \mp \eta) \psi_M(p) \right] + c(\xi_s^2 - 1) \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi\, G_o^N(p_1, p_s) \psi_M(p_s)$$

$$+ I_{M+1}^S(p_1) \; ; \qquad M = 0, 1, 2, \ldots \tag{4.13b}$$

4.2 The Recurrence Relations

The procedure we shall follow here is practically identical to that for the Dirichlet case. We assume $\psi_M(p_1)$ to be of the form

$$\psi_M(p_1) = \sum_{t=0}^{M} \sum_{r=0}^{M} \sum_{\ell=0}^{t} D_{r,t}^{M,\ell}(\xi_s) Q_r^\ell(\xi_1) P_t^\ell(\eta_1) \cos \ell \phi_1 \; , \qquad M = 0, 1, 2, \ldots \tag{4.14}$$

which we substitute in (4.13b) and solve for $\psi_{M+1}(p_1)$. If $\psi_{M+1}(p_1)$ turns out to be of the form (4.14), then because of $\psi_o(p_1)$ being zero we can conclude that (4.14) is true.

The volume integral of (4.13b) is practically identical to that of (3.10b) except for the Green's function. From equations (2.21) and (2.22) we see that these two functions are identical except for their dependence on the surface coordinate ξ_s. If we denote the volume integral of (4.13b) by $I_{M+1}^V(p_1)$, we can use the result of (3.19) and write

$$I^{v}_{M+1}(p_1) = -2 \sum_{t=0}^{M+1} \sum_{r=0}^{M+1} \sum_{\ell=0}^{t} (-1)^{\ell} \frac{(t-\ell)!}{(t+\ell)!} P^{\ell}_{t}(\eta_1) \cos \ell \phi_1$$

$$\cdot \left\{ \left[\frac{r(r-\ell)}{2r-1} D^{M,\ell}_{r-1,t}(\xi_s) \mp \frac{t(t-\ell)}{2t-1} D^{M,\ell}_{r,t-1}(\xi_s) \pm \frac{(t+1)(t+\ell+1)}{2t+3} D^{M,\ell}_{r,t+1}(\xi_s) \right. \right.$$

$$\left. \left. - \frac{(r+1)(r+\ell+1)}{2r+3} D^{M,\ell}_{r+1,t}(\xi_s) \right] \right\} \int_{\xi_s}^{\infty} d\xi \, K^{\ell}_{t}(\xi, \xi_1, \xi_s) Q^{\ell}_{r} \qquad (4.15)$$

where

$$K^{\ell}_{t}(\xi, \xi_1, \xi_s) = \left\{ \begin{matrix} P^{\ell}_{t}(\xi_1) Q^{\ell}_{t}(\xi) \\ \\ P^{\ell}_{t}(\xi) Q^{\ell}_{t}(\xi_1) \end{matrix} \right\} - \frac{P^{\ell}_{t}(\xi_s)'}{Q^{\ell}_{t}(\xi_s)'} Q^{\ell}_{t}(\xi_1) Q^{\ell}_{t}(\xi) \qquad \begin{matrix} \xi > \xi_1 \\ \\ \xi < \xi_1 \end{matrix} \qquad (4.16)$$

It is understood in (4.15) that $D^{M,\ell}_{r,t}(\xi_s)$ is identically zero whenever r, t or ℓ is greater than M.

From Appendix G,

$$\int_{\xi_s}^{\infty} d\xi \, K^{\ell}_{t}(\xi, \xi_1, \xi_s) Q^{\ell}_{r}(\xi) = \frac{(-1)^{\ell}}{r(r+1) - t(t+1)} \frac{(t+\ell)!}{(t-\ell)!} \left[\frac{Q^{\ell}_{r}(\xi_s)'}{Q^{\ell}_{t}(\xi_s)'} Q^{\ell}_{t}(\xi_1) - Q^{\ell}_{r}(\xi_1) \right] ,$$

$$r \neq t . \qquad (4.17)$$

Furthermore, through an inductive argument identical to that given in Appendix D for the Dirichlet case, we can show that

$$D^{M,\ell}_{r,t}(\xi_s) = (\pm 1)^{r+t} D^{M,\ell}_{t,r}(\xi_s) . \qquad (4.18)$$

Employing (4.17) in (4.15), we obtain

$$I^{V}_{M+1}(p_1) = -2 \sum_{t=0}^{M+1} \sum_{r=0}^{M+1} {\sum_{\ell=0}^{t}}' \frac{1}{r(r+1)-t(t+1)} \left[\frac{r(r-\ell)}{2r-1} D^{M,\ell}_{r-1,t}(\xi_s) \right.$$

$$\mp \frac{t(t-\ell)}{2t-1} D^{M,\ell}_{r,t-1}(\xi_s) \pm \frac{(t+1)(t+\ell+1)}{2t+3} D^{M,\ell}_{r,t+1}(\xi_s)$$

$$\left. - \frac{(r+1)(r+\ell+1)}{2r+3} D^{M,\ell}_{r+1,t}(\xi_s) \right] \left[\frac{Q^{\ell}_{r}(\xi_s)'}{Q^{\ell}_{t}(\xi_s)'} Q^{\ell}_{t}(\xi_1) - Q^{\ell}_{r}(\xi_1) \right] P^{\ell}_{t}(\eta_1) \cos \ell \phi_1 \;, \qquad (4.19)$$

Having evaluated the volume integral of equation (4.13b) we now turn to the surface integral of the same equation and we denote it by $I(p_1)$.

$$I(p_1) = c(\xi_s^2 - 1) \int_{-1}^{+1} d\eta \int_{0}^{2\pi} d\phi \, G^{N}_{o}(p_1, p_s) \psi_M(p_s) \;. \qquad (4.20)$$

Substituting equations (2.22) and (4.14) in (4.20), we obtain

$$I(p_1) = -\frac{1}{4\pi} (\xi_s^2 - 1) \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^m \epsilon_m (2n+1) \left[\frac{(n-m)!}{(n+m)!} \right]^2 P^{m}_{n}(\eta_1)$$

$$\cdot \sum_{s=0}^{M} \sum_{r=0}^{M} \sum_{\ell=0}^{s} D^{M,\ell}_{r,t}(\xi_s) Q^{\ell}_{r}(\xi_s) Q^{m}_{n}(\xi_1) \left[P^{m}_{n}(\xi_s) - \frac{P^{m}_{n}(\xi_s)'}{Q^{m}_{n}(\xi_s)'} Q^{m}_{n}(\xi_s) \right]$$

$$\times \int_{-1}^{+1} d\eta \, P^{m}_{n}(\eta) P^{\ell}_{t}(\eta) \int_{0}^{2\pi} d\phi \cos m(\phi - \phi_1) \cos \ell \phi \;.$$

Using equation (B.4) for the Wronskian and at the same time performing first the integration with respect to ϕ and then the integration with respect to η according to (3.18), we obtain

$$I(p_1) = \sum_{t=0}^{M} \sum_{r=0}^{M} \sum_{\ell=0}^{t} D_{r,t}^{M,\ell}(\xi_s) \frac{Q_r^\ell(\xi_s)}{Q_t^\ell(\xi_s)'} Q_t^\ell(\xi_1) P_t^\ell(\eta_1) \cos \ell\phi_1 \ . \qquad (4.21)$$

From equations (4.19) and (4.21), it is clear that $\psi_{M+1}(p_1)$ of (4.13b) is of the form given by (4.14). At this point, then, we not only have concluded the inductive argument that the representation (4.14) of $\psi_M(p_1)$ is correct, but in exactly the same fashion as in the Dirichlet problem we end up with the following recurrence relationships:

$$D_{r,t}^{M+1,\ell}(\xi_s) = \frac{2}{r(r+1)-t(t+1)} \left[\frac{r(r-\ell)}{2r-1} D_{r-1,t}^{M,\ell}(\xi_s) \mp \frac{t(t-\ell)}{2t-1} D_{r,t-1}^{M,\ell}(\xi_s) \right.$$

$$\left. \pm \frac{(t+1)(t+\ell+1)}{2t+3} D_{r,t+1}^{M,\ell}(\xi_s) - \frac{(r+1)(r+\ell+1)}{2r+3} D_{r+1,t}^{M,\ell}(\xi_s) \right] ,$$

$$M = 0, 1, 2, \ldots$$

$$r \neq t \qquad (4.22a)$$

$$D_{t,t}^{M+1,\ell}(\xi_s) = -\sum_{r=0}^{M+1}{}' \frac{Q_r^\ell(\xi_s)'}{Q_t^\ell(\xi_s)'} D_{r,t}^{M+1,\ell}(\xi_s) + \sum_{r=0}^{M} \frac{Q_r^\ell(\xi_s)}{Q_t^\ell(\xi_s)'} D_{r,t}^{M,\ell}(\xi_s) + A_t^{M+1,\ell}(\xi_s) ,$$

$$M = 0, 1, 2, \ldots \qquad (4.22b)$$

with

$$D_{o,o}^{o,o}(\xi_s) = 0 \ , \qquad (4.22c)$$

where $A_t^{M,\ell}(\xi_s)$ is given by (4.9a, b). The prime on the first summation in (4.22b) denotes that the term $r = t$ must be deleted.

The scattered field $u^s(p_1)$ for the Neumann problem is given by the same expressions as for the Dirichlet problem (equations (3.27) and (3.28)) with the understanding of course that the coefficients $D^{M,\ell}_{r,t}(\xi_s)$ are this time given by equations (3.22). The same is true for the far field and the scattering cross section expressions (see Section 3.3).

4.3 Nose-on Incidence

When $\theta_o = 0$, (4.9a) becomes zero unless $m = 0$. This is so because of the definition of the Legendre function $P^m_n(\mu)$ for $|\mu - 1| < 2$, equation (2.24). Consequently, when the incoming plane wave propagates along the z-axis, there is no dependence of the scattered field on the azimuthal angle ϕ, a result we should expect since the z-axis is the axis of symmetry of the prolate spheroid. Due to the substantial amount of simplification, we redefine our results for the Neumann problem as follows:

Equation (4.8) becomes

$$I^s_M(p_1) = \sum_{t=0}^{M} A^M_t(\xi_s) P_t(\eta_1) Q_t(\xi_1) , \qquad M = 1, 2, 3, \ldots \tag{4.23}$$

where

$$A^M_t(\xi_s) = \begin{cases} -\sqrt{\pi}\, M \dfrac{(2t+1)(\xi_s \pm 1)^{M-1}}{2^{M+1}\left(\frac{M-t}{2}\right)!\, \Gamma\left(\frac{M+t}{2} + \frac{3}{2}\right) Q_t(\xi_s)'} , & M+t \text{ even} \\ 0 , & M+t \text{ odd.} \end{cases} \tag{4.24}$$

The scattered field $u^s(p_1)$ is given by (3.36) with $D^M_{r,t}(\xi_s)$ given by

$$D^{M+1}_{r,t}(\xi_s) = \frac{2}{r(r+1)-t(t+1)}\left[\frac{r^2}{2r-1} D^M_{r-1,t}(\xi_s) \mp \frac{t^2}{2t-1} D^M_{r,t-1}(\xi_s) \right.$$

$$\left. \pm \frac{(t+1)^2}{2t+3} D^M_{r,t+1}(\xi_s) - \frac{(r+1)^2}{2r+3} D^M_{r+1,t}(\xi_s)\right], \quad \begin{matrix} M = 0,1,2,\ldots \\ r \neq t \end{matrix} \tag{4.25a}$$

$$D^{M+1}_{t,t}(\xi_s) = -\sum_{r=0}^{M+1}{}' \frac{Q_r(\xi_s)'}{Q_t(\xi_s)'} D^{M+1}_{r,t}(\xi_s) + \sum_{r=0}^{M} \frac{Q_r(\xi_s)}{Q_t(\xi_s)'} D^M_{r,t}(\xi_s) + A^{M+1}_t(\xi_s) ,$$

$$M = 0,1,2,\ldots \tag{4.25b}$$

with

$$D^o_{o,o}(\xi_s) = 0 . \tag{4.25c}$$

The expression for the far field is the same as the one for the Dirichlet problem (equation (3.37)) with $D^M_{o,t}(\xi_s)$ as above. The same is true for the coefficient of the scattering cross section, $u^{sf}_n(p_1)$, which is given by (3.38).

V
THE OBLATE SPHEROID AND THE DISC

The method employed in the preceding sections to determine the field scattered by a prolate spheroid can be employed in a straightforward manner to determine the field scattered by an oblate spheroid. This is not necessary, however, since we can transform the prolate spheroid into an oblate spheroid and utilize the results already obtained to determine the field scattered by the oblate spheroid. Specifically, if we let $\xi \rightarrow i\xi$ and $c \rightarrow -ic$, the prolate spheroid is transformed into an oblate one with the axis of revolution (minor axis 2b) coincident with the z-axis of a rectangular coordinate systems (cf. Morse and Feshbach, 1953, p. 1502). The ranges of the variables now are $0 \leqslant \xi < \infty$, $-1 \leqslant \eta \leqslant +1$, and $0 \leqslant \phi \leqslant 2\pi$. Moreover,

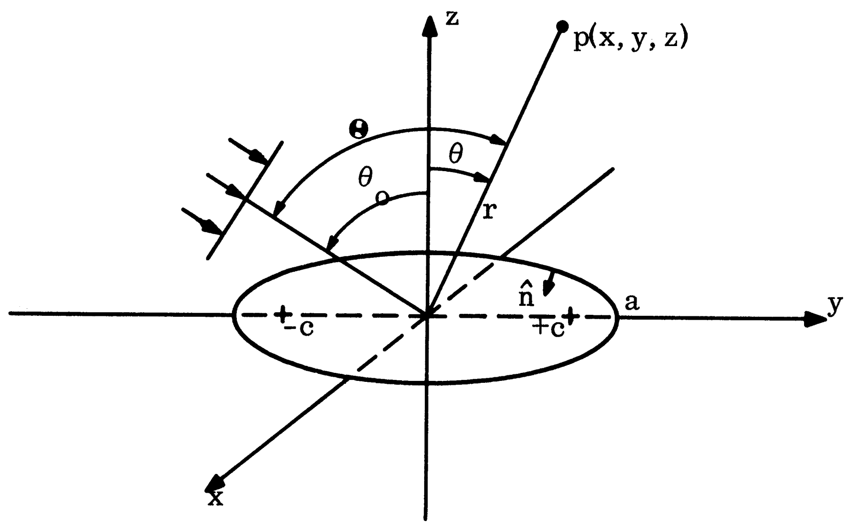

FIG. 5-1.

if we let $\xi_s \rightarrow 0$, the oblate spheroid degenerates into a disc of infinitesimal thickness, radius c (the semifocal distance), and coplanar with the x-y plane. In the remainder of this section we shall treat each body separately.

5.1 The Oblate Spheroid

With the incident field given by (see equation (3.3)),

$$u^i(p) = e^{-ikr\cos\Theta} = e^{-ikc\left[\cos\theta_o \xi\eta + \sin\theta_o \sqrt{\xi^2-1}\sqrt{1-\eta^2}\cos\phi\right]},$$

the scattered field is given by (3.28) with $\xi = i\xi$ and $c = -ic$ and can be written as follows

$$u^s(p_1) = e^{ikc\xi_1}\sum_{n=0}^{\infty}(-ikc)^n\sum_{M=0}^{n}(-i)^n\frac{(i\xi_s \mp \eta_1)^{n-M}}{(n-M)!}\sum_{t=0}^{M}\sum_{r=0}^{M}\sum_{\ell=0}^{t} D_{r,t}^{M,\ell}(i\xi_s)$$

$$\cdot\, Q_r^\ell(i\xi_1)P_t^\ell(\eta_1)\cos\ell\phi_1\,, \qquad (5.1)$$

where, for the Dirichlet problem, the recurrence relations (3.26a, b, c) hold among the coefficients $D_{r,t}^{M,\ell}(i\xi_s)$, with

$$A_t^{M,\ell}(i\xi_s) = \begin{cases} -\epsilon_\ell\sqrt{\pi}\,\dfrac{(i\xi_s \pm \cos\theta_o)^M}{2^{M+1}}(2t+1)\dfrac{(t-\ell)!}{(t+\ell)!}\,\dfrac{P_t^\ell\left(\dfrac{i\xi_s\cos\theta_o \pm 1}{i\xi_s \pm \cos\theta_o}\right)}{\left(\dfrac{M-t}{2}\right)!\,\Gamma\left(\dfrac{M+t}{2}+\dfrac{3}{2}\right)Q_t^\ell(i\xi_s)}\,, & M+t \text{ even} \\ 0\,, & M+t \text{ odd} \end{cases} \qquad (5.2)$$

For the Neumann problem the coefficients $D_{r,t}^{M,\ell}(i\xi_s)$ are related through equations (4.22a, b, c) with

$$A_t^{M,\ell}(i\xi_s) = -\epsilon_\ell \sqrt{\pi}\; \frac{(i\xi_s \pm \cos\theta_o)^{M-1}}{2^{M+1}\left(\frac{M-t}{2}\right)!\,\Gamma\left(\frac{M+t}{2}+\frac{3}{2}\right) Q_t^\ell(i\xi_s)'}\,(2t+1)\frac{(t-\ell)!}{(t+\ell)!}$$

$$\cdot\left\{ M P_t^\ell\left(\frac{i\xi_s\cos\theta_o \pm 1}{i\xi_s \pm \cos\theta_o}\right) \mp \frac{1}{\xi_s^2+1}\left[t(i\xi_s\cos\theta_o \pm 1) P_t^\ell\left(\frac{i\xi_s\cos\theta_o \pm 1}{i\xi_s \pm \cos\theta_o}\right)\right.\right.$$

$$\left.\left. - (t+\ell)(i\xi_s \pm \cos\theta_o) P_{t-1}^\ell\left(\frac{i\xi_s\cos\theta_o \pm 1}{i\xi_s \pm \cos\theta_o}\right)\right]\right\},$$

M+t even, (5.3)

and

$$A_t^{M,\ell}(i\xi_s) = 0\,, \qquad M+t \quad \text{odd} \tag{5.4}$$

The prime on $Q_t^\ell(i\xi_s)$ in (5.3) implies differentiation with respect to $i\xi_s$.

The Legendre function $Q_r^\ell(i\xi_1)$ in (5.1) must now be redefined since ξ_1 can now assume values between 0 and 1 as well as values greater than 1. This has been done in Appendix H where we show that

$$Q_n^m(i\xi) = \frac{(-1)^m}{i^{n+1}2^m}\,\frac{\Gamma(n+m+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})}\;\frac{(\xi^2+1)^{-\frac{1}{2}m}}{\left(\xi+\sqrt{\xi^2+1}\right)^{n-m+1}}$$

$$\cdot\, {}_2F_1\left(n-m+1,\ \frac{1}{2}-m;\ n+\frac{3}{2};\ -\frac{1}{(\xi+\sqrt{\xi^2+1})^2}\right), \quad \xi \geqslant 0\,. \tag{5.5}$$

The far field is given by (5.1) by letting $\xi_1 \to \infty$,

$$u^{sf}(p_1) = \frac{e^{ikc\xi_1}}{\xi_1} \sum_{n=0}^{\infty} (-ikc)^n \sum_{M=0}^{n} (-i)^{n-1} \frac{(i\xi_s \mp \eta_1)^{n-M}}{(n-M)!}$$

$$\cdot \sum_{t=0}^{M} \sum_{\ell=0}^{t} (-1)^{\ell} \ell!\, D_{o,t}^{M,\ell}(i\xi_s) P_t^{\ell}(\eta_1) \cos \ell\phi_1 \quad . \tag{5.6}$$

The scattering cross section is given by (3.30) and, in the present case, can be written in the form

$$\sigma = 4\pi c^2 \sum_{n=0}^{\infty} (kc)^{2n} \sum_{m=0}^{2n} (-1)^{n+m} \overline{u_{2n-m}^{sf}(p_1) u_m^{sf}(p_1)} \ , \tag{5.7}$$

where we have taken k to be real, and

$$u_n^{sf}(p_1) = (-i)^{n-1} \sum_{M=0}^{n} \frac{(i\xi_s \mp \eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} \sum_{\ell=0}^{t} (-1)^{\ell} \ell!\, D_{o,t}^{M,\ell}(i\xi_s) P_t^{\ell}(\eta_1) \cos \ell\phi_1 \ . \tag{5.8}$$

Nose-on Incidence:

When $\theta_o = 0$, we can rewrite (5.1) as follows:

$$u^s(p_1) = e^{ikc\xi_1} \sum_{n=0}^{\infty} (-ikc)^n \sum_{M=0}^{n} (-i)^n \frac{(i\xi_s \mp \eta_1)}{(n-M)!} \sum_{r=0}^{M} \sum_{t=0}^{M} D_{r,t}^{M}(i\xi_s) Q_r(i\xi_1) P_t(\eta_1) \ , \tag{5.9}$$

where, for the Dirichlet problem,

$$D^{M+1}_{r,t}(i\xi_s) = \frac{2}{r(r+1)-t(t+1)}\left[\frac{r^2}{2r-1}D^M_{r-1,t}(i\xi_s) \mp \frac{t^2}{2t-1}D^M_{r,t-1}(i\xi_s)\right.$$

$$\left. \pm \frac{(t+1)^2}{2t+3}D^M_{r,t+1}(i\xi_s) - \frac{(r+1)^2}{2r+3}D^M_{r+1,t}(i\xi_s)\right] \; ; \; \begin{array}{l} r \neq t \\ M = 0,1,2,\ldots \end{array} \qquad (5.10a)$$

$$D^{M+1}_{t,t}(i\xi_s) = -\sum_{r=0}^{M+1}{}' \frac{Q_r(i\xi_s)}{Q_t(i\xi_s)} D^{M+1}_{r,t}(i\xi_s) + A^{M+1}_t(i\xi_s) \; ; \; M = 0,1,2,\ldots \qquad (5.10b)$$

$$D^o_{o,o}(i\xi_s) = A^o_o(i\xi_s) \; , \qquad (5.10c)$$

with

$$A^M_t(i\xi_s) = \begin{cases} -\sqrt{\pi}\,\dfrac{(2t+1)(i\xi_s \pm 1)^M}{2^{M+1}\left(\frac{M-t}{2}\right)!\,\Gamma\left(\frac{M+t}{2}+\frac{3}{2}\right)Q_t(i\xi_s)} \, , & M+t \text{ even} \\ 0 \, , & M+t \text{ odd} \end{cases} \qquad (5.11)$$

The corresponding expressions for the Neumann problem are,

$$D^{M+1}_{r,t}(i\xi_s) = \frac{2}{r(r+1)-t(t+1)}\left[\frac{r^2}{2r-1}D^M_{r-1,t}(i\xi_s) \mp \frac{t^2}{2t-1}D^M_{r,t-1}(i\xi_s)\right.$$

$$\left. \pm \frac{(t+1)^2}{2t+3}D^M_{r,t+1}(i\xi_s) - \frac{(r+1)^2}{2r+3}D^M_{r+1,t}(i\xi_s)\right] \; ; \; \begin{array}{l} r \neq t \\ M = 0,1,2,\ldots \end{array} \qquad (5.12a)$$

$$D^{M+1}_{t,t}(i\xi_s) = -\sum_{r=0}^{M+1}{}' \frac{Q_r(i\xi_s)'}{Q_t(i\xi_s)'} D^{M+1}_{r,t}(i\xi_s) + \sum_{r=0}^{M} \frac{Q_r(i\xi_s)}{Q_t(i\xi_s)'} D^M_{r,t}(i\xi_s) + A^{M+1}_t(i\xi_s) \; ;$$

$$M = 0,1,2,\ldots \qquad (5.12b)$$

$$D^o_{o,o}(i\xi_s) = 0 \; , \qquad (5.12c)$$

with

$$A_t^M(i\xi_s) = \begin{cases} -\sqrt{\pi}\, M \dfrac{(2t+1)(i\xi_s \pm 1)^{M-1}}{2^{M+1}\left(\frac{M-t}{2}\right)!\,\Gamma\left(\frac{M+t}{2}+\frac{3}{2}\right) Q_t(i\xi_s)'} , & M+t \text{ even} \\ 0 , \quad M+t \text{ odd} . \end{cases} \tag{5.13}$$

In both the Dirichlet and Neumann problems the far field is given by

$$u^{sf}(p_1) = \frac{e^{ikc\xi_1}}{\xi_1} \sum_{n=0}^{\infty} (-ikc)^n \sum_{M=0}^{n} (-i)^{n-1} \frac{(i\xi_s \mp \eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} D_{o,t}^{M}(i\xi_s) P_t(\eta_1) , \tag{5.14}$$

and the scattering cross section be equation (5.7) where in the present case,

$$u_n^{sf}(p_1) = (-i)^{n-1} \sum_{M=0}^{n} \frac{(i\xi_s \mp \eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} D_{o,t}^{M}(i\xi_s) P_t(\eta_1) . \tag{5.15}$$

5.2 <u>The Disc</u>

As we mentioned earlier, when $\xi_s = 0$ the oblate spheroid degenerates to a disc of radius c in the x, y plane, with center at the origin (Morse and Feshbach, 1953, p. 1292). It is easy to verify from the corresponding formulas for the oblate spheroid that the scattered field due to the presence of the disc is given by

$$u^s(p_1) = e^{ikc\xi_1} \sum_{n=0}^{\infty} (-ikc)^n \sum_{M=0}^{n} (-i)^n \frac{(\mp \eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} \sum_{r=0}^{M} \sum_{\ell=0}^{t} D_{r,t}^{M,\ell}(0) \cdot Q_r^{\ell}(i\xi_1) P_t^{\ell}(\eta_1) \cos \ell\phi_1 , \tag{5.16}$$

where, for the Dirichlet problem, the recurrence relations (3.26a, b, c) hold among the coefficients $D_{r,t}^{M,\ell}(0)$, with

$$A_t^{M,\ell}(0) = \begin{cases} -\epsilon_\ell\sqrt{\pi}\, \dfrac{(\pm\cos\theta_o)^M}{2^{M+1}}\,(2t+1)\,\dfrac{(t-\ell)!}{(t+\ell)!}\,\dfrac{P_t^\ell\left(\dfrac{1}{\cos\theta_o}\right)}{\left(\dfrac{M-t}{2}\right)!\;\Gamma\left(\dfrac{M+t}{2}+\dfrac{3}{2}\right)Q_t^\ell(0)}\,, & M+t \text{ even} \\ 0\,, \quad M+t \text{ odd} \end{cases} \tag{5.17}$$

For the Neumann problem, the coefficients $D_{r,t}^{M,\ell}(0)$ are related through equations (4.22a, b, c) with

$$A_t^{M,\ell}(0) = -\epsilon_\ell\sqrt{\pi}\cos\theta_o\,\frac{(\pm\cos\theta_o)^{M-1}}{2^{M+1}\left(\dfrac{M-t}{2}\right)!\;\Gamma\left(\dfrac{M+t}{2}+\dfrac{3}{2}\right)Q_t^{\ell}(0)'}\cdot\frac{(t-\ell)!}{(t+\ell)!}$$

$$\cdot\left[(M-t)(t-\ell+1)P_{t+1}^\ell\left(\frac{1}{\cos\theta_o}\right)+(M+t+1)(t+\ell)P_{t-1}^\ell\left(\frac{1}{\cos\theta_o}\right)\right],$$

$$M+t \text{ even}\,, \tag{5.18}$$

$$A_t^{M,\ell}(0) = 0\,, \qquad M+t \text{ odd}\,. \tag{5.19}$$

From (5.16) the far field is

$$u^{sf}(p_1) = \frac{e^{ikc\xi_1}}{\xi_1}\sum_{n=0}^{\infty}(-ikc)^n\sum_{M=0}^{n}(-i)^{n-1}\frac{(\mp\eta_1)^{n-M}}{(n-M)!}\sum_{t=0}^{M}\sum_{\ell=0}^{t}(-1)^\ell\ell!\,D_{o,t}^{M,\ell}(0)$$

$$\cdot\, P_t^\ell(\eta_1)\cos\ell\phi_1\,. \tag{5.20}$$

The scattering cross section is given by (5.7) with

$$u_n^{sf}(p_1) = (-i)^{n-1} \sum_{M=0}^{n} \frac{(\mp\eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} \sum_{\ell=0}^{t} (-1)^{\ell} \ell !\, D_{o,t}^{M,\ell}(0) P_t^{\ell}(\eta_1) \cos \ell \phi_1 \tag{5.21}$$

Normal Incidence:

When $\theta_o = 0$, we can write, as we did in section 5.1,

$$u^s(p_1) = e^{ikc\xi_1} \sum_{n=0}^{\infty} (-ikc)^n \sum_{M=0}^{n} (-i)^n \frac{(\mp\eta_1)^{n-M}}{(n-M)!} \sum_{r=0}^{M} \sum_{t=0}^{M} D_{r,t}^{M}(0) Q_r(i\xi_1) P_t(\eta_1) \tag{5.22}$$

where, for the Dirichlet problem, the coefficients $D_{r,t}^{M}(0)$ are related through equations (5.10a, b, c), with

$$A_t^M(0) = \begin{cases} -\sqrt{\pi}\,(\pm 1)^M \dfrac{2t+1}{2^{M+1}\left(\frac{M-t}{2}\right)!\,\Gamma\left(\frac{M+t}{2}+\frac{3}{2}\right) Q_t(0)}, & M+t \text{ even} \\ 0, & M+t \text{ odd} \end{cases} \tag{5.23}$$

For the Neumann problem the coefficients $D_{r,t}^{M}(0)$ in (5.23) are related through (5.12a, b, c), with

$$A_t^M(0) = \begin{cases} -\sqrt{\pi}\,(\pm 1)^{M-1} M \dfrac{2t+1}{2^{M+1}\left(\frac{M-t}{2}\right)!\,\Gamma\left(\frac{M+t}{2}+\frac{3}{2}\right) Q_t(0)'}, & M+t \text{ even} \\ 0, & M+t \text{ odd}. \end{cases} \tag{5.24}$$

Equations (5.23) and (5.24) were derived from (5.17) and (5.18), (5.19), respectively, by letting $\theta_o = 0$ and $\ell = 0$, and they are in agreement with the corresponding equations (5.11) and (5.13) for the oblate spheroid.

For both the Dirichlet and Neumann problems the far field is given by

$$u^{sf}(p_1) = \frac{e^{ikc\xi_1}}{\xi_1} \sum_{n=0}^{\infty} (-ikc)^n \sum_{M=0}^{n} (-i)^{n-1} \frac{(\mp\eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} D^M_{o,t}(0) P_t(\eta_1) \ , \tag{5.25}$$

while the scattering cross section is given by (5.7) with

$$u_n^{sf}(p_1) = (-i)^{n-1} \sum_{M=0}^{n} \frac{(\mp\eta_1)^{n-M}}{(n-M)!} \sum_{t=0}^{M} D^M_{o,t}(0) P_t(\eta_1) \ . \tag{5.26}$$

In Appendix I, we give the first six terms of the far field expansion for both the Dirichlet and Neumann problems.

VI
NUMERICAL CALCULATIONS

As a demonstration of their usefulness, the theoretical results have been employed to calculate the scattering cross section of some representative prolate spheroids for both Dirichlet and Neumann boundary conditions. The prolate spheroids considered had major to minor axis ratios of 10:1, 5:1, and 2:1. Back scattered and forward scattered cross sections were determined as functions of wavelength, and complete polar diagrams of bistatic cross section were obtained for a few special values of kc. All calculations were carried out for a plane wave incident along the axis of symmetry of the spheroid.

The expressions employed for this calculation, which we repeat here for convenience, were equation (3.32)

$$\sigma = 4\pi c^2 \sum_{n=0}^{\infty} (kc)^{2n} \sum_{m=0}^{2n} (-1)^{n+m} u^{sf}_{2n-m}(\eta_1) u^{sf}_m(\eta_1) \tag{6.1}$$

and equation (3.38)

$$u^{sf}_m(\eta_1) = \sum_{j=0}^{m} \sum_{i=0}^{j} \frac{(\xi_s \mp \eta_1)^{m-j}}{(m-j)!} D^j_{o,i}(\xi_s) P_i(\eta_1) \tag{6.2}$$

where $D^j_{o,i}$ in (6.2) is given by (3.35) for the Dirichlet problem and (4.25) for the Neumann problem.

The series in (6.1) was terminated at $n = 10$ for the 2:1 and 5:1 spheroids and at $n = 9$ for the 10:1 spheroid. Thus the cross section values included terms up to and including $(kc)^{20}$ and $(kc)^{18}$ respectively. The back and forward scattering results were also obtained for smaller values of n so as to reveal the manner in which the inclusion of higher order terms improves the Rayleigh approximation.

Figures 6-1 and 6-2 present the back scattering ($\eta_1 = 1$) cross sections of soft and hard spheroids respectively. The cross section values are normalized with respect to the geometric optics value

$$\sigma_{g.o.} = \frac{\pi b^4}{a^2} = \pi c^2 \frac{(\xi_s^2 - 1)^2}{\xi_s^2} \quad . \tag{6.3}$$

The number associated with each curve indicates the value of n at which equation (6.1) was terminated. The Rayleigh curve (the curve obtained by terminating (6.1) at the first nonvanishing power of kc) is denoted by n = 0 for the soft spheroid and by n = 2 for the hard spheroid. The exact result shown in Figures 6-1 and 6-2 was computed from the prolate spheroidal function series (Senior, 1966). Also included in each figure is the maximum value of ka ($= kc\xi_s$) for which the series in (6.1) converges, i.e. the radius of convergence, as estimated by Darling and Senior (1965). The present low frequency calculations have no precedent except in the case of the 10:1 hard spheroid where similar calculations (though not as extensive) were reported by Sleator (1964).

Figures 6-3 and 6-4 present the forward scattering ($\eta_1 = -1$) cross sections of the same spheroids. The cross section values are normalized with respect to the limiting form of the bistatic geometric optics value

$$\sigma_{g.o.} = \pi a^2 = \pi c^2 \xi_s^2 \quad . \tag{6.4}$$

As before, the number associated with each curve designates the value of n at which the series in equation (6.1) was terminated. No exact results were available for comparison in this case.

Figures 6-5 through 6-9 present polar diagrams of the bistatic cross sections of the same spheroids. Since the polar diagram is symmetric for nose-on incidence, which is the only case considered, each figure includes data for both hard and soft spheroids. The back and forward scattered cross sections lie on the heavy vertical line bisecting the figure with the back scattering ($\theta_1 = 0$) value on the upper part and the forward scattering ($\theta_1 = \pi$) value on the lower. The values of the cross section are normalized with respect to the geometric optics cross section, viz.,

$$\sigma_{g.o.} = 4\pi b^4 a^2 \left[a^2(1+\eta_1) + b^2(1-\eta_1) \right]^{-2}$$

$$= 4\pi c^2 \frac{\xi_s^2(\xi_s^2 - 1)^2}{(2\xi_s^2 - 1 + \eta_1)^2} \qquad (6.5)$$

with $\eta_1 = \cos\theta_1$.

As noted previously, the values presented for the 2:1 and 5:1 spheroids were obtained after terminating the series in (6.1) at $n = 10$ while for the 10:1 spheroid the series was terminated at $n = 9$.

Similar calculations have been carried out by Spence and Granger (1951) for hard spheroids though the values of ξ_s and kc were different from those employed here. In the few cases where comparison was possible ($kc = 1$, $a/b = 5, 10$), good agreement was obtained.

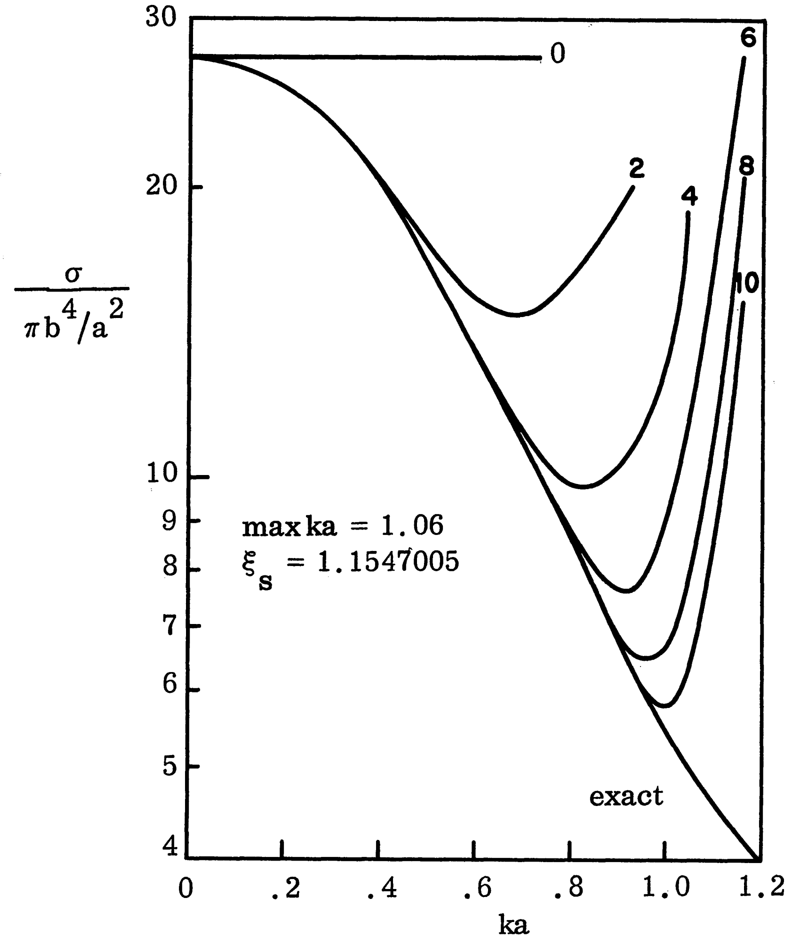

FIG. 6-1a: BACK SCATTERING CROSS SECTION OF SOFT, 2:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

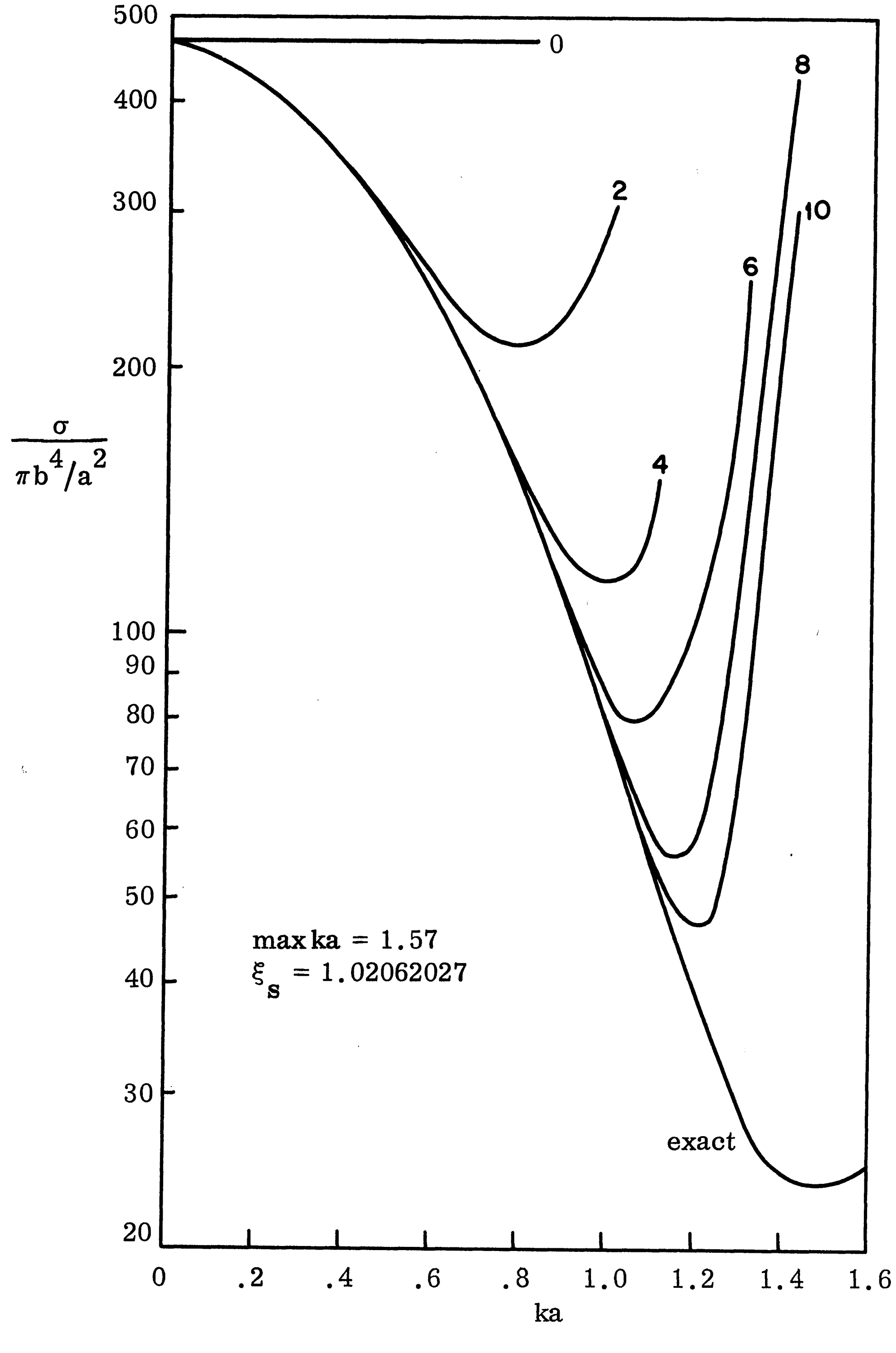

FIG. 6-1b: BACK SCATTERING CROSS SECTION OF SOFT, 5:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

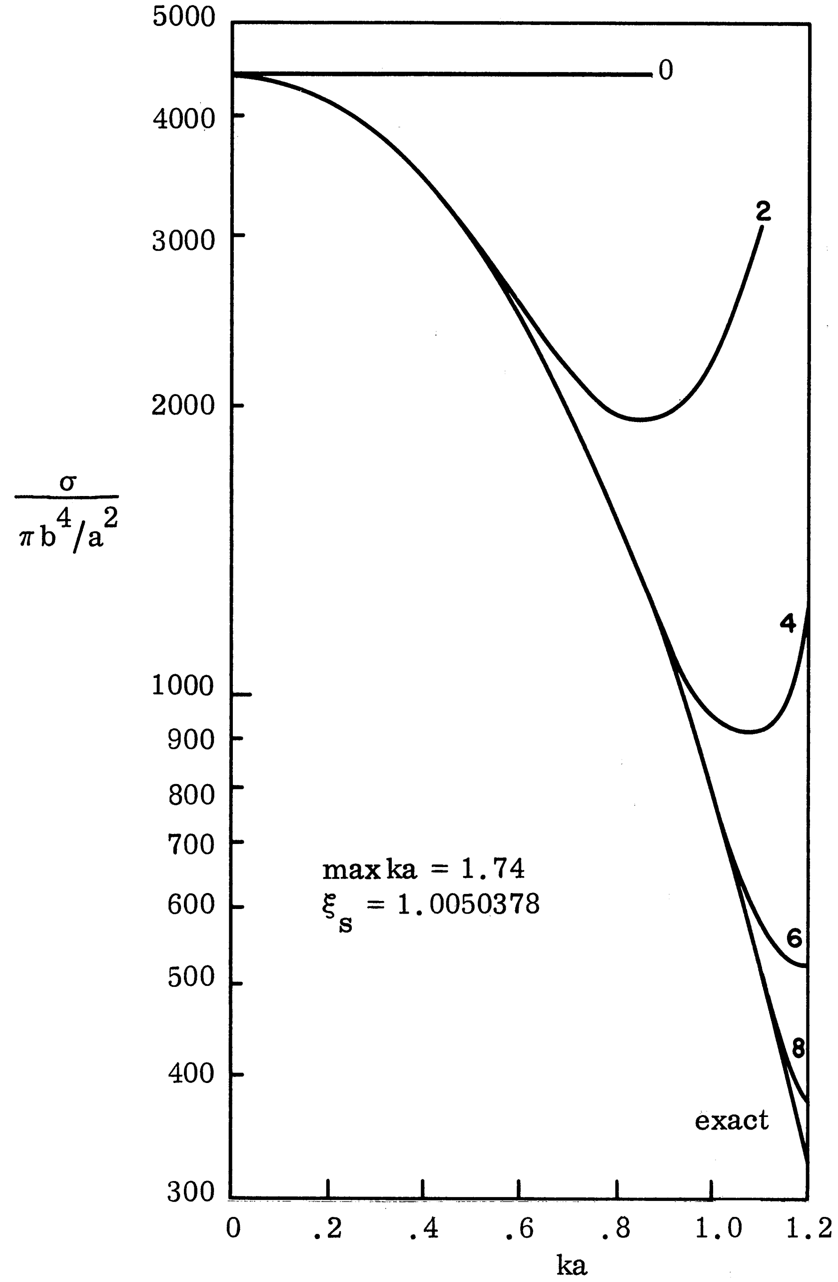

FIG. 6-1c: BACK SCATTERING CROSS SECTION OF SOFT, 10:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

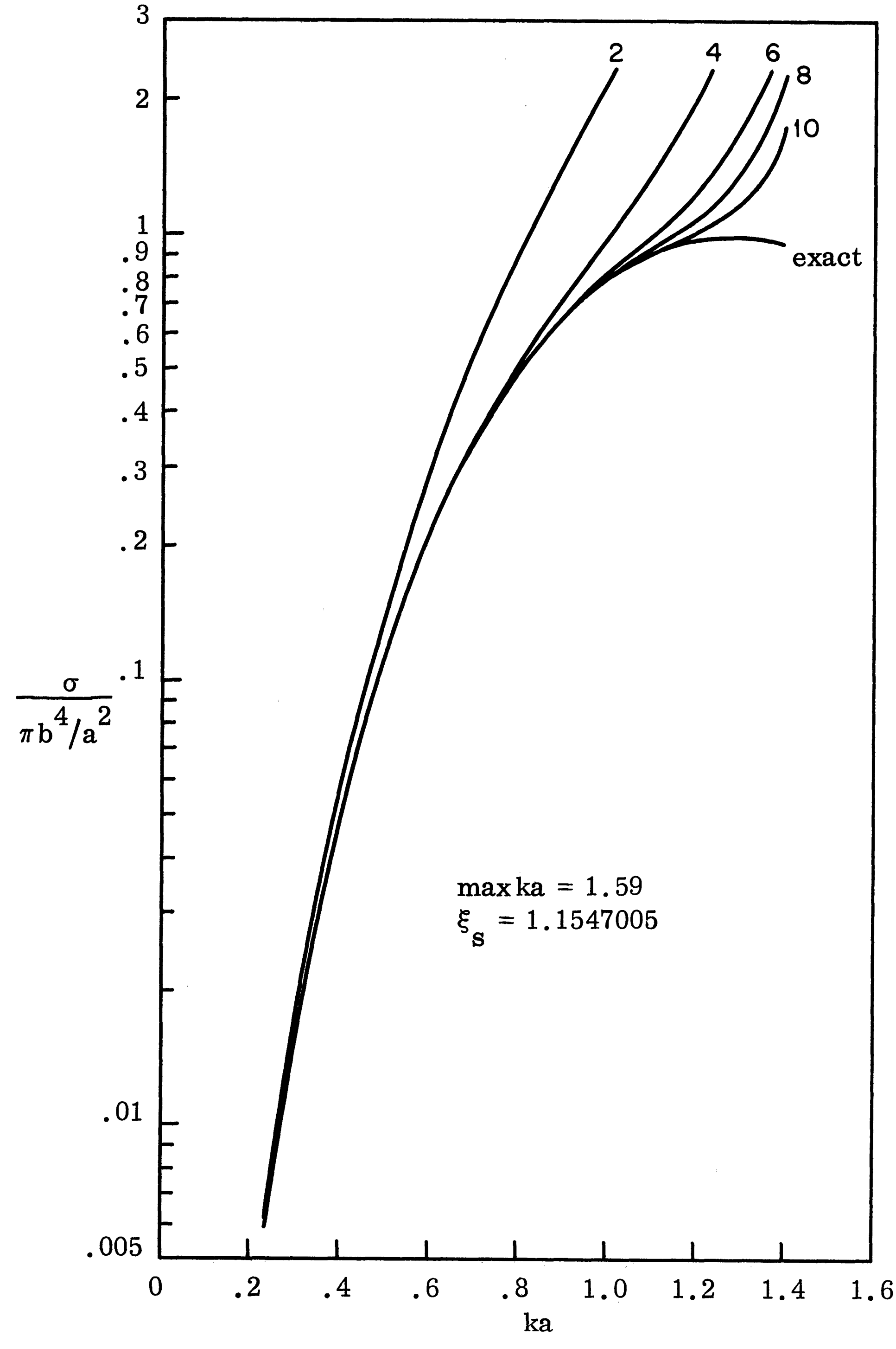

FIG. 6-2a: BACK SCATTERING CROSS SECTION OF HARD, 2:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

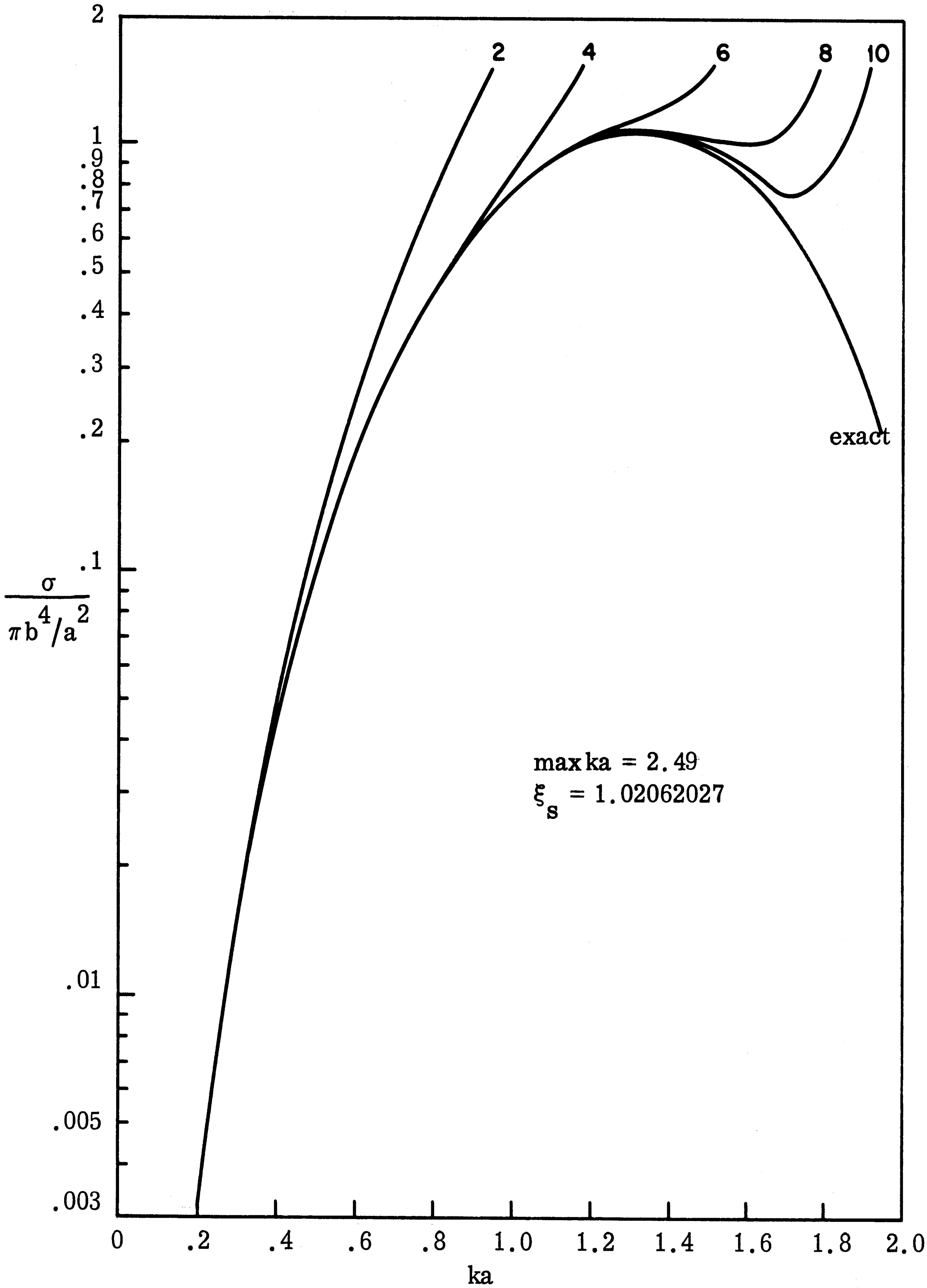

FIG. 6-2b: BACK SCATTERING CROSS SECTION OF HARD, 5:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

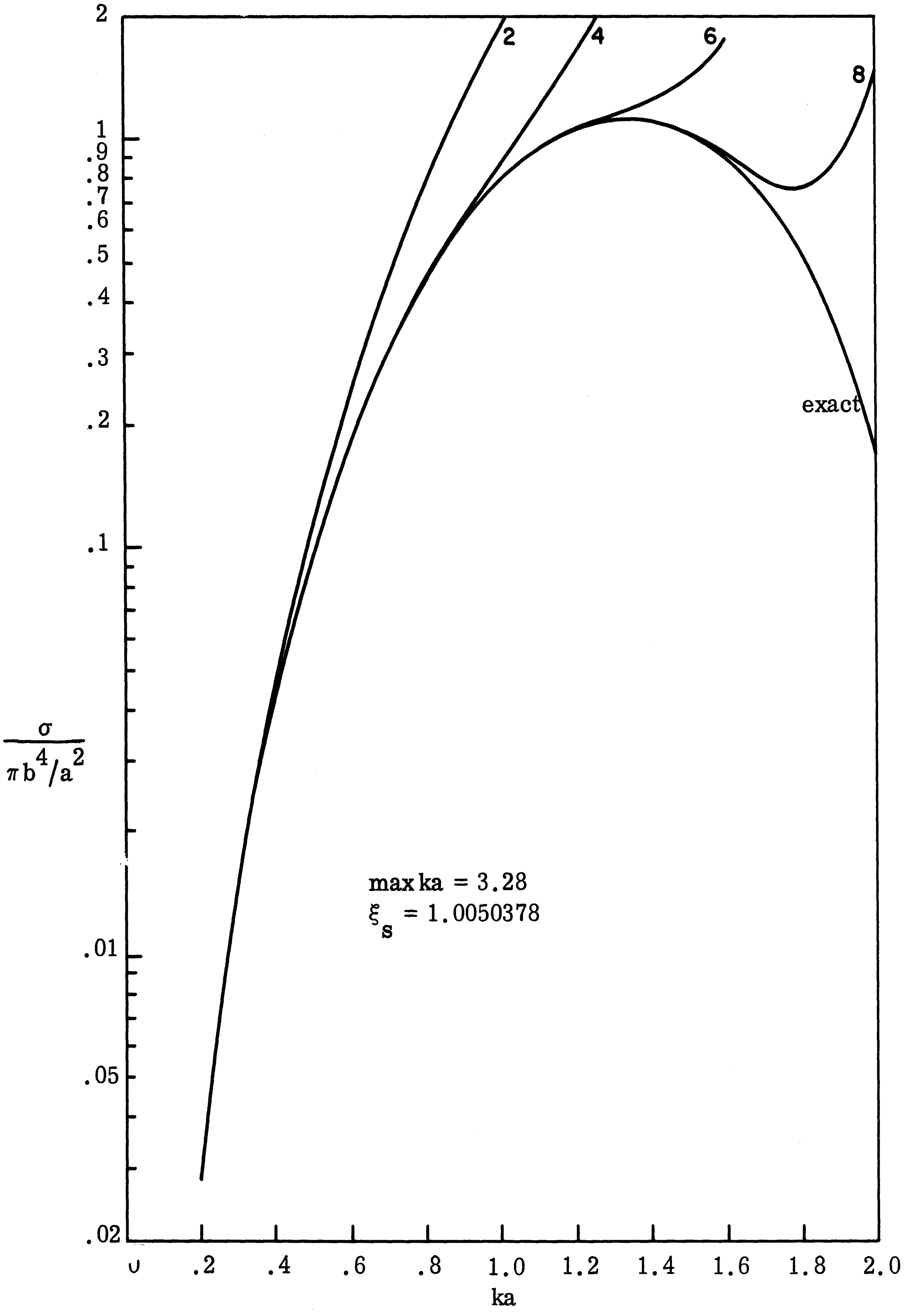

FIG. 6-2c: BACK SCATTERING CROSS SECTION OF HARD, 10:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

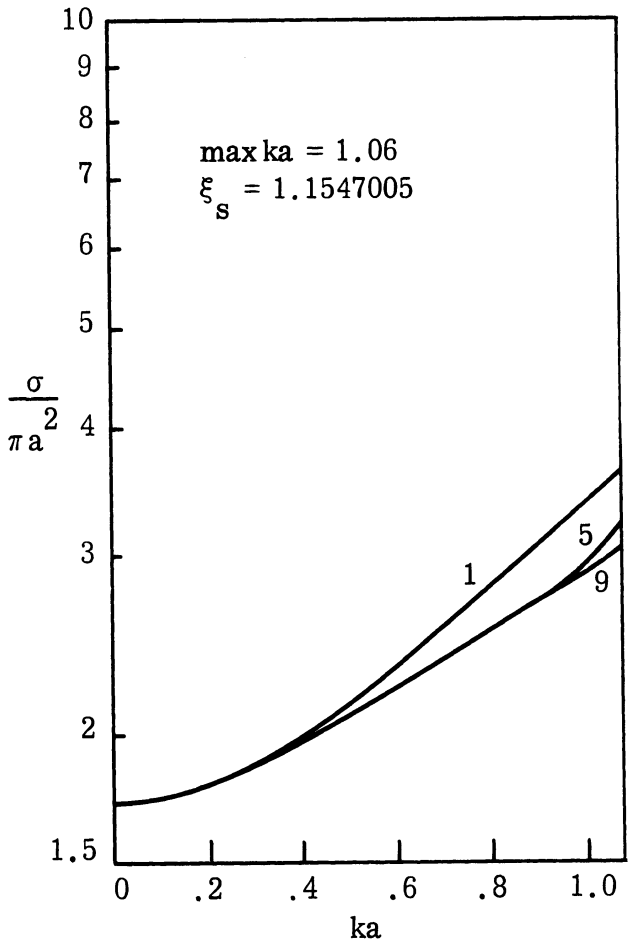

FIG. 6-3a: FORWARD SCATTERING CROSS SECTION OF SOFT, 2:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

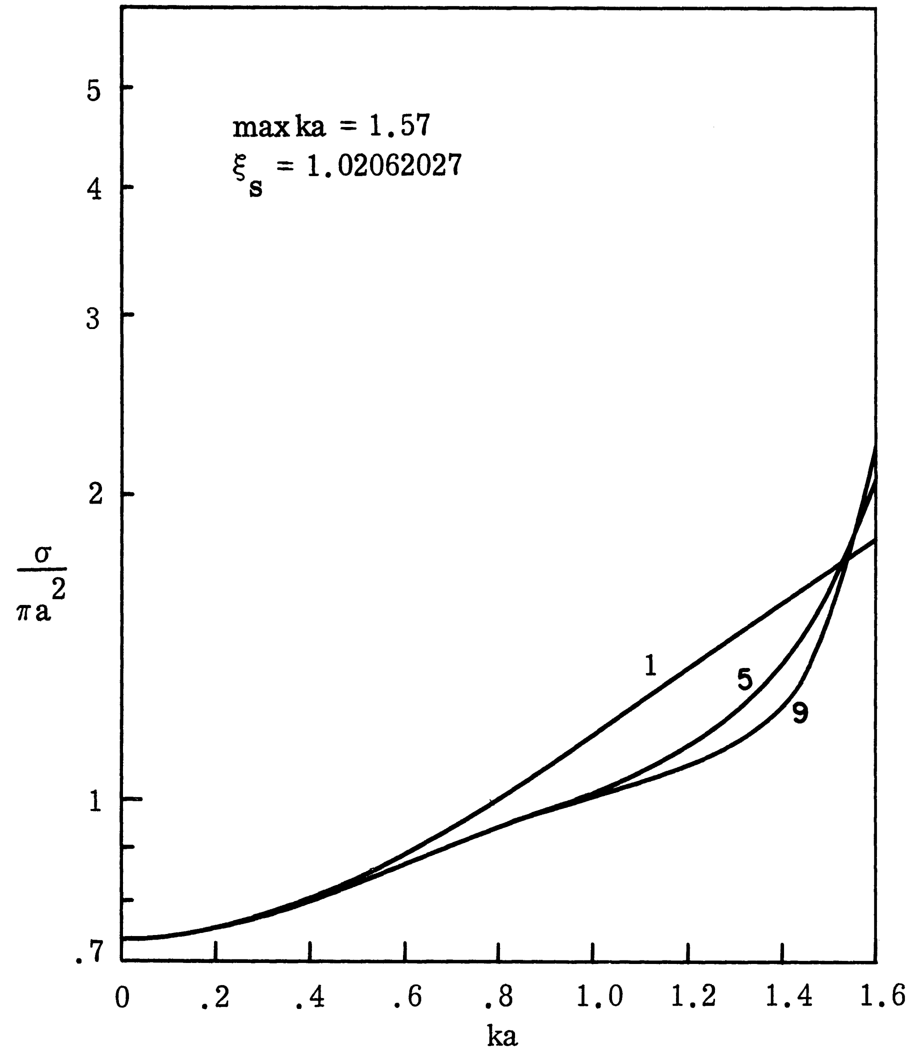

FIG. 6-3b: FORWARD SCATTERING CROSS SECTION OF SOFT, 5:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

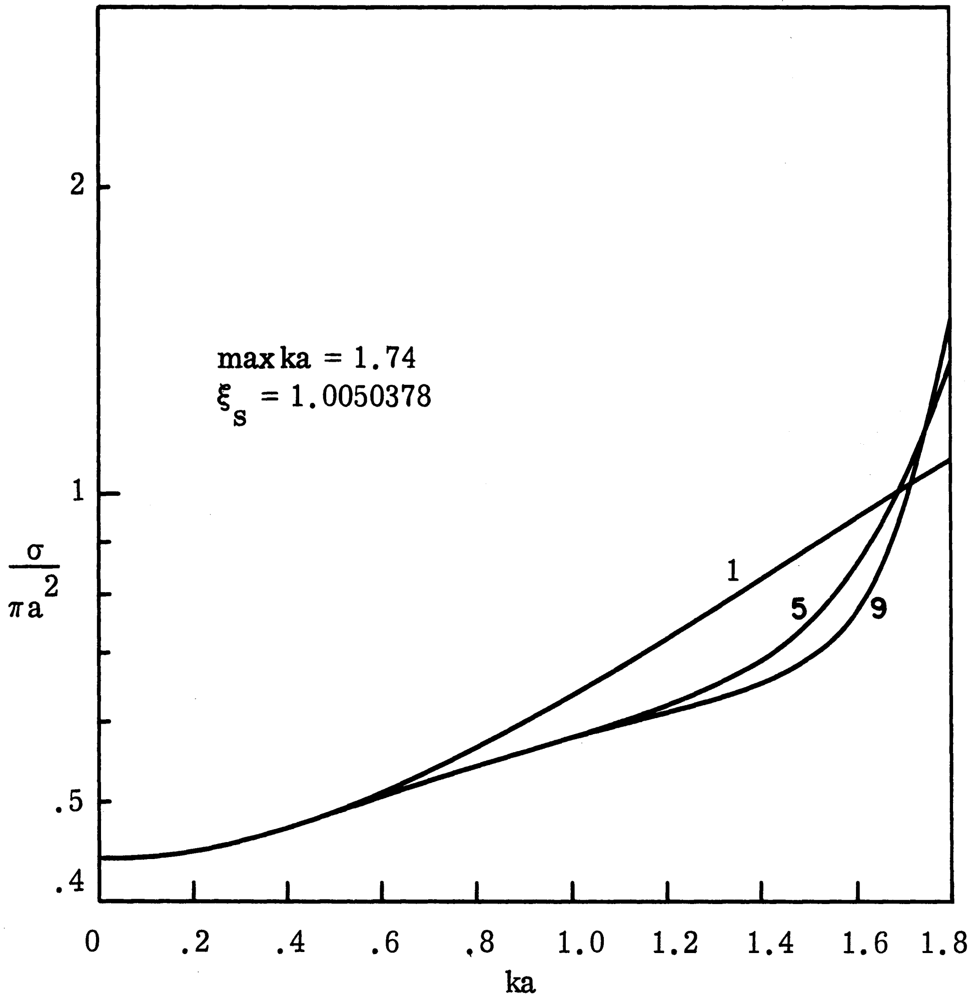

FIG. 6-3c: FORWARD SCATTERING CROSS SECTION OF SOFT, 10:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

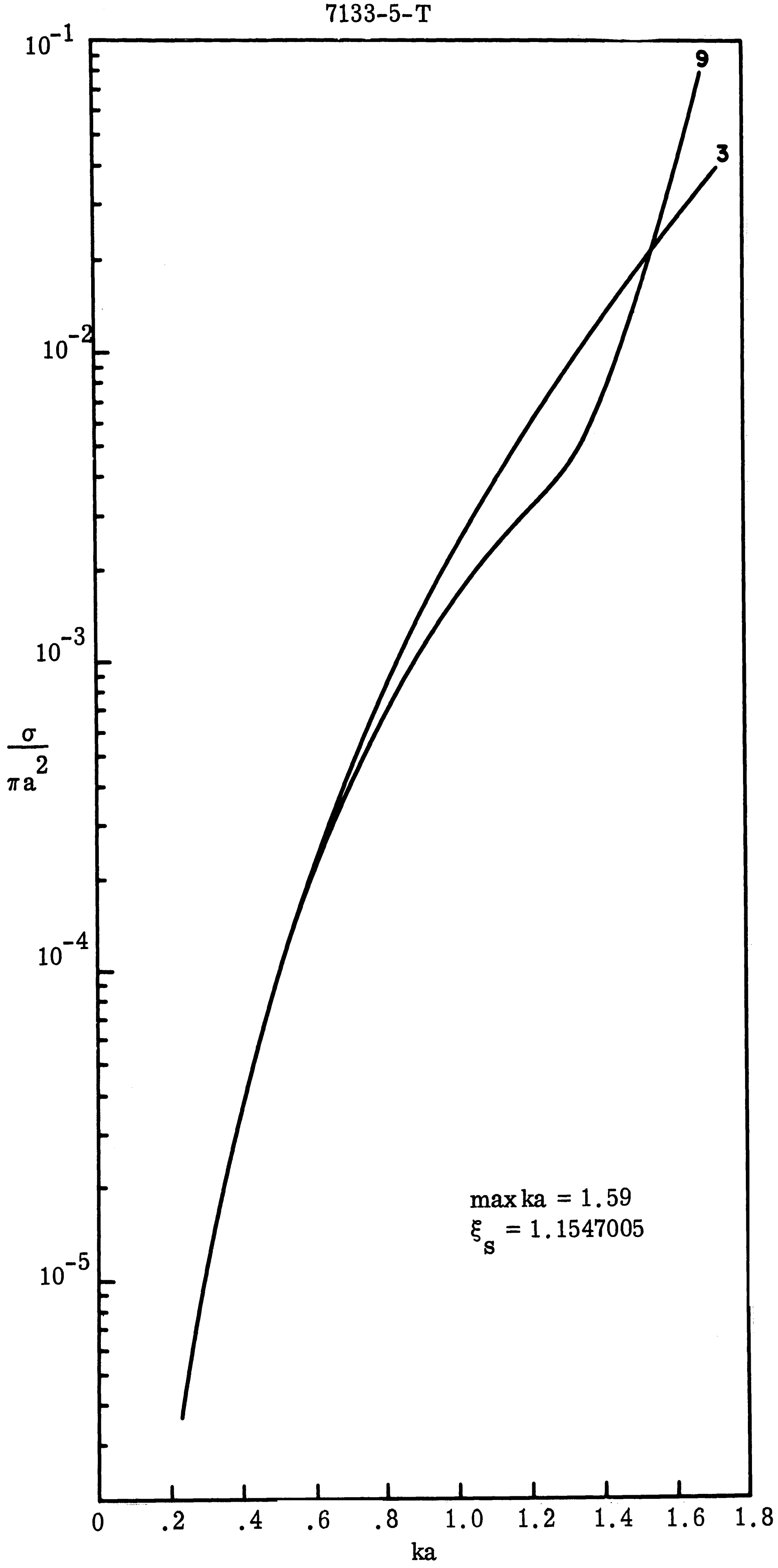

FIG. 6.4a: FORWARD SCATTERING CROSS SECTION OF HARD, 2:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

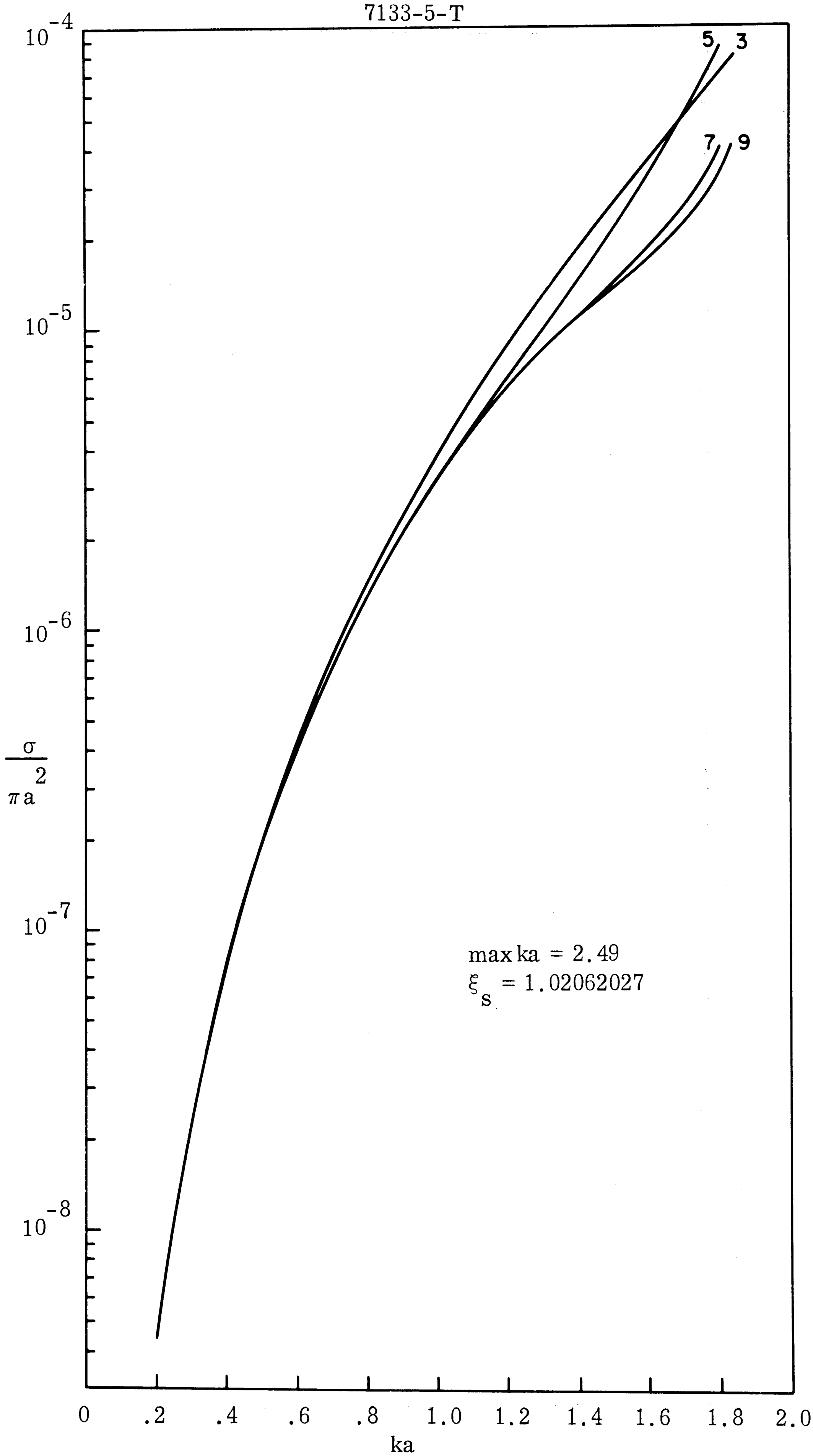

FIG. 6-4b: FORWARD SCATTERING CROSS SECTION OF HARD, 5:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

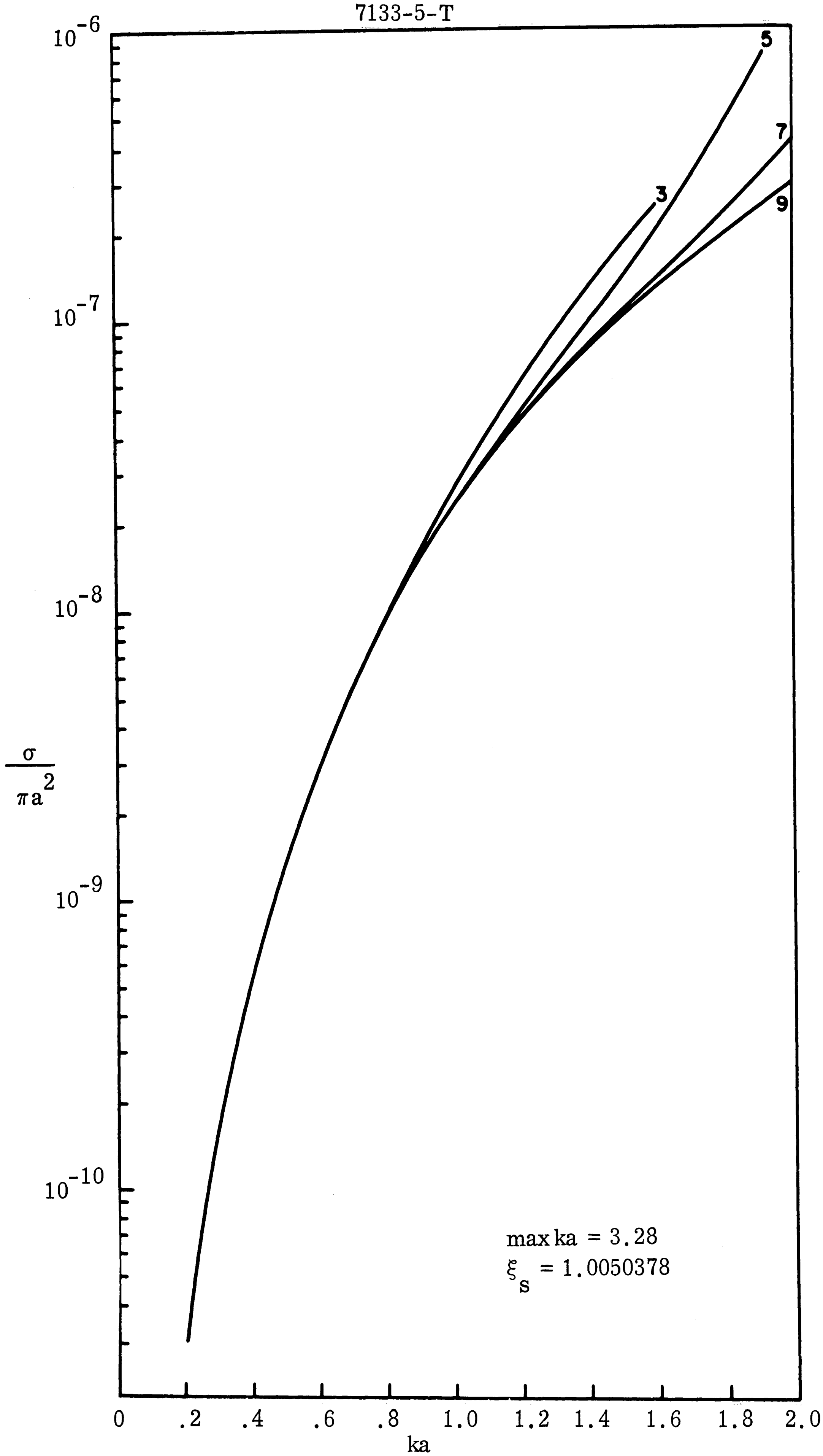

FIG. 6-4c: FORWARD SCATTERING CROSS SECTION OF HARD, 10:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

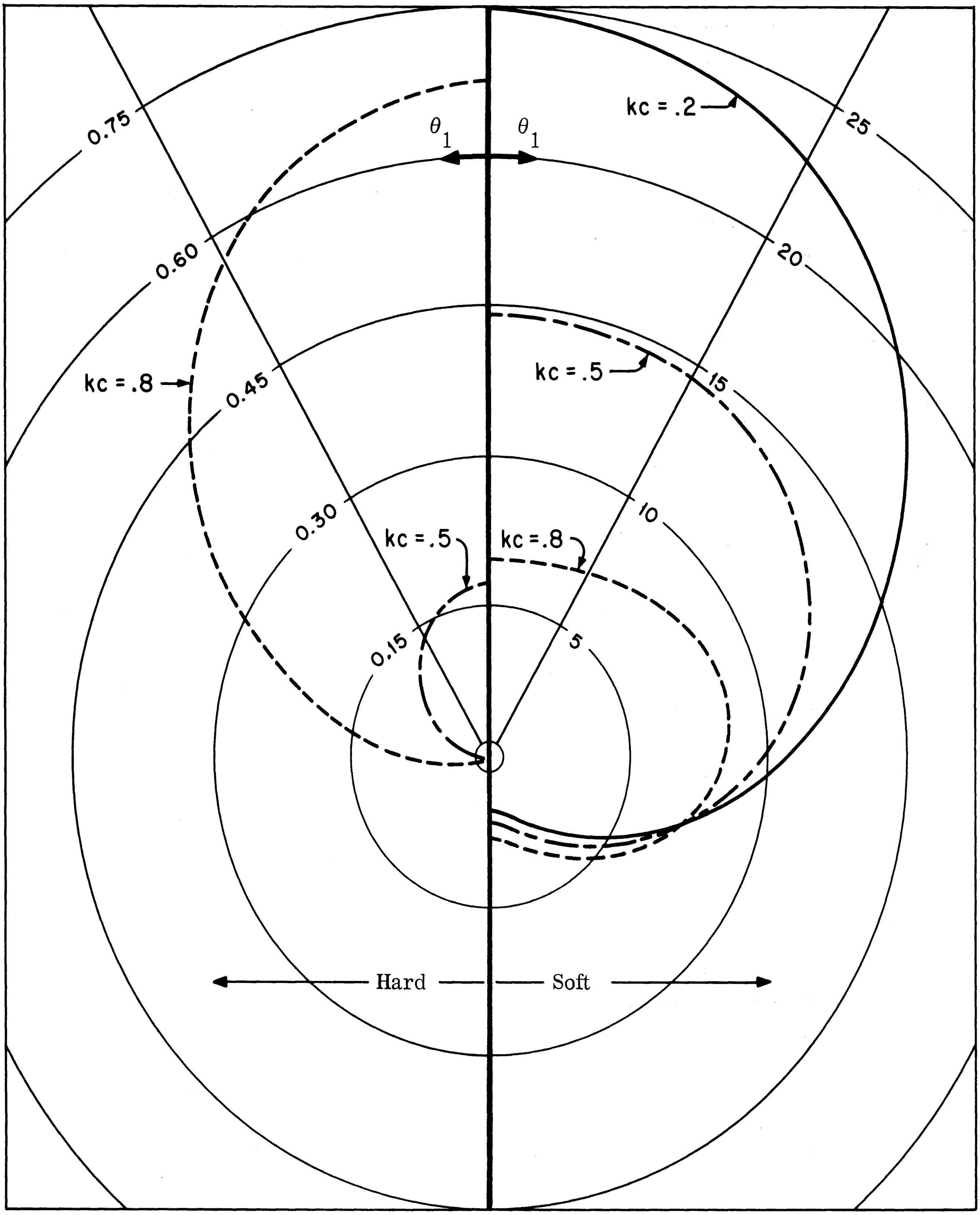

FIG. 6-5: BISTATIC CROSS SECTION OF 2:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

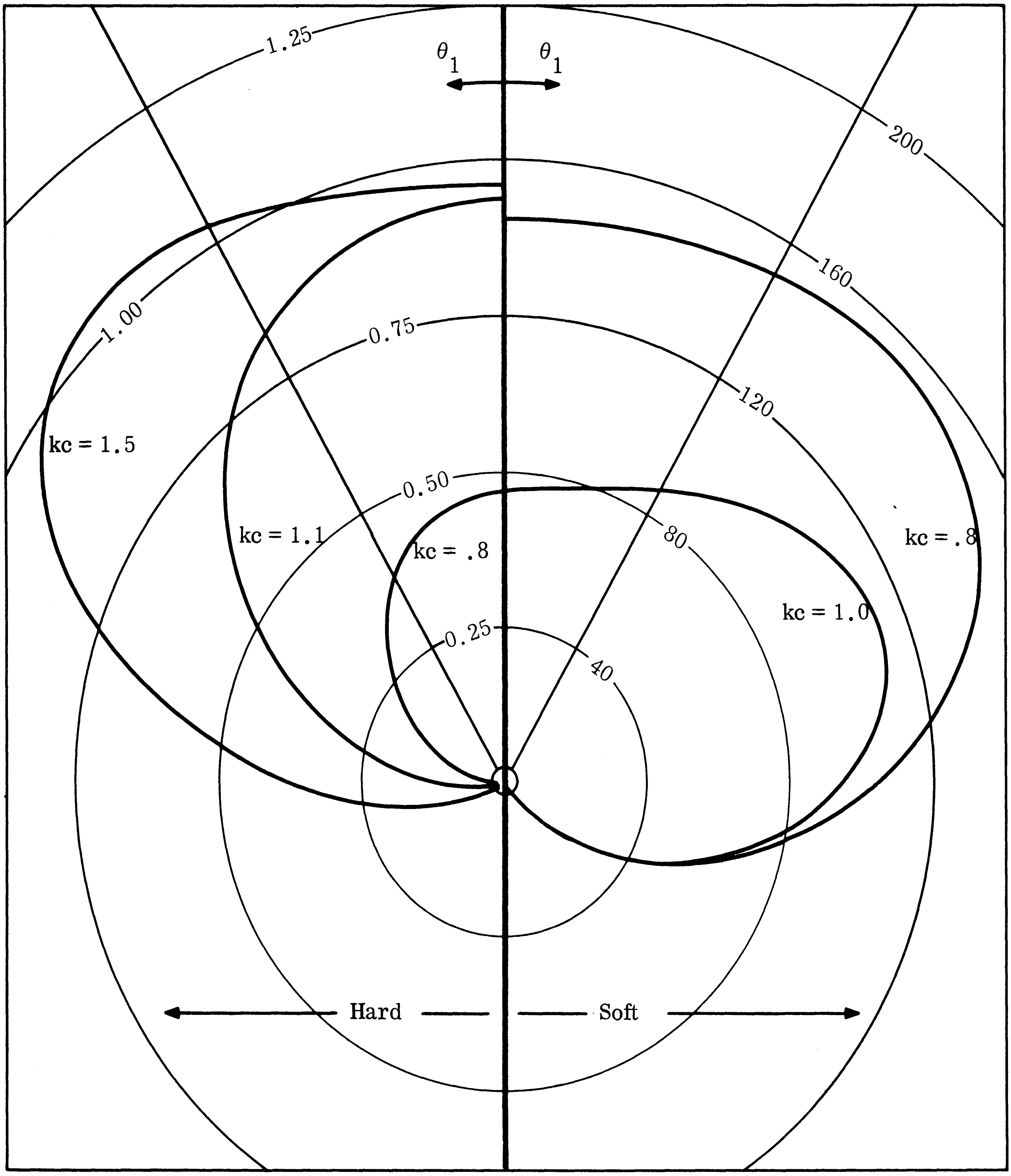

FIG. 6-6: BISTATIC CROSS SECTION OF 5:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

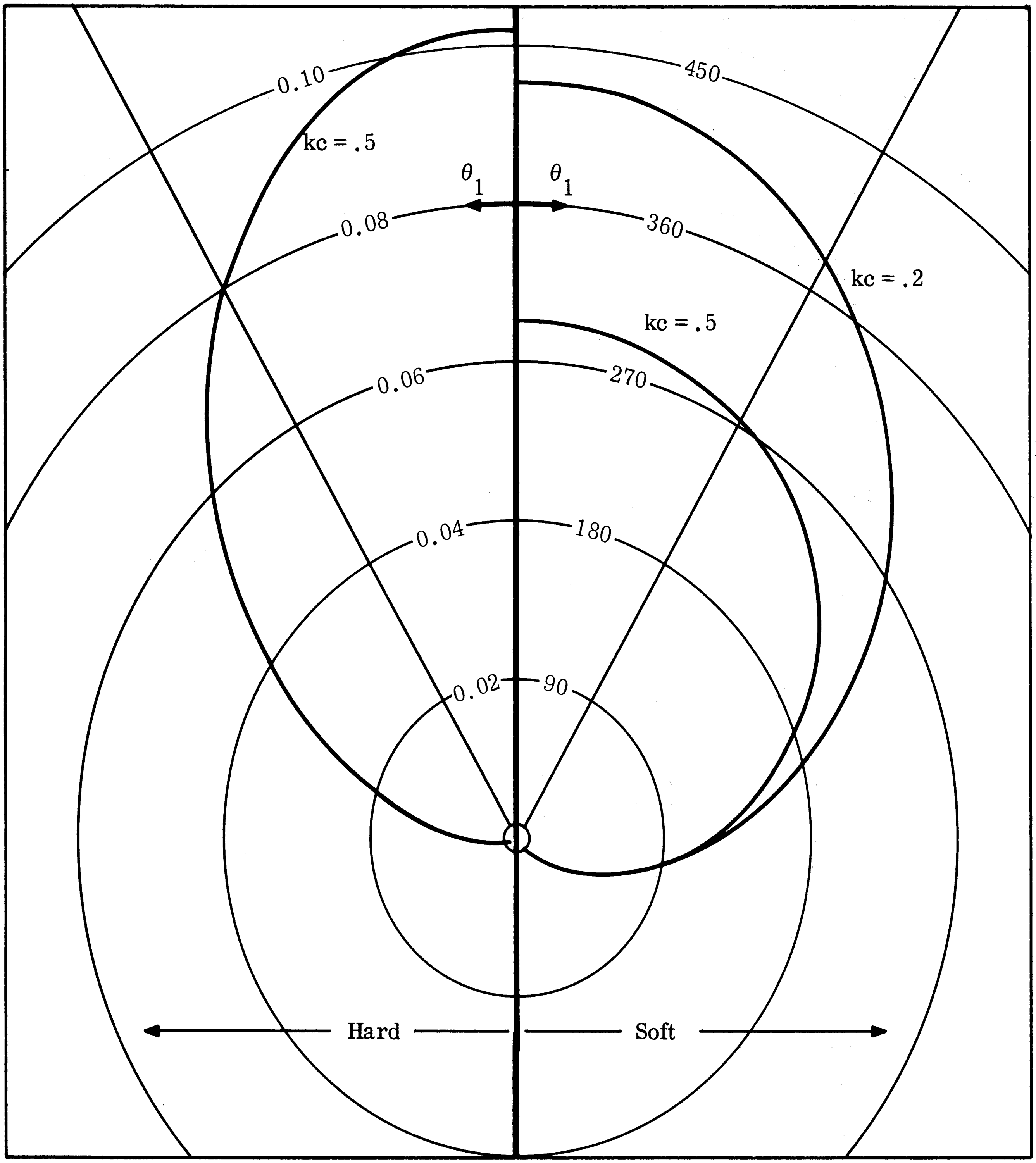

FIG. 6-7: BISTATIC CROSS SECTION OF 5:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

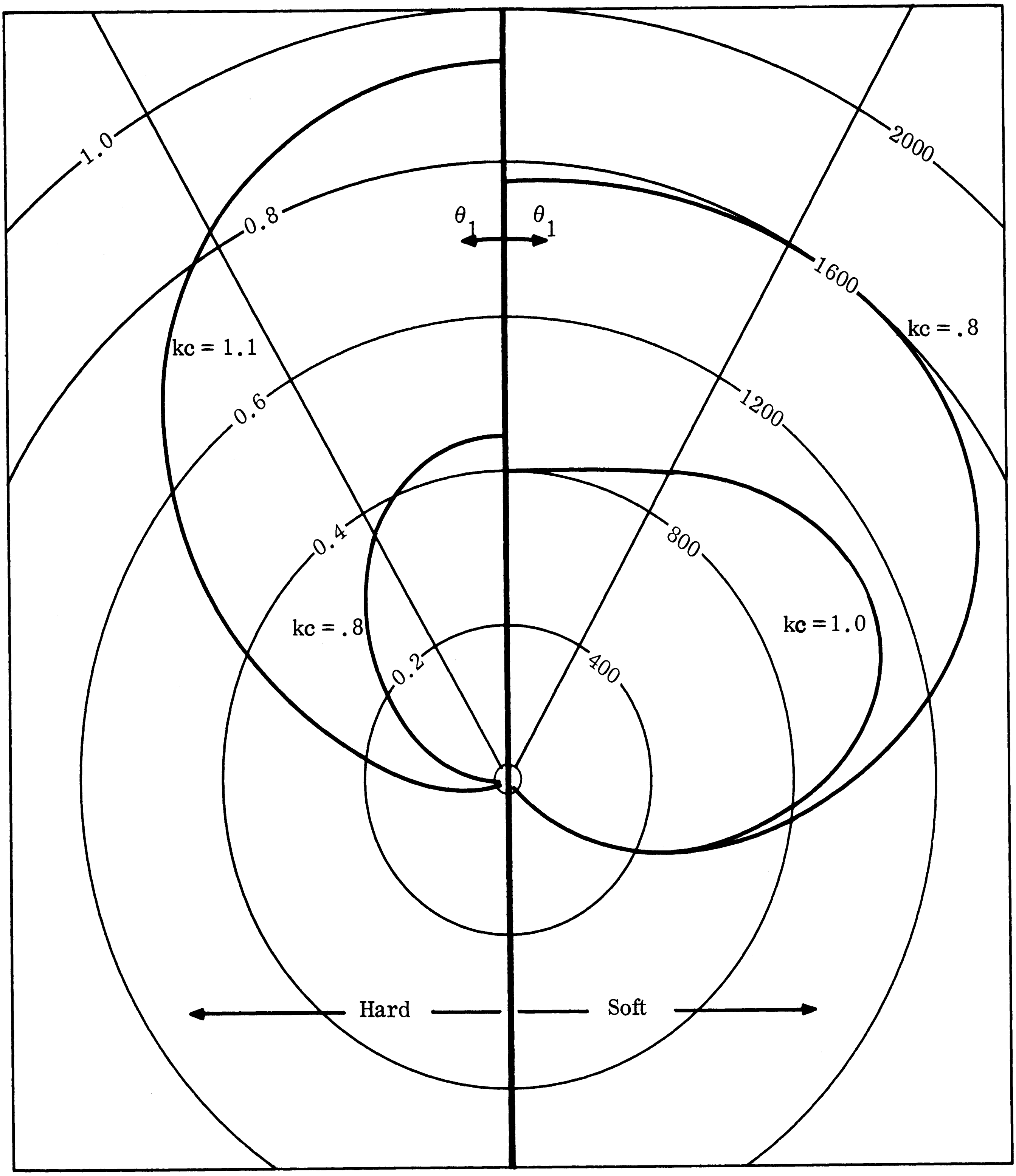

FIG. 6-8: BISTATIC CROSS SECTION OF 10:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

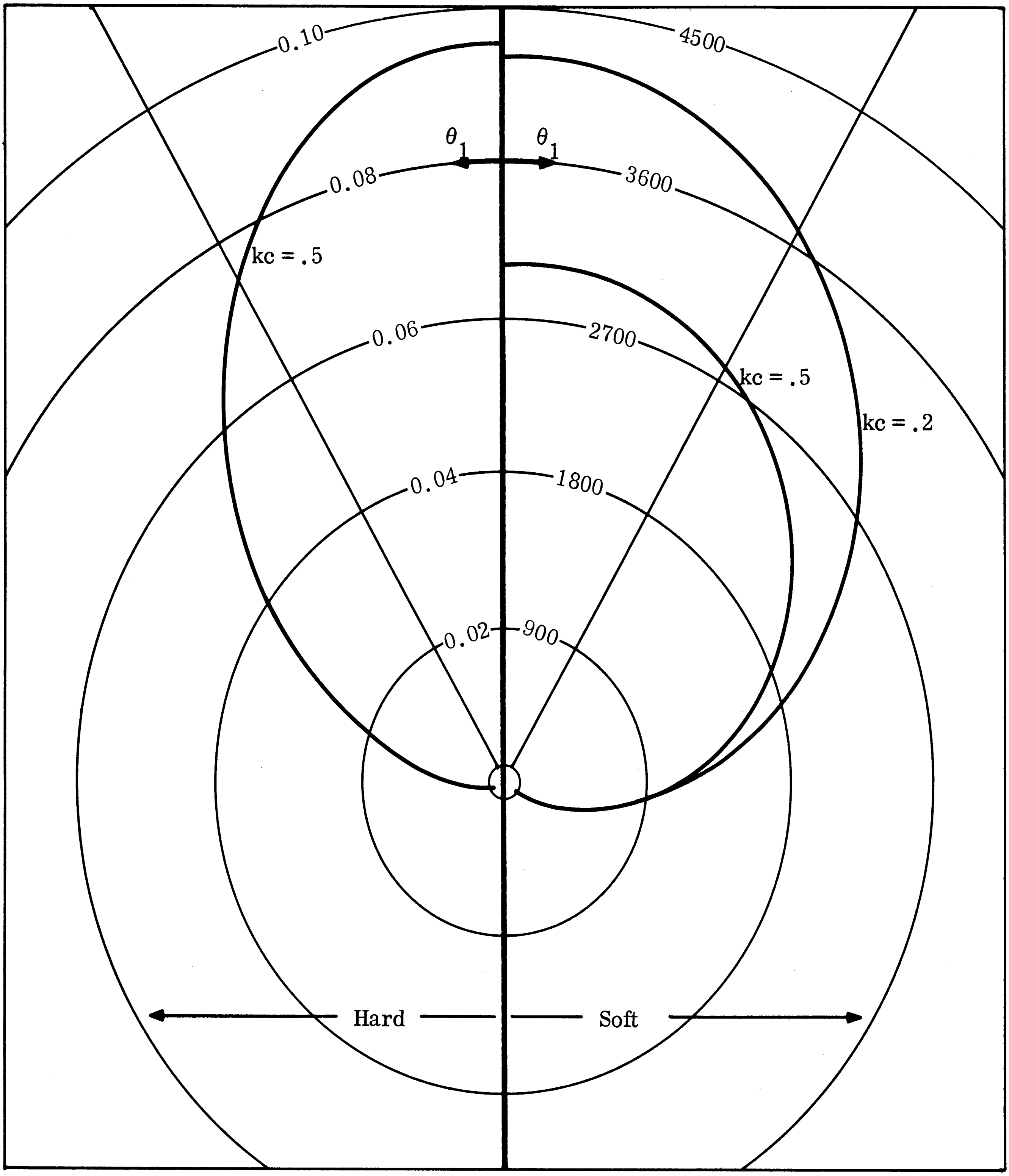

FIG. 6-9: BISTATIC CROSS SECTION OF 10:1 PROLATE SPHEROID FOR NOSE-ON INCIDENCE.

APPENDIX A
THE REGULARITY OF THE FUNCTION $\omega(p) = e^{-ikc(\xi \pm \eta)} u^s(p)$

In this appendix we offer a proof that the function $\omega(p)$ in (2.14) is regular at infinity in the sense of Kellogg, that is

$$\lim_{r \to \infty} \left| r\omega(p) \right| < \infty \quad \text{and} \quad \lim_{r \to \infty} \left| r^2 \frac{\partial \omega(p)}{\partial r} \right| < \infty , \quad \begin{array}{l} 0 \leqslant \theta \leqslant \pi \\ 0 \leqslant \phi \leqslant 2\pi \end{array} . \tag{A.1}$$

The proof is based on an expansion theorem (Wilcox, 1956) which guarantees that the field scattered by the prolate spheroid may be written in the form

$$u^s(p) = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{f_n(\theta, \phi)}{r^n} , \qquad r > a \tag{A.2}$$

where the series is uniformly and absolutely convergent for all r, θ, ϕ provided $r > a$, a being the radius of the smallest sphere completely enclosing the prolate spheroid.

From (A.2) it is clear that $u^s(p)$ satisfies the first of conditions (A.1) but not the second and, consequently, is not regular at infinity. The function $\omega(p)$, however, which by (2.14) and (A.2) may be written

$$\omega(p) = e^{-ik(c\xi - r \pm c\eta)} \frac{1}{r} \sum_{n=0}^{\infty} \frac{f_n(\theta, \phi)}{r^n} \tag{A.3}$$

can be shown to satisfy the Kellogg conditions. The proof is as follows:

The variables ξ and η are related to the spheroidal coordinates by the equations

$$\xi = \frac{1}{2c} \left[\sqrt{r^2 + 2cr\cos\theta + c^2} + \sqrt{r^2 - 2cr\cos\theta + c^2} \right] ,$$

$$\eta = \frac{1}{2c} \left[\sqrt{r^2 + 2cr\cos\theta + c^2} - \sqrt{r^2 - 2cr\cos\theta + c^2} \right] .$$

The factor $c(\xi \pm \eta)$ appearing in the exponential of (A.3) can now be written

$$c(\xi \pm \eta) = \sqrt{r^2 \pm 2cr\cos\theta + c^2} = r\sqrt{1 \pm 2\cos\theta(c/r) + (c/r)^2} \quad , \tag{A.4}$$

and, if r is large,

$$c(\xi \pm \eta) = r \pm c\cos\theta + O(1/r), \qquad r \to \infty \; . \tag{A.5}$$

We can then write for the first Kellogg condition

$$\lim_{r\to\infty} \left| r\omega(p) \right| = \lim_{r\to\infty} \left| e^{\mp ick\cos\theta + O(1/r)} \sum_{n=0}^{\infty} \frac{f_n(\theta,\phi)}{r^n} \right| =$$

$$= \left| e^{\mp ikc\cos\theta} f_o \right| < \infty \; . \tag{A.6}$$

To show that the second condition is satisfied we need the derivative of $\omega(p)$ with respect to r

$$\frac{\partial\omega(p)}{\partial r} = e^{-ik\left[c(\xi\pm\eta) - r\right]} \left\{ ik\left[1 - \frac{1 \pm \cos\theta(c/r)}{\sqrt{1 \pm 2\cos\theta(c/r) + (c/r)^2}}\right] \sum_{n=0}^{\infty} \frac{f_n(\theta,\phi)}{r^{n+1}} \right.$$

$$\left. - \sum_{n=0}^{\infty} \frac{(n+1) f_n(\theta,\phi)}{r^{n+2}} \right\} \; . \tag{A.7}$$

For r large

$$\frac{1}{\sqrt{1 \pm 2\cos\theta(c/r) + (c/r)^2}} = 1 \mp \cos\theta(c/r) + O(1/r^2) \; , \qquad r \to \infty \tag{A.8}$$

so that for the bracketed expression in (A.7) we can write

$$1 - \frac{1 \pm \cos\theta(c/r)}{\sqrt{1 \pm 2\cos\theta(c/r) + (c/r)^2}} = 1 - \left[1 \pm \cos\theta(c/r)\right]\left[1 \mp \cos\theta(c/r) + O(1/r^2)\right]$$

$$= O(1/r^2) \; . \tag{A.9}$$

Employing (A. 5) and (A. 9) in (A. 7) we have that

$$\lim_{r\to\infty}\left| r^2 \frac{\partial \omega(p)}{\partial r}\right| = \lim_{r\to\infty}\left| e^{\mp ikc\cos\theta + O(1/r)} \left\{ O(1) \sum_{n=0}^{\infty} \frac{f_n(\theta,\phi)}{r^{n+1}} - \sum_{n=0}^{\infty} \frac{f_n(\theta,\phi)}{r^n} \right\} \right| = \left| e^{\mp ik\cos\theta} f_o \right| < \infty , \qquad (A.10)$$

which shows that the second Kellogg condition holds also.

APPENDIX B
THE SURFACE INTEGRAL FOR THE DIRICHLET PROBLEM

In this appendix we evaluate the surface integral of (3.4). Repeating the expression,

$$I^s(p_1) = -c(\xi_s^2 - 1)\int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi\, \omega(p_s)\frac{\partial}{\partial \xi_s} G_o^D(p_1, p_s) \, . \tag{B.1}$$

From equations (3.2) and (3.3),

$$\omega(p_s) = -e^{-ikc(\xi_s \pm \eta)}\, e^{-ikc(\cos\theta_o \xi_s \eta + \sin\theta_o \sqrt{\xi_s^2 - 1}\,\sqrt{1-\eta^2}\cos\phi)} \tag{B.2}$$

and from (2.21), with $\xi_s < \xi_1$,

$$\frac{\partial}{\partial \xi_s} G_o^D(p_1, p_s) = -\frac{1}{4\pi c}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^m \epsilon_m (2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2 \cos m(\phi - \phi_1)$$

$$\cdot P_n^m(\eta_1)P_n^m(\eta)\left[P_n^m(\xi_s)' Q_n^m(\xi_1) - \frac{P_n^m(\xi_s)}{Q_n^m(\xi_s)} Q_n^m(\xi_s)' Q_n^m(\xi_1)\right]$$

$$= -\frac{1}{4\pi c(\xi_s^2 - 1)}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m (2n+1)\frac{(n-m)!}{(n+m)!}\cos m(\phi - \phi_1)$$

$$\cdot P_n^m(\eta_1)P_n^m(\eta)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)} \, , \tag{B.3}$$

where, above, we used the Wronskian relation

$$W\left[Q_n^m(\xi), P_n^m(\xi)\right] = P_n^m(\xi)' Q_n^m(\xi) - Q_n^m(\xi)' P_n^m(\xi) = \frac{(-1)^m}{\xi^2 - 1}\,\frac{(n+m)!}{(n-m)!} \, . \tag{B.4}$$

Substituting (B.2) and (B.3) in the integral (B.1), we have

$$I^s(p_1) = -\frac{e^{-ikc\xi_s}}{4\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m(2n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)}$$

$$\cdot\int_{-1}^{+1}d\eta\, e^{-ikc(\xi_s\cos\theta_o\pm 1)\eta}P_n^m(\eta)\int_0^{2\pi}d\phi\, e^{-ikc\sin\theta_o\sqrt{\xi_s^2-1}\sqrt{1-\eta^2}\cos\phi}\cos m(\phi-\phi_1). \tag{B.5}$$

The functions involved in the integrands are continuous in the intervals of integration and the only assumption we made in interchanging integration and summation is the uniformity of convergence of the series (cf. Whittaker and Watson, 1952, p. 78).

We now use the expansion (Magnus and Oberhettinger, 1949, p. 155)

$$e^{ik\rho\cos\phi} = \sum_{m=0}^{\infty} i^m\epsilon_m J_m(k\rho)\cos m\phi\ . \tag{B.6}$$

Utilizing (B.6) in (B.5), with $k\rho = -kc\sin\theta_o\sqrt{\xi_s^2-1}\sqrt{1-\eta^2}$, results in

$$I^s(p_1) = -\frac{e^{-ikc\xi_s}}{4\pi}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m(-i)^{\ell}\epsilon_{\ell}(2n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)}$$

$$\cdot\int_{-1}^{+1}d\eta\, e^{-ikc(\xi_s\cos\theta_o\pm 1)\eta}P_n^m(\eta)J_{\ell}\left(kc\sin\theta_o\sqrt{\xi_s^2-1}\sqrt{1-\eta^2}\right)\int_0^{2\pi}d\phi\cos m(\phi-\phi_1)\cos\ell\phi\ . \tag{B.7}$$

The integration with respect to ϕ can be simply performed, while to integrate with respect to η we use the relation (Morse and Feshbach, 1953, p. 1325),

$$\int_0^{\pi} d\mu\, e^{iz\cos\nu\cos\mu} P_n^m(\cos\mu) J_m(z\sin\nu\sin\mu)\sin\mu = i^{n-m}\sqrt{\frac{2\pi}{z}}\; P_n^m(\cos\nu) J_{n+1/2}(z) \; . \tag{B.8}$$

In this expression we let

$$\eta = \cos\mu$$

$$z\cos\nu = -kc(\xi_s\cos\theta_o \pm 1)$$

$$z\sin\nu = kc\sin\theta_o\sqrt{\xi_s^2 - 1}$$

so that

$$z^2 = \left[kc(\xi_s \pm \cos\theta_o)\right]^2 \; ; \qquad \cos\nu = -\frac{\xi_s\cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o} \; .$$

We can then write

$$\int_{-1}^{+1} d\eta\; e^{-ikc(\xi_s\cos\theta_o \pm 1)\eta}\; P_n^m(\eta) J_m\left(kc\sin\theta_o\sqrt{(\xi_s^2 - 1)(1-\eta^2)}\right)$$

$$= i^{n-m}\sqrt{\frac{2\pi}{kc(\xi_s \pm \cos\theta_o)}}\; P_n^m\left(-\frac{\xi_s\cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o}\right) J_{n+1/2}\left[kc(\xi_s \pm \cos\theta_o)\right] . \tag{B.9}$$

Performing the ϕ integration in (B.7) and using (B.9), we obtain

$$I^s(p_1) = -\frac{e^{-ikc\xi_s}}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m(-i)^n(2n+1)\frac{(n-m)!}{(n+m)!}\; P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)}\cos m\phi_1$$

$$\cdot\, P_n^m\left(\frac{\xi_s\cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o}\right) J_{n+1/2}\left[kc(\xi_s \pm \cos\theta_o)\right]\sqrt{\frac{2\pi}{kc(\xi_s \pm \cos\theta_o)}} \; , \tag{B.10}$$

where, above, we used the relation (Magnus and Oberhettinger, 1949, p. 63)

$$P_n^m(-x) = (-1)^{n+m} P_n^m(x) , \quad -1 \leqslant x \leqslant 1 . \tag{B.11}$$

We now expand the Bessel function in (B.10) according to (Magnus and Oberhettinger, 1949, p. 16),

$$J_m(z) = (z/2)^m \sum_{\ell=0}^{\infty} \frac{(iz/2)^{2\ell}}{\ell!\,\Gamma(m+\ell+1)} , \qquad |\arg z| < \pi \tag{B.12}$$

to get

$$I^s(p_1) = -e^{-ikc\xi_s} \frac{\sqrt{\pi}}{2} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \epsilon_m (-1)^n (2n+1) \frac{(n-m)!}{(n+m)!} \left[-\frac{ikc(\xi_s \pm \cos\theta_o)}{2} \right]^{n+2\ell}$$

$$\cdot P_n^m\left(\frac{\xi_s \cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o} \right) \frac{1}{\ell!\,\Gamma(n+\ell+\frac{3}{2})} P_n^m(\eta_1) \frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)} \cos m\phi_1$$

$$= -e^{-ikc\xi_s} \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \sum_{\ell=0}^{[n/2]} \sum_{m=0}^{n-2\ell} \epsilon_m \left[2(n-2\ell)+1 \right] \frac{(n-2\ell-m)!}{(n-2\ell+m)!}$$

$$\cdot \left[-\frac{ikc(\xi_s \pm \cos\theta_o)}{2} \right]^n P_{n-2\ell}^m \left(\frac{\xi_s \cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o} \right) \frac{1}{\ell!\,\Gamma(n-\ell+\frac{3}{2})}$$

$$\cdot P_{n-2\ell}^m(\eta_1) \frac{Q_{n-2\ell}^m(\xi_1)}{Q_{n-2\ell}^m(\xi_s)} \cos m\phi_1 .$$

This last expression may be written as follows:

$$I^s(p_1) = -e^{-ikc\xi_s} \frac{\sqrt{\pi}}{2} \sum_{M=0}^{\infty} (-ikc)^M (\xi_s \pm \cos\theta_o)^M \sum_{\ell=0}^{M} \sum_{m=0}^{\ell} \epsilon_m (2\ell+1) \frac{(\ell-m)!}{(\ell+m)!}$$

$$\cdot \frac{P_\ell^m\left(\frac{\xi_s \cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o}\right)}{2^M \left(\frac{M-\ell}{2}\right)! \, \Gamma\left(\frac{M+\ell}{2} + \frac{3}{2}\right)} P_\ell^m(\eta_1) \frac{Q_\ell^m(\xi_1)}{Q_\ell^m(\xi_s)} \cos m\phi_1 \quad , \tag{B.13}$$

where in the above series in k the only nonzero contributions to the coefficients are made by terms for which $M+\ell$ is an even integer.

We have then written the surface integral (B.1) as a power series in kc of the form

$$I^s(p_1) = e^{-ikc\xi_s} \sum_{M=0}^{\infty} (-ikc)^M I_M^s(p_1) \tag{B.14}$$

where $I_M^s(p_1)$ is given by

$$I_M^s(p_1) = \sum_{\ell=0}^{M} \sum_{m=0}^{\ell} A_\ell^{M,m}(\xi_s) P_\ell^m(\eta_1) Q_\ell^m(\xi_1) \cos m\phi_1 \quad , \tag{B.15}$$

with

$$A_\ell^{M,m}(\xi_s) = \begin{cases} -\epsilon_m \sqrt{\pi} \, \dfrac{(\xi_s \pm \cos\theta_o)^M}{2^{M+1}} (2\ell+1) \dfrac{(\ell-m)!}{(\ell+m)!} \dfrac{P_\ell^m\left(\frac{\xi_s\cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o}\right)}{\left(\frac{M-\ell}{2}\right)! \, \Gamma\left(\frac{M+\ell}{2}+\frac{3}{2}\right) Q_\ell^m(\xi_s)} \, , & M+\ell \text{ even} \\ 0 \quad , & M+\ell \text{ odd} \quad . \end{cases} \tag{B.16}$$

APPENDIX C
EVALUATION OF THE INTEGRAL $\int_{\xi_s}^{\infty} C_t^{\ell}(\xi,\xi_1,\xi_s)Q_r^{\ell}(\xi)d\xi$.

$$\int_{\xi_s}^{\infty} d\xi\, C_t^{\ell}(\xi,\xi_1,\xi_s)Q_r^{\ell}(\xi) = \int_{\xi_s}^{\xi_1} d\xi\, Q_t^{\ell}(\xi_1)P_t^{\ell}(\xi)Q_r^{\ell}(\xi) + \int_{\xi_1}^{\infty} d\xi\, P_t^{\ell}(\xi_1)Q_t^{\ell}(\xi)Q_r^{\ell}(\xi)$$

$$- \frac{P_t^{\ell}(\xi_s)}{Q_t^{\ell}(\xi_s)} \int_{\xi_s}^{\infty} d\xi\, Q_t^{\ell}(\xi_1)Q_t^{\ell}(\xi)Q_r^{\ell}(\xi) \; . \qquad (C.1)$$

From Legendre's associated equation we have that

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dQ_r^{\ell}(\xi)}{d\xi}\right] + \left[r(r+1) - \frac{\ell^2}{1-\xi^2}\right] Q_r^{\ell}(\xi) = 0$$

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dQ_t^{\ell}(\xi)}{d\xi}\right] + \left[t(t+1) - \frac{\ell^2}{1-\xi^2}\right] Q_t^{\ell}(\xi) = 0$$

Multiplying the first of these equations by $Q_t^{\ell}(\xi)$ and the second by $Q_r^{\ell}(\xi)$ and subtracting the second from the first we obtain the following:

$$Q_t^{\ell}(\xi)\frac{d}{d\xi}\left[(1-\xi^2)\frac{dQ_r^{\ell}(\xi)}{d\xi}\right] - Q_r^{\ell}(\xi)\frac{d}{d\xi}\left[(1-\xi^2)\frac{dQ_t^{\ell}(\xi)}{d\xi}\right]$$

$$+ \left[r(r+1) - t(t+1)\right] Q_r^{\ell}(\xi)Q_t^{\ell}(\xi) = 0$$

Integrating this expression we have that

$$\int_{\xi_s,\,\xi_1}^{\xi_1,\,\infty} d\xi\, Q_r^{\ell}(\xi) Q_t^{\ell}(\xi) = \frac{-1}{r(r+1)-t(t+1)} \int_{\xi_s,\,\xi_1}^{\xi_1,\,\infty} d\xi \left\{ Q_t^{\ell}(\xi) \frac{d}{d\xi}\left[(1-\xi^2)\frac{dQ_r^{\ell}(\xi)}{d\xi}\right] \right.$$

$$\left. -Q_r^{\ell}(\xi)\frac{d}{d\xi}\left[(1-\xi^2)\frac{dQ_t^{\ell}(\xi)}{d\xi}\right]\right\}$$

$$= \frac{-1}{r(r+1)-t(t+1)} \left\{ Q_t^{\ell}(\xi)(1-\xi^2)\frac{dQ_r^{\ell}(\xi)}{d\xi}\Bigg|_{\xi_s,\,\xi_1}^{\xi_1,\,\infty} -Q_r^{\ell}(\xi)(1-\xi^2)\frac{dQ_t^{\ell}(\xi)}{d\xi}\Bigg|_{\xi_s,\,\xi_1}^{\xi_1,\,\infty}\right\}$$

$$= \frac{1}{r(r+1)-t(t+1)}\left[\frac{Q_t^{\ell}(\xi)}{2r+1}\left[(r-\ell+1)Q_{r+1}^{\ell}(\xi)-(r+1)(r+\ell)Q_{r-1}^{\ell}(\xi)\right]\right.$$

$$\left. -\frac{Q_r^{\ell}(\xi)}{2t+1}\left[(t-\ell+1)Q_{t+1}^{\ell}(\xi)-(t+1)(t+\ell)Q_{t-1}^{\ell}(\xi)\right]\right]_{\xi_s,\,\xi_1}^{\xi_1,\,\infty} \; ; \quad r \neq t \neq 0 . \qquad \text{(C.2)}$$

Similarly,

$$\int_{\xi_s}^{\xi_1} d\xi\, P_t^{\ell}(\xi) Q_r^{\ell}(\xi) = \frac{(\xi^2-1)}{r(r+1)-t(t+1)}\left[P_t^{\ell}(\xi)\frac{dQ_r^{\ell}(\xi)}{d\xi} - Q_r^{\ell}(\xi)\frac{dP_t^{\ell}(\xi)}{d\xi}\right]_{\xi_s}^{\xi_1}$$

$$= \frac{1}{r(r+1)-t(t+1)}\left[\frac{P_t^{\ell}(\xi)}{2r+1}\left[(r-\ell+1)Q_{r+1}^{\ell}(\xi)-(r+1)(r+\ell)Q_{r-1}^{\ell}(\xi)\right]\right.$$

$$\left. -\frac{Q_r^{\ell}(\xi)}{2t+1}\left[(t-\ell+1)P_{t+1}^{\ell}(\xi)-(t+1)(t+\ell)P_{t-1}^{\ell}(\xi)\right]\right]_{\xi_s}^{\xi_1} \; ; \quad r \neq t \neq 0 . \qquad \text{(C.3)}$$

Substituting (C.2) and (C.3) in (C.1), we obtain

$$\int_{\xi_s}^{\infty} d\xi\, C_t^{\ell}(\xi, \xi_1, \xi_s) Q_r^{\ell}(\xi) = \frac{Q_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)} \left\{ (\xi_1^2 - 1) \left[P_t^{\ell}(\xi_1) \frac{dQ_r^{\ell}(\xi_1)}{d\xi_1} \right. \right.$$

$$\left. \left. - Q_r^{\ell}(\xi_1) \frac{dP_t^{\ell}(\xi_1)}{d\xi_1} \right] - (\xi_s^2 - 1) \left[P_t^{\ell}(\xi_s) \frac{dQ_r^{\ell}(\xi_s)}{d\xi_s} - Q_r^{\ell}(\xi_s) \frac{dP_t^{\ell}(\xi_s)}{d\xi_s} \right] \right\}$$

$$+ \frac{P_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)} \left\{ -(\xi_1^2 - 1) \left[Q_t^{\ell}(\xi_1) \frac{dQ_r^{\ell}(\xi_1)}{d\xi_1} - Q_r^{\ell}(\xi_1) \frac{dQ_t^{\ell}(\xi_1)}{d\xi_1} \right] \right\}$$

$$- \frac{Q_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)} \frac{P_t^{\ell}(\xi_s)}{Q_t^{\ell}(\xi_s)} \left\{ -(\xi_s^2 - 1) \left[Q_t^{\ell}(\xi_s) \frac{dQ_r^{\ell}(\xi_s)}{d\xi_s} - Q_r^{\ell}(\xi_s) \frac{dQ_t^{\ell}(\xi_s)}{d\xi_s} \right] \right\}$$

$$= \frac{Q_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)} \left\{ -(\xi_1^2 - 1) Q_r^{\ell}(\xi_1) \frac{dP_t^{\ell}(\xi_1)}{d\xi_1} + (\xi_s^2 - 1) Q_r^{\ell}(\xi_s) \frac{dP_t^{\ell}(\xi_s)}{d\xi_s} \right\}$$

$$+ \frac{P_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)} \left\{ (\xi_1^2 - 1) Q_r^{\ell}(\xi_1) \frac{dQ_t^{\ell}(\xi_1)}{d\xi_1} \right\} - \frac{Q_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)}$$

$$\cdot \left\{ (\xi_s^2 - 1) \frac{P_t^{\ell}(\xi_s)}{Q_t^{\ell}(\xi_s)} Q_r^{\ell}(\xi_s) \frac{dQ_t^{\ell}(\xi_s)}{d\xi_s} \right\}$$

$$= \frac{-(\xi_1^2 - 1) Q_r^{\ell}(\xi_1)}{r(r+1)-t(t+1)} \left\{ Q_t^{\ell}(\xi_1) \frac{dP_t^{\ell}(\xi_1)}{d\xi_1} - P_t^{\ell}(\xi_1) \frac{dQ_t^{\ell}(\xi_1)}{d\xi_1} \right\}$$

$$+ \frac{(\xi_s^2 - 1) Q_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)} \cdot \frac{Q_r^{\ell}(\xi_s)}{Q_t^{\ell}(\xi_s)} \left\{ Q_t^{\ell}(\xi_s) \frac{dP_t^{\ell}(\xi_s)}{d\xi_s} - P_t^{\ell}(\xi_s) \frac{dQ_t^{\ell}(\xi_s)}{d\xi_s} \right\}$$

$$= - \frac{Q_r^{\ell}(\xi_1)}{r(r+1)-t(t+1)} \cdot \frac{(t+\ell)!}{(t-\ell)!} (-1)^{\ell} + \frac{Q_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)} \cdot \frac{Q_r^{\ell}(\xi_s)}{Q_t^{\ell}(\xi_s)} \cdot \frac{(t+\ell)!}{(t-\ell)!} (-1)^{\ell}$$

(cont'd)

$$= \frac{(-1)^{\ell}}{r(r+1)-t(t+1)} \cdot \frac{(t+\ell)!}{(t-\ell)!} \left\{ \frac{Q_r^{\ell}(\xi_s)}{Q_t^{\ell}(\xi_s)} Q_t^{\ell}(\xi_1) - Q_r^{\ell}(\xi_1) \right\} \; ; \quad \begin{matrix} r \neq t \\ \ell \leqslant t \end{matrix} \quad . \tag{C.4}$$

Also,

$$\int_{\xi_s}^{\infty} C_t^{\ell}(\xi, \xi_1, \xi_s) Q_r^{\ell}(\xi) d\xi = 0 \qquad \text{for} \qquad \ell > t \tag{C.5}$$

since $C_t^{\ell} = 0$ for $\ell > t$.

APPENDIX D
DERIVATION OF RELATION (3.21)

In this appendix we give a proof of the statement of equation (3.21), that is

$$D^{M,\ell}_{r,t}(\xi_s) = (\pm 1)^{r+t} D^{M,\ell}_{t,r}(\xi_s)\ , \quad \ell, r, t \leqslant M\ . \tag{D.1}$$

The choice of sign is determined by the sign chosen in (2.18).

The proof follows an inductive argument. First we show that (D.1) is true for $M = 0$ and then that if it is true for any M it is true for $M+1$.

Since $r = t = 0$ when $M = 0$, equation (D.1) is certainly true for $M = 0$. Assume next that it is true for M. We can then integrate (3.19) and, following the same procedure as we did there, end up with the recurrence relations (3.26a, b, c). We are interested mainly in (3.26a) since for $r = t$ equation (D.1) is obviously true. Repeating here (3.26a) and subsequently employing it in (D.1) which is assumed to hold for M, we obtain

$$D^{M+1,\ell}_{r,t}(\xi_s) = \frac{2}{r(r+1)-t(t+1)}\left[\frac{r(r-\ell)}{2r-1} D^{M,\ell}_{r-1,t}(\xi_s) \mp \frac{t(t-\ell)}{2t-1} D^{M,\ell}_{r,t-1}(\xi_s) \pm \frac{(t+1)(t+\ell+1)}{2t+3} D^{M,\ell}_{r,t+1}(\xi_s) - \frac{(r+1)(r+\ell+1)}{2r+3} D^{M,\ell}_{r+1,t}(\xi_s)\right]$$

$$= \frac{2}{r(r+1)-t(t+1)}\left[\frac{r(r-\ell)}{2r-1}(\pm 1)^{r+t-1} D^{M,\ell}_{t,r-1}(\xi_1) \mp \frac{t(t-\ell)}{2t-1}(\pm 1)^{r+t-1} D^{M,\ell}_{t-1,r}(\xi_s) \pm \frac{(t+1)(t+\ell+1)}{2t+3}(\pm 1)^{r+t+1} D^{M,\ell}_{t+1,r}(\xi_s) - \frac{(r+1)(r+\ell+1)}{2r+3}(\pm 1)^{r+t+1} D^{M,\ell}_{t,r+1}(\xi_s)\right]$$

(cont'd)

$$= (\pm 1)^{r+t} \frac{2}{t(t+1) - r(r+1)} \left[\frac{t(t-\ell)}{2t-1} D^{M,\ell}_{t-1,r}(\xi_s) \mp \frac{r(r-\ell)}{2r-1} D^{M,\ell}_{t,r-1}(\xi_s) \right.$$

$$\left. \pm \frac{(r+1)(r+\ell+1)}{2r+3} D^{M,\ell}_{t,r+1}(\xi_s) - \frac{(t+1)(t+\ell+1)}{2t+3} D^{M,\ell}_{t+1,r}(\xi_s) \right]$$

$$= (\pm 1)^{r+t} D^{M+1,\ell}_{t,r}(\xi_s) \ . \qquad \text{(D.2)}$$

So (D.1) is true for M+1 if it is true for M. Since it is true for M = 0, it is true for all M (M = 0, 1, 2, ...).

APPENDIX E
DERIVATION OF EQUATIONS (3.24a, b)

In order to arrive at equations (3.24a, b), we start with (3.22) and (3.23) which we repeat here

$$I^v_{M+1}(p_1) = -2\sum_{t=0}^{M+1}\sum_{r=0}^{M+1}{}'\sum_{\ell=0}^{t}\frac{1}{r(r+1)-t(t+1)}\left[\frac{r(r-\ell)}{2r-1}D^{M,\ell}_{r-1,t}(\xi_s)\right.$$

$$\mp\frac{t(t-\ell)}{2t-1}D^{M,\ell}_{r,t-1}(\xi_s)\pm\frac{(t+1)(t+\ell+1)}{2t+3}D^{M,\ell}_{r,t+1}(\xi_s)$$

$$\left.-\frac{(r+1)(r+\ell+1)}{2r+3}D^{M,\ell}_{r+1,t}(\xi_s)\right]\left[\frac{Q^\ell_r(\xi_s)}{Q^\ell_t(\xi_s)}Q^\ell_t(\xi_1)-Q^\ell_r(\xi_1)\right]P^\ell_t(\eta_1)\cos\ell\phi_1\ , \qquad (E.1)$$

$$I^v_{M+1}(p_1) = \sum_{t=0}^{M+1}\sum_{r=0}^{M+1}{}'\sum_{\ell=0}^{t}E^{M+1,\ell}_{r,t}(\xi_s)Q^\ell_r(\xi_1)P^\ell_t(\eta_1)\cos\ell\phi_1\ . \qquad (E.2)$$

When $r \neq t$, a comparison of these two equations gives (3.24a). When $r = t$, we rewrite the above equations as follows.

Equation (E.1):

$$I^v_{M+1}(p_1) = 2\sum_{t=0}^{M+1}\sum_{r=0}^{M+1}{}'\sum_{\ell=0}^{t}\frac{1}{r(r+1)-t(t+1)}\left[\frac{r(r-\ell)}{2r-1}D^{M,\ell}_{r-1,t}(\xi_s)\right.$$

$$\mp\frac{t(t-\ell)}{2t-1}D^{M,\ell}_{r,t-1}(\xi_s)\pm\frac{(t+1)(t+\ell+1)}{2t+3}D^{M,\ell}_{r,t+1}(\xi_s)$$

$$\left.-\frac{(r+1)(r+\ell+1)}{2r+3}D^{M,\ell}_{r+1,t}(\xi_s)\right]Q^\ell_r(\xi_1)P^\ell_t(\eta_1)\cos\ell\phi_1$$

$$-2\sum_{t=0}^{M+1}\sum_{r=0}^{M+1}{}'\sum_{\ell=0}^{t}\frac{1}{r(r+1)-t(t+1)}\left[\frac{r(r-\ell)}{2r-1}D^{M,\ell}_{r-1,t}(\xi_s)\right.$$

(cont'd)

$$\mp \frac{t(t-\ell)}{2t-1} D^{M,\ell}_{r,t-1}(\xi_s) \pm \frac{(t+1)(t+\ell+1)}{2t+3} D^{M,\ell}_{r,t+1}(\xi_s)$$

$$\left. - \frac{(r+1)(r+\ell+1)}{2r+3} D^{M,\ell}_{r+1,t}(\xi_s) \right] \frac{Q^{\ell}_r(\xi_s)}{Q^{\ell}_t(\xi_s)} Q^{\ell}_t(\xi_1) P^{\ell}_t(\eta_1) \cos \ell\phi_1 \ . \qquad (E.3)$$

Equation (E.2):

$$I^{v}_{M+1}(p_1) = \sum_{t=0}^{M+1} \sum_{\ell=0}^{t} E^{M+1,\ell}_{t,t}(\xi_s) Q^{\ell}_t(\xi_1) P^{\ell}_t(\eta_1) \cos \ell\phi_1 \ . \qquad (E.4)$$

A comparison of (E.3) and (E.4) gives

$$E^{M+1,\ell}_{t,t}(\xi_s) = \sum_{r=0}^{M+1}{}' \frac{-2}{r(r+1) - t(t+1)} \left[\frac{r(r-\ell)}{2r-1} D^{M,\ell}_{r-1,t}(\xi_s) \mp \frac{t(t-\ell)}{2t-1} D^{M,\ell}_{r,t-1}(\xi_s) \right.$$

$$\left. \pm \frac{(t+1)(t+\ell+1)}{2t+3} D^{M,\ell}_{r,t+1}(\xi_s) - \frac{(r+1)(r+\ell+1)}{2r+3} D^{M,\ell}_{r+1,t}(\xi_s) \right] \frac{Q^{\ell}_r(\xi_s)}{Q^{\ell}_t(\xi_s)} \ . \qquad (E.5)$$

Using (3.24a), the above expression can be written

$$E^{M+1,\ell}_{t,t}(\xi_s) = - \sum_{r=0}^{M+1}{}' \frac{Q^{\ell}_r(\xi_s)}{Q^{\ell}_t(\xi_s)} E^{M+1,\ell}_{r,t}(\xi_s) \ , \qquad (E.6)$$

which is equation (3.24b).

APPENDIX F
THE SURFACE INTEGRAL FOR THE NEUMANN PROBLEM

The integral to be evaluated in this appendix is that of equation (4.6)

$$I^s(p_1) = -c(\xi_s^2 - 1)\, e^{-ikc\xi_s} \int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi\, e^{\mp ikc\eta}\, G_o^N(p_1, p_s) \frac{\partial u^i(p_s)}{\partial \xi_s} \; . \qquad (F.1)$$

By equation (3.3)

$$\frac{\partial u^i(p_s)}{\partial \xi_s} = -ikc\left[\cos\theta_o \eta + \sin\theta_o \frac{\xi_s\sqrt{1-\eta^2}}{\sqrt{\xi_s^2 - 1}} \cos\phi\right] u^i(p_s)$$

$$= -ikc\left[\cos\theta_o \eta + \sin\theta_o \frac{\xi_s\sqrt{1-\eta^2}}{\sqrt{\xi_s^2 - 1}} \cos\phi\right] e^{-ikc\left[\cos\theta_o \xi_s \eta + \sin\theta_o \sqrt{\xi_s^2 - 1}\sqrt{1-\eta^2}\cos\phi\right]} .$$

Substituting the above expression together with the appropriate part ($\xi < \xi_1$) of (2.22) in (F.1) we have

$$I^s(p_1) = -\frac{ikc}{4\pi}(\xi_s^2 - 1)\, e^{-ikc\xi_s} \sum_{n=0}^{\infty}\sum_{m=0}^{n} (-1)^m \epsilon_m (2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2 P_n^m(\eta_1) Q_n^m(\xi_1)$$

$$\cdot \left[P_n^m(\xi_s) - \frac{P_n^m(\xi_s)'}{Q_n^m(\xi_s)'} Q_n^m(\xi_s)\right] \int_{-1}^{+1} d\eta\, P_n^m(\eta)\, e^{-ikc\left[\xi_s \cos\theta_o \pm 1\right]\eta}$$

$$\cdot \int_0^{2\pi} d\phi \left[\cos\theta_o \eta + \sin\theta_o \frac{\xi_s\sqrt{1-\eta^2}}{\sqrt{\xi_s^2 - 1}} \cos\phi\right] e^{-ikc\sin\theta_o \sqrt{\xi_s^2 - 1}\sqrt{1-\eta^2}\cos\phi}$$

$$\times \cos m(\phi - \phi_1) \; . \qquad (F.2)$$

Using (B.4) for the Wronskian and the expansion (B.6) with $k\rho = -kc\sin\theta_o\sqrt{\xi_s^2-1}\sqrt{1-\eta^2}$, we can write for equation (F.2)

$$e^{ikc\xi_s} I^s(p_1) = \frac{ikc}{4\pi}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-i)^\ell \epsilon_\ell \epsilon_m (2n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}$$

$$\cdot \cos m\phi_1 \int_{-1}^{+1} d\eta\, P_n^m(\eta) J_\ell\left(kc\sin\theta_o\sqrt{\xi_s^2-1}\sqrt{1-\eta^2}\right) e^{-ick(\xi_s\cos\theta_o \pm 1)\eta}$$

$$\cdot \int_0^{2\pi} d\phi\left(\cos\theta_o\eta + \sin\theta_o \frac{\xi_s\sqrt{1-\eta^2}}{\sqrt{\xi_s^2-1}}\cos\phi\right)\cos\ell\phi\cos m\phi \ . \tag{F.3}$$

To perform the integration with respect to ϕ we employ the identify

$$\cos\phi\cos\ell\phi = \frac{1}{2}\left[\cos(\ell+1)\phi + \cos(\ell-1)\phi\right]$$

the result being

$$e^{ikc\xi_s} I^s(p_1) = \frac{ikc}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m(-i)^m(2n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}\cos m\phi_1$$

$$\cdot\int_{-1}^{+1} d\eta\, e^{-ikc(\xi_s\cos\theta_o \pm 1)\eta}\,\eta\, P_n^m(\eta) J_m\left(kc\sin\theta_o\sqrt{\xi_s^2-1}\sqrt{1-\eta^2}\right)$$

$$+\frac{ikc}{2}\frac{\xi_s\sin\theta_o}{\sqrt{\xi_s^2-1}}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-i)^{m+1}(2n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}\cos m\phi_1$$

$$\cdot\int_{-1}^{+1} d\eta\, e^{-ikc(\xi_s\cos\theta_o \pm 1)\eta}\sqrt{1-\eta^2}\, P_n^m(\eta) J_{m+1}\left(kc\sin\theta_o\sqrt{\xi_s^2-1}\sqrt{1-\eta^2}\right)$$

(cont'd)

$$+\frac{ikc}{2}\frac{\xi_s \sin\theta_o}{\sqrt{\xi_s^2-1}}\sum_{n=1}^{\infty}\sum_{m=1}^{n}(-i)^{m+1}(2n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}\cos m\phi_1$$

$$\cdot\int_{-1}^{+1} d\eta\, e^{-ikc(\xi_s\cos\theta_o \pm 1)\eta}\sqrt{1-\eta^2}\,P_n^m(\eta)J_{m-1}\left(kc\sin\theta_o\sqrt{\xi_s^2-1}\,\sqrt{1-\eta^2}\right). \qquad (F.4)$$

To perform the integration with respect to η we employ the following recurrence relations (Magnus and Oberhettinger, 1949, p. 62)

$$(2n+1)\eta P_n^m(\eta) = (n-m+1)P_{n+1}^m(\eta)+(n+m)P_{n-1}^m(\eta), \quad m \leqslant n, \quad n=0,1,\ldots \qquad (F.5)$$

$$(2n+1)\sqrt{1-\eta^2}\,P_n^m(\eta) = P_{n-1}^{m+1}(\eta)-P_{n+1}^{m+1}(\eta), \quad m\leqslant n, \quad n=0,1,2,\ldots \qquad (F.6)$$

$$(2n+1)\sqrt{1-\eta^2}\,P_n^m(\eta) = (n-m+1)(n-m+2)P_{n+1}^{m-1}(\eta)-(n-m-1)(n+m)P_{n-1}^{m-1}(\eta),$$

$$m\leqslant n, \quad n=0,1,\ldots \qquad (F.7)$$

Substitution of these expressions in (F.4) and a simple rearrangement of the terms leads to

$$e^{ikc\xi_s}\,I^s(p_1) = \frac{ikc}{2}\cos\theta_o\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m(-i)^m\left\{\frac{(n-m)!}{(n+m)!}P_{n-1}^m(\eta_1)\frac{Q_{n-1}^m(\xi_1)}{Q_{n-1}^m(\xi_s)'}\right.$$

$$\left.+\frac{(n-m+1)!}{(n+m)!}P_{n+1}^m(\eta_1)\frac{Q_{n+1}^m(\xi_1)}{Q_{n+1}^m(\xi_s)'}\right\}\cos m\phi_1$$

$$\cdot\int_{-1}^{+1} d\eta\, e^{-ikc(\xi_s\cos\theta_o \pm 1)\eta}\,P_n^m(\eta)J_m\left(kc\sin\theta_o\sqrt{\xi_s^2-1}\,\sqrt{1-\eta^2}\right)$$

(cont'd)

$$+\frac{ikc}{2}\,\frac{\xi_s \sin\theta_o}{\sqrt{\xi_s^2-1}} \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (-i)^{m+1} \left\{ \frac{(n-m+1)!}{(n+m+1)!} P_{n+1}^m(\eta_1) \frac{Q_{n+1}^m(\xi_1)}{Q_{n+1}^m(\xi_s)'} \right.$$

$$\left. -\frac{(n-m-1)!}{(n+m-1)!} P_{n-1}^m(\eta_1) \frac{Q_{n-1}^m(\xi_1)}{Q_{n-1}^m(\xi_s)'} \right\} \cos m\phi_1 \int_{-1}^{+1} d\eta\, e^{-ikc(\xi_s \cos\theta_o \pm 1)\eta} P_n^{m+1}(\eta)$$

$$\cdot J_{m+1}\left(kc \sin\theta_o \sqrt{\xi_s^2-1}\,\sqrt{1-\eta^2}\right)$$

$$+\frac{ikc}{2}\,\frac{\xi_s \sin\theta_o}{\sqrt{\xi_s^2-1}} \sum_{n=0}^{\infty} \sum_{m=1}^{n+1} (-i)^{m-1} \left\{ \frac{(n-m+1)!}{(n+m-1)!} P_{n-1}^m(\eta_1) \frac{Q_{n-1}^m(\xi_1)}{Q_{n-1}^m(\xi_s)'} \right.$$

$$\left. -\frac{(n-m+1)!}{(n+m-1)!} P_{n+1}^m(\eta_1) \frac{Q_{n+1}^m(\xi_1)}{Q_{n+1}^m(\xi_s)'} \right\} \cos m\phi_1 \int_{-1}^{+1} d\eta\, e^{-ikc(\xi_s \cos\theta_o \pm 1)\eta}$$

$$\cdot P_n^{m-1}(\eta) J_{m-1}\left(kc \sin\theta_o \sqrt{\xi_s^2-1}\,\sqrt{1-\eta^2}\right) .$$

To perform the integration with respect to η we employ (B.9) in Appendix B, the result being

$$e^{ikc\xi_s} I^S(p_1) = \frac{ikc}{2} \cos\theta_o \sum_{n=0}^{\infty} \sum_{m=0}^{n} \epsilon_m (-i)^n \sqrt{\frac{2\pi}{z}}\; J_{n+1/2}(z) P_n^m(\beta)$$

$$\cdot \left\{ \frac{(n-m)!}{(n+m-1)!} P_{n-1}^m(\eta_1) \frac{Q_{n-1}^m(\xi_1)}{Q_{n-1}^m(\xi_s)'} + \frac{(n-m+1)!}{(n+m)!} P_{n+1}^m(\eta_1) \frac{Q_{n+1}^m(\xi_1)}{Q_{n+1}^m(\xi_s)'} \right\} \cos m\phi_1$$

$$+\frac{ikc}{2} \sin\theta_o \frac{\xi_s}{\sqrt{\xi_s^2-1}} \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-i)^n \sqrt{\frac{2\pi}{z}}\; J_{n+1/2}(z) P_n^{m+1}(\beta)$$

(cont'd)

$$\cdot \left\{ \frac{(n-m+1)!}{(n+m+1)!} P_{n+1}^{m}(\eta_1) \frac{Q_{n+1}^{m}(\xi_1)}{Q_{n+1}^{m}(\xi_s)'} - \frac{(n-m-1)!}{(n+m-1)!} P_{n-1}^{m}(\eta_1) \frac{Q_{n-1}^{m}(\xi_1)}{Q_{n-1}^{m}(\xi_s)'} \right\} \cos m\phi_1$$

$$+ \frac{ikc}{2} \sin\theta_o \frac{\xi_s}{\sqrt{\xi_s^2 - 1}} \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} (-i)^n \sqrt{\frac{2\pi}{z}}\, J_{n+1/2}(z) P_n^{m-1}(\beta)$$

$$\cdot \left\{ \frac{(n-m+1)!}{(n+m-1)!} P_{n-1}^{m}(\eta_1) \frac{Q_{n-1}^{m}(\xi_1)}{Q_{n-1}^{m}(\xi_s)'} - \frac{(n-m+1)!}{(n+m-1)!} P_{n+1}^{m}(\eta_1) \frac{Q_{n+1}^{m}(\xi_1)}{Q_{n+1}^{m}(\xi_s)'} \right\} \cos m\phi_1 , \tag{F.8}$$

where

$$z = kc(\xi_s \pm \cos\theta_o) \tag{F.9}$$

$$\beta = \frac{\xi_s \cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o} \tag{F.10}$$

Equation (F.8) is now put in the following form

$$e^{ikc\xi_s} I^s(p_1) = \frac{ikc}{2} \cos\theta_o \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \epsilon_m (-i)^{n+1} \sqrt{\frac{2\pi}{z}}\, J_{n+3/2}(z) P_{n+1}^{m}(\beta) \frac{(n-m+1)!}{(n+m)!} \right.$$

$$\cdot P_n^m(\eta_1) \frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'} \cos m\phi_1$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \epsilon_m (-i)^{n-1} \sqrt{\frac{2\pi}{z}}\, J_{n-1/2}(z) P_{n-1}^{m}(\beta) \frac{(n-m)!}{(n+m-1)!} P_n^m(\eta_1) \frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}$$

$$\left. \times \cos m\phi_1 \right\}$$

(cont'd)

$$+\frac{ikc}{2}\sin\theta_o \frac{\xi_s}{\sqrt{\xi_s^2-1}} \left\{ \left[\sum_{n=2}^{\infty} \sum_{m=0}^{n-2} (-i)^{n-1} \sqrt{\frac{2\pi}{z}}\, J_{n-1/2}(z) P_{n-1}^{m+1}(\beta) \frac{(n-m)!}{(n+m)!} \right. \right.$$

$$\cdot P_n^m(\eta_1) \frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'} \cos m\phi_1$$

$$\left. - \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-i)^{n+1} \sqrt{\frac{2\pi}{z}}\, J_{n+3/2}(z) P_{n+1}^{m+1}(\beta) \frac{(n-m)!}{(n+m)!} P_n^m(\eta_1) \frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'} \cos m\phi_1 \right]$$

$$+\frac{ikc}{2}\sin\theta_o \frac{\xi_s}{\sqrt{\xi_s^2-1}} \left[\sum_{n=1}^{\infty} \sum_{m=1}^{n+2} (-i)^{n+1} \sqrt{\frac{2\pi}{z}}\, J_{n+3/2}(z) P_{n+1}^{m-1}(\beta) \frac{(n-m+2)!}{(n+m)!} \right.$$

$$\cdot P_n^m(\eta_1) \frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'} \cos m\phi_1$$

$$\left. \left. - \sum_{n=1}^{\infty} \sum_{m=1}^{n} (-i)^{n-1} \sqrt{\frac{2\pi}{z}}\, J_{n-1/2}(z) P_{n-1}^{m-1}(\beta) \frac{(n-m)!}{(n+m-2)!} P_n^m(\eta_1) \frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'} \cos m\phi_1 \right] \right\} . \tag{F.11}$$

Substitution of the relations (Magnus and Oberhettinger, 1949, p. 16)

$$J_{n-1/2}(z) = \frac{2n+1}{2z} J_{n+1/2}(z) + \frac{d}{dz} J_{n+1/2}(z) \tag{F.12}$$

$$J_{n+3/2}(z) = \frac{2n+1}{2z} J_{n+1/2}(z) - \frac{d}{dz} J_{n+1/2}(z) \tag{F.13}$$

in (F.11) and a regrouping of the terms leads to

$$e^{ikc\xi_s} I^s(p_1) = -\frac{kc}{2}\cos\theta_o\sqrt{\frac{2\pi}{z}}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m(-i)^n\frac{2n+1}{2z}\frac{(n-m)!}{(n+m)!}J_{n+1/2}(z)\right.$$

$$\cdot\left[(n+m)P_{n-1}^m(\beta)-(n-m+1)P_{n+1}^m(\beta)\right]P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}\cos m\phi_1$$

$$\left.+\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m(-i)^n(2n+1)\beta P_n^m(\beta)J_{n+1/2}(z)'\frac{(n-m)!}{(n+m)!}P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}\cos m\phi_1\right\}$$

$$-\frac{kc}{2}\sin\theta_o\frac{\xi_s}{\sqrt{\xi_s^2-1}}\sqrt{\frac{2\pi}{z}}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{n}(-i)^n\frac{(2n+1)}{2z}J_{n+1/2}(z)\frac{(n-m)!}{(n+m)!}\right.$$

$$\cdot\left[P_{n-1}^{m+1}(\beta)+P_{n+1}^{m+1}(\beta)\right]P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}\cos m\phi_1$$

$$+2\sum_{n=1}^{\infty}\sum_{m=1}^{n}(-i)^n(2n+1)\frac{(n-m)!}{(n+m)!}J_{n+1/2}(z)'\sqrt{1-\beta^2}\,P_n^m(\beta)P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}$$

$$\cdot\cos m\phi_1-\sum_{n=1}^{\infty}\sum_{m=1}^{n}(-i)^n\frac{(2n+1)}{2z}J_{n+1/2}(z)\left[\frac{(n-m+2)!}{(n+m)!}P_{n+1}^{m-1}(\beta)\right.$$

$$\left.\left.+\frac{(n-m)!}{(n+m-2)!}P_{n-1}^{m-1}(\beta)\right]P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}\cos m\phi_1\right\}. \qquad (F.14)$$

This expression can be simplified using the properties of the Legendre functions mentioned above. After simplifying and collecting terms in $J_{n+1/2}(z)$ and its derivative, we have

$$e^{ikc\xi_s} I^s(p_1) = -\frac{kc}{2}\sqrt{\frac{2\pi}{z}}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m(-i)^n(2n+1)\frac{(n-m)!}{(n+m)!}J_{n+1/2}(z)'\right.$$

$$\cdot\left[\beta\cos\theta_o + \frac{\xi_s\sqrt{1-\beta^2}}{\sqrt{\xi_s^2-1}}\sin\theta_o\right] P_n^m(\beta)P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}\cos m\phi_1\Bigg\}$$

$$-\frac{kc}{2}\sqrt{\frac{2\pi}{z}}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m(-i)^n\ \frac{2n+1}{2z}\ \frac{(n-m)!}{(n+m)!}J_{n+1/2}(z)\left[\cos\theta_o(n+m)P_{n-1}^m(\beta)\right.\right.$$

$$-\cos\theta_o(n-m+1)P_{n+1}^m(\beta) - \sin\theta_o\frac{\xi_s}{\sqrt{\xi_s^2-1}}\sqrt{1-\beta^2}\,P_n^m(\beta)$$

$$\left.+\sin\theta_o\frac{\xi_s}{\sqrt{\xi_s^2-1}}\beta P_n^{m+1}(\beta) - (n+m)(n-m+1)\sin\theta_o\frac{\xi_s}{\sqrt{\xi_s^2-1}}\beta P_n^{m-1}(\beta)\right]$$

$$\left.\cdot P_n^m(\eta_1)\frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'}\cos m\phi_1\right\}. \tag{F.15}$$

But by (F.10)

$$\beta\cos\theta_o + \frac{\xi_s\sqrt{1-\beta^2}}{\sqrt{\xi_s^2-1}}\sin\theta_o = 1\ . \tag{F.16}$$

Moreover, from the definition of β and the recurrence relations (Magnus and Oberhettinger, 1949, p. 62)

$$P_n^{m+1}(\beta) = \frac{1}{\sqrt{1-\beta^2}}\left[(n-m)\beta P_n^m(\beta) - (n+m)P_{n-1}^m(\beta)\right], \tag{F.17}$$

$$P_n^{m-1}(\beta) = \frac{1}{(n+m)\sqrt{1-\beta^2}} \left[\beta P_n^m(\beta) - P_{n+1}^m(\beta)\right], \tag{F.18}$$

we have that

$$\cos\theta_o (n+m) P_{n-1}^m(\beta) - \cos\theta_o (n-m+1) P_{n+1}^m(\beta) - \sin\theta_o \frac{\xi_s\sqrt{1-\beta^2}}{\sqrt{\xi_s^2-1}} P_n^m(\beta)$$

$$+ \sin\theta_o \frac{\xi_s}{\sqrt{\xi_s^2-1}} \beta P_n^{m+1}(\beta) - (n+m)(n-m+1)\sin\theta_o \frac{\xi_s}{\sqrt{\xi_s^2-1}} \beta P_n^{m-1}(\beta)$$

$$= \frac{\pm(\xi_s \pm \cos\theta_o)}{\xi_s^2 - 1} \left[(n-m+1)P_{n+1}^m(\beta) \mp \xi_s P_n^m(\beta) - (n+m)P_{n-1}^m(\beta)\right] \tag{F.19}$$

Substituting (F.16) and (F.19) in (F.15) we have

$$e^{ikc\xi_s} I^s(p_1) = -\frac{kc}{2}\sqrt{\frac{2\pi}{z}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \epsilon_m (-i)^n (2n+1) \frac{(n-m)!}{(n+m)!} J_{n+1/2}(z)' P_n^m(\beta)$$

$$\cdot P_n^m(\eta_1) \frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'} \cos m\phi_1$$

$$-\frac{kc}{2}\sqrt{\frac{2\pi}{z}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \epsilon_m (-i)^n \frac{2n+1}{2z} \frac{(n-m)!}{(n+m)!} J_{n+1/2}(z) \frac{(\pm 1)(\xi_s \pm \cos\theta_o)}{\xi_s^2 - 1}$$

$$\cdot \left[(n-m+1)P_{n+1}^m(\beta) \mp \xi_s P_n^m(\beta) - (n+m)P_{n-1}^m(\beta)\right] P_n^m(\eta_1) \frac{Q_n^m(\xi_1)}{Q_n^m(\xi_s)'} \cos m\phi_1 . \tag{F.20}$$

According to equation (B.12),

$$\sqrt{\frac{2\pi}{z}}\, J_{n+\frac{1}{2}}(z) = \sqrt{\pi} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} z^{2\ell+n}}{2^{2\ell+n}\,\ell!\,\Gamma(n+\ell+\frac{3}{2})}$$

$$\sqrt{\frac{2\pi}{z}}\, J_{n+\frac{1}{2}}(z)' = \frac{\sqrt{\pi}}{z} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}(2\ell+n+\frac{1}{2}) z^{2\ell+n}}{2^{2\ell+n}\,\ell!\,\Gamma(n+\ell+\frac{3}{2})} .$$

Substituting these expressions in (F.20) and using the formula

$$\left(\sum_{n=0}^{\infty} a_n z^{2n}\right)\left(\sum_{n=0}^{\infty} b_n z^{n}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} a_k b_{n-2k} z^n$$

we obtain

$$e^{ikc\xi_s} I^s(p_1) = -\sqrt{\pi}\,\frac{kc}{2z} \sum_{n=0}^{\infty} \sum_{\ell=0}^{[n/2]} \sum_{m=0}^{n-2\ell} \epsilon_m (-i)^n \frac{z^n(2n+1)}{2^{n+1}\,\ell!\,\Gamma(n-\ell+\frac{3}{2})} \left[2(n-2\ell)+1\right]$$

$$\cdot \frac{(n-2\ell-m)!}{(n-2\ell+m)!} P^m_{n-2\ell}(\beta) P^m_{n-2\ell}(\eta_1) \frac{Q^m_{n-2\ell}(\xi_1)}{Q^m_{n-2\ell}(\xi_s)'} \cos m\phi_1$$

$$-\sqrt{\pi}\,\frac{kc}{2z} \sum_{n=0}^{\infty} \sum_{\ell=0}^{[n/2]} \sum_{m=0}^{n-2\ell} \epsilon_m (-i)^n \frac{z^n}{2^{n+1}\,\ell!\,\Gamma(n-\ell+\frac{3}{2})} \left[2(n-2\ell)+1\right] \frac{(n-2\ell-m)!}{(n-2\ell+m)!}$$

$$\cdot \frac{(\pm 1)(\xi_s \pm \cos\theta_o)}{\xi_s^2 - 1} \Big[(n-2\ell-m+1) P^m_{n-2\ell+1}(\beta)$$

$$\mp \xi_s P^m_{n-2\ell}(\beta) - (n-2\ell+m) P^m_{n-2\ell-1}(\beta) \Big] P^m_{n-2\ell}(\eta_1) \frac{Q^m_{n-2\ell}(\xi_1)}{Q^m_{n-2\ell}(\xi_s)'} \cos m\phi_1 .$$

A simple inspection of this expression reveals that it is zero for $n=0$. We can therefore write

$$I^s(p_1) = e^{-ikc\xi_s} \sum_{M=1}^{\infty} (-ikc)^M I_M^s(p_1) \tag{F.21}$$

where

$$I_M^s(p_1) = \frac{-\sqrt{\pi}}{2^{M+2}} (\xi_s \pm \cos\theta_o)^{M-1} \sum_{\ell=0}^{M} \sum_{m=0}^{\ell} \epsilon_m \frac{2\ell+1}{\left(\frac{M-\ell}{2}\right)!\, \Gamma\left(\frac{M+\ell}{2}+\frac{3}{2}\right)} \frac{(\ell-m)!}{(\ell+m)!}$$

$$\cdot \left\{(2M+1)P_\ell^m(\beta) + \frac{(\pm)(\xi_s \pm \cos\theta_o)}{\xi_s^2 - 1}\left[(\ell-m+1)P_{\ell+1}^m(\beta) \mp \xi_s P_\ell^m(\beta)\right.\right.$$

$$\left.\left. -(\ell+m)P_{\ell-1}^m(\beta)\right]\right\} P_\ell^m(\eta_1) \frac{Q_\ell^m(\xi_1)}{Q_\ell^m(\xi_s)'} \cos m\phi_1 \,, \qquad M+\ell \text{ even}\,, \tag{F.22a}$$

$$I_M^s(p_1) = 0\,, \qquad M+\ell \text{ odd}\,, \tag{F.22b}$$

where above we have substituted (F.9) for z and we have rearranged the series. Equation (F.22a) can be further simplified by taking into consideration (F.10) for β and the relation (F.5). In this way we can write

$$I_M^s(p_1) = \sum_{\ell=0}^{M} \sum_{m=0}^{\ell} A_\ell^{M,m}(\xi_s) P_\ell^m(\eta_1) Q_\ell^m(\xi_1) \cos m\phi_1 \tag{F.23}$$

where

$$A_\ell^{M,m}(\xi_s) = -\epsilon_m\sqrt{\pi}\ \frac{(\xi_s \pm \cos\theta_o)^{M-1}}{2^{M+1}}\ (2\ell+1)\ \frac{(\ell-m)!}{(\ell+m)!}\ \frac{1}{\left(\frac{M-\ell}{2}\right)!\ \Gamma\left(\frac{M+\ell}{2}+\frac{3}{2}\right) Q_\ell^m(\xi_s)'}$$

$$\cdot \left\{ M P_\ell^m\left(\frac{\xi_s\cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o}\right) \pm \frac{1}{\xi_s^2 - 1}\left[\ell(\xi_s\cos\theta_o \pm 1) P_\ell^m\left(\frac{\xi_s\cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o}\right)\right.\right.$$

$$\left.\left. - (\ell+m)(\xi_s \pm \cos\theta_o) P_{\ell-1}^m\left(\frac{\xi_s\cos\theta_o \pm 1}{\xi_s \pm \cos\theta_o}\right)\right]\right\}, \qquad M+\ell \text{ even}, \tag{F.24}$$

$$A_\ell^{M,m}(\xi_s) = 0\ , \qquad M+\ell \text{ odd}\ . \tag{F.25}$$

APPENDIX G
THE INTEGRAL $\int_{\xi_s}^{\infty} d\xi\, K_t^{\ell}(\xi,\xi_1,\xi_s) Q_r^{\ell}(\xi)$

According to the results of Appendix C,

$$\int_{\xi_s}^{\infty} d\xi\, K_t^{\ell}(\xi,\xi_1,\xi_s) Q_r^{\ell}(\xi) = Q_t^{\ell}(\xi_1)\int_{\xi_s}^{\xi_1} d\xi\, P_t^{\ell}(\xi) Q_r^{\ell}(\xi) + P_t^{\ell}(\xi_1)\int_{\xi_1}^{\infty} d\xi\, Q_t^{\ell}(\xi) Q_r^{\ell}(\xi)$$

$$- \frac{P_t^{\ell}(\xi_s)'}{Q_t^{\ell}(\xi_s)'} Q_t^{\ell}(\xi_1)\int_{\xi_s}^{\infty} d\xi\, Q_t^{\ell}(\xi) Q_r^{\ell}(\xi)$$

$$= \frac{Q_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)} \left\{(\xi_1^2-1)\left[P_t^{\ell}(\xi_1)Q_r^{\ell}(\xi_1)' - Q_r^{\ell}(\xi_1)P_t^{\ell}(\xi_1)'\right]\right.$$

$$\left. -(\xi_s^2-1)\left[P_t^{\ell}(\xi_s)Q_r^{\ell}(\xi_s)' - Q_r^{\ell}(\xi_s)P_t^{\ell}(\xi_s)'\right]\right\} + \frac{P_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)}(\xi_1^2-1)$$

$$\cdot\left[Q_r^{\ell}(\xi_1)Q_t^{\ell}(\xi_1)' - Q_t^{\ell}(\xi_1)Q_r^{\ell}(\xi_1)'\right] + \frac{P_t^{\ell}(\xi_s)'}{Q_t^{\ell}(\xi_s)'}\,\frac{Q_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)}(\xi_s^2-1)$$

$$\cdot\left[Q_t^{\ell}(\xi_s)Q_r^{\ell}(\xi_s)' - Q_r^{\ell}(\xi_s)Q_t^{\ell}(\xi_s)'\right]$$

$$= \frac{(\xi_1^2-1)Q_r^{\ell}(\xi_1)}{r(r+1)-t(t+1)}\left[P_t^{\ell}(\xi_1)Q_t^{\ell}(\xi_1)' - Q_t^{\ell}(\xi_1)P_t^{\ell}(\xi_1)'\right]$$

$$+ \frac{(\xi_s^2-1)Q_t^{\ell}(\xi_1)}{r(r+1)-t(t+1)}\left[Q_t^{\ell}(\xi_s)P_t^{\ell}(\xi_s)' - P_t^{\ell}(\xi_s)Q_t^{\ell}(\xi_s)'\right]\frac{Q_r^{\ell}(\xi_s)'}{Q_t^{\ell}(\xi_s)'}$$

(cont'd)

$$= - \frac{(-1)^{\ell}}{r(r+1)-t(t+1)} \frac{(t+\ell)!}{(t-\ell)!} Q_r^{\ell}(\xi_1) + \frac{(-1)^{\ell}}{r(r+1)-t(t+1)} \frac{(t+\ell)!}{(t-\ell)!}$$

$$\cdot \frac{Q_r^{\ell}(\xi_s)'}{Q_t^{\ell}(\xi_s)'} Q_t^{\ell}(\xi_1)$$

$$= (-1)^{\ell} \frac{1}{r(r+1)-t(t+1)} \frac{(t+\ell)!}{(t-\ell)!} \left\{ \frac{Q_r^{\ell}(\xi_s)'}{Q_t^{\ell}(\xi_s)'} Q_t^{\ell}(\xi_1) - Q_r^{\ell}(\xi_1) \right\} \; ; \quad \begin{matrix} r \neq t \\ \ell < t \end{matrix} \qquad \text{(G.1)}$$

Also

$$\int_{\xi_s}^{\infty} K_t^{\ell}(\xi, \xi_1, \xi_s) Q_r^{\ell}(\xi) d\xi = 0 \qquad \text{if} \qquad \ell > t \; . \qquad \text{(G.2)}$$

APPENDIX H
REDEFINITION OF Q_n^m

The original definition of $Q_n^m(\mu)$ as given by (2.26) is

$$Q_n^m(\mu) = \frac{(-1)^m}{2^{n+1}} \frac{\Gamma(n+m+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} \frac{(\mu^2-1)^{m/2}}{\mu^{n+m+1}} \; {}_2F_1\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}; n+\frac{3}{2}; \frac{1}{\mu^2}\right),$$

$$|\mu| > 1, \quad |\arg(\mu-1)| < \pi . \tag{H.1}$$

Hobson (1953, pp 233-234) has shown that if

$$z = \mu + \sqrt{\mu^2 - 1} \ , \tag{H.2}$$

then the function

$$u(\mu) = \frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}} \; {}_2F_1\left(\frac{1}{2}+m, n+m+1; n+\frac{3}{2}; \frac{1}{z^2}\right) ,$$

$$|z| > 1, \quad |\arg(\mu - 1)| < \pi \tag{H.3}$$

satisfies the associated Legendre equation. Using this expression we can define a new function $Q_n^m(\mu)$ which holds for $|z| > 1$ or equivalently $|\mu| > 0$, which is identical to $Q_n^m(\mu)$ given by (H.1) in their common domain of definition, $|\mu| > 1$. To do this it is sufficient to compare (H.1) and (H.3) for large values of $|\mu|$. The resulting relation between the two functions is

$$Q_n^m(\mu) = (-1)^m 2^m \cdot \frac{\Gamma(n+m+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} \; u(\mu) \tag{H.4}$$

or

$$Q_n^m(\mu) = (-1)^m 2^m \frac{\Gamma(n+m+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} \frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}} {}_2F_1\left(\frac{1}{2}+m, n+m+1; n+\frac{3}{2}; \frac{1}{z^2}\right),$$

$$|z|>1, \quad |\arg(\mu-1)|<\pi. \qquad \text{(H.5)}$$

Letting $\mu = i\xi$, $\xi > 0$, we have $z = i\left(\xi + \sqrt{\xi^2+1}\right)$, and

$$Q_n^m(i\xi) = \frac{(-2)^m}{i^{n+1}} \frac{\Gamma(n+m+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} \frac{(\xi^2+1)^{m/2}}{\left(\xi+\sqrt{\xi^2+1}\right)^{n+m+1}}$$

$$\times \; {}_2F_1\left(\frac{1}{2}+m, n+m+1; n+\frac{3}{2}; -\frac{1}{\left(\xi+\sqrt{\xi^2+1}\right)^2}\right),$$

$$\xi > 0 \quad . \qquad \text{(H.6)}$$

Using the relation (Magnus and Oberhettinger, 1949, p. 8)

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z)$$

we can write

$${}_2F_1\left(\frac{1}{2}+m, n+m+1; n+\frac{3}{2}; \frac{1}{z^2}\right) = \left(1-\frac{1}{z^2}\right)^{-2m} {}_2F_1\left(n-m+1, \frac{1}{2}-m; n+\frac{3}{2}; \frac{1}{z^2}\right).$$

(H.7)

Letting $z = i\left(\xi+\sqrt{\xi^2+1}\right)$ and substituting in (H.6) we obtain

$$Q_n^m(i\xi) = \frac{(-2)^m}{i^{n+1}} \frac{\Gamma(n+m+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} \frac{(\xi^2+1)^{m/2}}{\left[\left(\xi+\sqrt{\xi^2+1}\right)^2+1\right]^{2m}\left(\xi+\sqrt{\xi^2+1}\right)^{n-3m+1}}$$

$$\cdot \; {}_2F_1\left(n-m+1, \frac{1}{2}-m; n+\frac{3}{2}; -\frac{1}{\left(\xi+\sqrt{\xi^2+1}\right)^2}\right), \quad \xi \geqslant 0 \qquad \text{(H.8)}$$

which holds at $\xi = 0$ also since $(n-m+1)+(\frac{1}{2}-m)-(n+\frac{3}{2}) = -2m \leqslant 0$ for $m = 0, 1, 2, \ldots$ (Magnus and Oberhettinger, 1949, p. 7).

Equation (H.8) can be rewritten to read

$$Q_n^m(i\xi) = \frac{(-1)^m}{i^{n+1}\, 2^m} \, \frac{\Gamma(n+m+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} \, \frac{(\xi^2+1)^{-\frac{1}{2}m}}{\left(\xi+\sqrt{\xi^2+1}\right)^{n-m+1}}$$

$$\cdot \, {}_2F_1\left(n-m+1, \frac{1}{2}-m; n+\frac{3}{2}; -\frac{1}{\left(\xi+\sqrt{\xi^2+1}\right)^2}\right), \quad \xi \geqslant 0 . \tag{H.9}$$

APPENDIX I
THE FAR FIELD FOR THE DISC

In this appendix we give the first six terms in the far field expansion for the disc for both Dirichlet and Neumann boundary conditions with the incident wave at normal incidence.

The far field in both cases is given by (5.25). This expansion involves the coefficients $A_t^M(0)$ given by (5.23) and (5.24) which contain the Legendre functions of the second kind and their first derivatives with respect to $i\xi_s$ evaluated at $\xi_s = 0$. Their values are determined as follows.

From equation (5.5),

$$Q_n(i\xi) = \frac{1}{i^{n+1}} \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} \frac{1}{\left(\xi+\sqrt{\xi^2+1}\right)^{n+1}} \, {}_2F_1\left(n+1, \frac{1}{2}; n+\frac{3}{2}; -\frac{1}{\left(\xi+\sqrt{\xi^2+1}\right)^2}\right), \qquad \xi \geq 0 \tag{I.1}$$

Letting $\xi = 0$, we have

$$Q_n(0) = \frac{1}{i^{n+1}} \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} \, {}_2F_1(n+1, \frac{1}{2}; n+\frac{3}{2}; -1) . \tag{I.2}$$

Now,

$$ {}_2F_1(n+1, \frac{1}{2}; n+\frac{3}{2}; -1) = \frac{\sqrt{\pi}}{2^{n+1}} \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}+1\right)} \quad (*) . \tag{I.3}$$

Then (I.2) becomes

$$Q_n(0) = \frac{\pi n!}{(2i)^{n+1}\left[\Gamma(\frac{n}{2}+1)\right]^2} . \tag{I.4}$$

---

* See, for example, Handbook of Mathematical Functions, National Bureau of Standards, Applied Math. Series No. 55, p. 557 (June 1964).

Or, we can write

$$Q_{2n}(0) = \frac{\pi(2n)!}{(2i)^{2n+1}\, n!\, n!} , \qquad n = 0, 1, \ldots \tag{I.5}$$

$$Q_{2n+1}(0) = \frac{\pi(2n+1)!}{(-4)^{n+1}\left[\Gamma(n+\frac{3}{2})\right]^2} , \qquad n = 0, 1, \ldots \tag{I.6}$$

Turning now to the derivative of Q_n, denote by $Q_n'(0)$ the derivative of $Q_n(i\xi)$ with respect to $i\xi$ evaluated at $\xi = 0$. From (I.1)

$$Q_n'(0) = \frac{-1}{i^n} \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+\frac{3}{2})} \left\{ -(n+1)\,{}_2F_1(n+1, \tfrac{1}{2}; n+\tfrac{3}{2}; -1) \right.$$

$$\left. + \frac{2(n+1)}{2n+3}\, {}_2F_1(n+2, \tfrac{3}{2}; n+\tfrac{5}{2}; -1) \right\} . \tag{I.7}$$

Employing (I.3), (I.4) and the relation

$${}_2F_1(a, b; a-b; -1) = 2^{-a}\sqrt{\pi}\,(b-1)^{-1}\,\Gamma(a-b+2)\left[\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{3}{2}+\frac{1}{2}a-b)} - \frac{1}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(1+\frac{1}{2}a-b)}\right], \tag{I.8}$$

which can be found in the same reference and page as (I.3), equation (I.7) becomes

$$Q_n'(0) = i(n+1)Q_n(0) - \frac{n!\,\pi}{i^n\, 2^{n+1}}\left[\frac{n+1}{\Gamma(\frac{n}{2}+1)\Gamma(\frac{n}{2}+1)} - \frac{2}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}\right] . \tag{I.9}$$

Finally, with the help of (I.5) and (I.6), we obtain for (I.9),

$$Q'_{2n}(0) = \frac{\pi\,(2n)!}{(-4)^n \left[\Gamma(n+\frac{1}{2})\right]^2}\,, \qquad n=0,1,\ldots \tag{I.10}$$

$$Q'_{2n+1}(0) = \frac{\pi(2n+1)!}{(2i)^{2n+1}\, n!\, n!}\,, \qquad n=0,1,\ldots\,. \tag{I.11}$$

We now employ equation (5.26) and write:

For the Dirichlet Case

$$u_o^{sf}(p_1) = \frac{2}{\pi}\, P_o(\eta_1)$$

$$u_1^{sf}(p_1) = \frac{4}{\pi^2}\, P_o(\eta_1)$$

$$u_2^{sf}(p_1) = -\frac{2}{9\pi}\, P_2(\eta_1) + \left(\frac{8}{\pi^3} - \frac{4}{9\pi}\right) P_o(\eta_1)$$

$$u_3^{sf}(p_1) = -\frac{4}{9\pi^2}\, P_2(\eta_1) - \left(-\frac{16}{\pi^4} + \frac{4}{3\pi^2}\right) P_o(\eta_1)$$

$$u_4^{sf}(p_1) = \frac{2}{525\pi}\, P_4(\eta_1) + \left(\frac{-8}{9\pi^3} + \frac{8}{105\pi}\right) P_2(\eta_1) + \left(\frac{32}{\pi^5} - \frac{32}{9\pi^3} + \frac{4}{75\pi}\right) P_o(\eta_1)$$

$$u_5^{sf}(p_1) = \frac{4}{525\pi^2}\, P_4(\eta_1) + \left(-\frac{16}{9\pi^4} + \frac{92}{567\pi^2}\right) P_2(\eta_1)$$

$$+ \left(\frac{64}{\pi^6} - \frac{80}{9\pi^4} + \frac{508}{2025\pi^2}\right) P_o(\eta_1)$$

Substituting these results in (5.25) we obtain

$$u^{sf}(p_1) = \frac{e^{ikc\xi_1}}{\xi_1}\left\{-\frac{2}{\pi}P_o(\eta_1) + ikc\frac{4}{\pi^2}P_o(\eta_1) + k^2c^2\left[-\frac{2}{9\pi}P_2(\eta_1)\right.\right.$$

$$\left.+\left(\frac{8}{\pi^3}-\frac{4}{9\pi}\right)P_o(\eta_1)\right] + ik^3c^3\left[\frac{4}{9}P_2(\eta_1) + \left(-\frac{16}{\pi^4}+\frac{4}{3\pi^2}\right)P_o(\eta_1)\right]$$

$$+k^4c^4\left[-\frac{2}{525\pi}P_4(\eta_1) + \left(\frac{8}{9\pi^3}+\frac{8}{105\pi}\right)P_2(\eta_1) + \left(-\frac{32}{\pi^5}+\frac{32}{9\pi^3}-\frac{4}{75\pi}\right)P_o(\eta_1)\right]$$

$$+ik^5c^5\left[\frac{4}{525\pi^2}P_4(\eta_1) + \left(-\frac{16}{9\pi^4}+\frac{92}{567\pi^2}\right)P_2(\eta_1)\right.$$

$$\left.\left.+\left(\frac{64}{\pi^6}-\frac{80}{9\pi^4}+\frac{508}{2025\pi^2}\right)P_o(\eta_1)\right] + O(k^6c^6)\right\} . \tag{I.12}$$

<u>For the Neumann Case</u>

$$u_o^{sf}(p_1) = 0$$

$$u_1^{sf}(p_1) = 0$$

$$u_2^{sf}(p_1) = -\frac{2}{3\pi}P_1(\eta_1)$$

$$u_3^{sf}(p_1) = 0$$

$$u_4^{sf}(p_1) = \frac{2}{75\pi}P_3(\eta_1) + \frac{8}{75\pi}P_1(\eta_1)$$

$$u_5^{sf}(p_1) = -\frac{4}{27\pi^2}P_1(\eta_1)$$

and

$$u^{sf}(p_1) = \frac{e^{ikc\xi_1}}{\xi_1}\left\{-k^2c^2\frac{2}{3\pi}P_1(\eta_1) - k^4c^4\left[\frac{2}{75\pi}P_3(\eta_1) + \frac{8}{75\pi}P_1(\eta_1)\right]\right.$$

$$\left. - ik^5c^5\frac{4}{27\pi^2}P_1(\eta_1) + O(k^6c^6)\right\} . \qquad (I.13)$$

The results given by (I.12) and (I, 13) are in complete agreement with those obtained by Senior (1960).

REFERENCES

Ar, E. and R.E. Kleinman (1966) "The Exterior Neumann Problem for the Three-Dimensional Helmholtz Equation", Arch. Rational Mech. Anal. 23, 218-236.

Darling, D.A. and T.B.A. Senior (1965) "Low-Frequency Expansions for Scattering by Separable and Nonseparable Bodies", J. Acoust. Soc. Amer. 37, 228-234.

Hobson, E.W. (1953) The Theory of Spherical and Ellipsoidal Harmonics, Chelsea Publishing Co., New York.

Kellogg, O.D. (1929) Foundations of Potential Theory, Springer, Berlin.

Kleinman, R.E. (1965) "The Dirichlet Problem for the Helmholtz Equation", Arch. Rational Mech. Anal. 18, 205-229.

Magnus, W. and F. Oberhettinger (1949) Formulas and Theorems for the Special Functions of Mathematical Physics, Chelsea Publishing Co., New York.

Morse, P.M. and H. Feshbach (1953) Methods of Theoretical Physics, McGraw-Hill Book Co., Inc., New York.

Senior, T.B.A. (1960) "Scalar Diffraction by a Prolate Spheroid at Low Frequencies", Can. J. Phys. 38, 1632-1641.

Senior, T.B.A. (1961) "The Convergence of Low Frequency Expansions in Scalar Scattering by Spheroids", The University of Michigan Radiation Laboratory Report No. 3648-4-T.

Senior, T.B.A. (1966) "The Scattering from Acoustically Hard and Soft Prolate Spheroids for Axial Incidence", Can. J. Phys. 44, 655-667.

Sleator, F.B. (1964) "Diffraction and Scattering by Regular Bodies III: The Prolate Spheroid", The University of Michigan Radiation Laboratory Report No. 3648-6-T.

Spence, R.D. and S. Granger (1951) "The Scattering of Sound from a Prolate Spheroid", J. Acoust. Soc.Amer. 23, 701-706.

Whittaker, E.T. and G.N. Watson (1952) A Course of Modern Analysis, Cambridge University Press, Cambridge.

Wilcox, C.H. (1956) "A Generalization of Theorems of Rellich and Atkinson", Proc. Amer. Math. Soc. 7, 271-276.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) The University of Michigan Dept. of Electrical Engineering, Radiation Laboratory Ann Arbor, Michigan 48108	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

LOW FREQUENCY SCATTERING BY SPHEROIDS AND DISCS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Interim Scientific Report

5. AUTHOR(S) (Last name, first name, initial)

Asvestas, John S.

Kleinman, Ralph E.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1967	102	14

8a. CONTRACT OR GRANT NO. AF 19(628)-4328 b. PROJECT NO. Task, Work Unit Nos. 5635 - 02 56350201 c. DoD Element 61445014 d. Dod Subelement 681305	9a. ORIGINATOR'S REPORT NUMBER(S) 7133-5-T Scientific Report No. 4 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) AFCRL-67-0360

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited. It may be released to the Clearinghouse, Department of Commerce, for sale to the general public.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Air Force Cambridge Research Laboratories L. G. Hanscom Field Bedford, Massachusetts 01730

13. ABSTRACT

The problem of scattering of a scalar plane wave by a spheroid of revolution is solved for either Dirichlet or Neumann boundary conditions, arbitrary major to minor axis ratio, and arbitrary incident direction. The solution is obtained using an iterative method applied to solutions of the corresponding potential problem and is expressed as a series of products of Legendre and trigonometric functions, and ascending powers of wave number. A recursion relation for the coefficients in this series is derived. These results are employed to calculate scattering cross sections for 2:1, 5:1 and 10:1 prolate spheroids.

DD FORM 1 JAN 64 1473

UNCLASSIFIED

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
scattering scalar plane wave spheroid of revolution iterative method cross section						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ______________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ______________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ______________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.