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TOTAL RADIANCES AND EQUIVALENT WIDTHS OF ISOLATED LINES WITH COMBINED DOPPLER AND COLLISION BROADENED PROFILES*

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Abstract—Analytical expressions for total radiances and equivalent widths of isolated spectral lines with combined Doppler and collision broadened profiles are presented for isothermal paths.

WHEN a spectral line is broadened by contributions from both the Doppler effect and Lorentz collision damping, the absorption coefficient, k, at frequency ν can be expressed by:[1]

$$k(\nu, a) = k_0 \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{a^2+(\omega-y)^2} \mathrm{d}y \tag{1}$$

where $k_0 = (\ln 2)^{1/2} S/\gamma_D$, $a = (\ln 2)^{1/2}(\gamma_L/\gamma_D)$, $\omega = (\ln 2)^{1/2}(\nu-\nu_0)/\gamma_D$, S is the line strength, γ_L is the Lorentz half-width at half-maximum intensity, γ_D is the Doppler half-width at half-maximum intensity, and ν_0 is the line center frequency. The absorption coefficient [equation (1)] is not a simple function compared to those for Lorentz and Doppler lines which lead to explicit expressions for the equivalent widths† in terms of known functions:[2,3]

$$W_L = 2\pi\gamma_L f(x_L) \tag{2}$$

where $x_L = SX/(2\pi\gamma_L)$, X is the optical depth, and $f(x_L) = x_L\, e^{-x_L}[I_0(x_L)+I_1(x_L)]$;

$$W_D = (\pi/\ln 2)^{1/2}\gamma_D g(x_D) \tag{3}$$

where

$$x_D = (\ln 2/\pi)^{1/2} SX/\gamma_D \quad \text{and} \quad g(x_D) = \sum_{n=0}^{\infty} \frac{(-1)^n x_D^{n+1}}{(n+1)!(n+1)^{1/2}}.$$

The subscripts L and D refer to Lorentz and Doppler, respectively. The subscript V will refer to lines with combined Doppler and collision broadened profiles, which are referred to as Voigt lines since the subject was originally discussed by VOIGT.[1] For isothermal

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† Equivalent width, W, is defined as: $W \equiv \int_0^\infty \alpha(\nu)\,\mathrm{d}\nu$, where $\alpha(\nu)$ is the absorptance.

lines, the total radiance $N = N^*_{\nu_0}(T)W$, where $N^*_{\nu_0}(T)$ is the Planck function of temperature T and line center frequency ν_0, and the Planck function is assumed to be invariant over the spectral interval occupied by a single line, so the analysis can be made in terms of W only.

Curves of growth for Voigt lines, i.e. plots of $\log_{10}\{N_V/[2\gamma_D N^*_{\nu_0}/(\ln 2)^{1/2}]\}$ versus $\log_{10}(10.6x_D)$, for various values of a have been obtained by VAN DER HELD[4] and PENNER and KAVANAGH.[5] The ordinate can also be expressed as $\log_{10}\{W_V/[2\gamma_D/(\ln 2)^{1/2}]\}$. Equations (2) and (3) can be expressed in terms of a and x_D and in forms which can readily be compared with the curves of growth:

$$W'_L = W_L/[2\gamma_D/(\ln 2)^{1/2}] = \pi a f\{x_D/[2\sqrt{(\pi)}a]\} \tag{4}$$

$$W'_D = W_D/[2\gamma_D/(\ln 2)^{1/2}] = [\sqrt{(\pi)}/2]g(x_D). \tag{5}$$

Similarly, we define

$$W'_V = W_V/[2\gamma_D/(\ln 2)^{1/2}]. \tag{6}$$

Figures 1–7 are plots of equations (4) and (5) on the Voigt curves of growth[4,5] for $a = 5\times10^{-4}$, 5×10^{-3}, 5×10^{-2}, 5×10^{-1}, 1, 2, and 10. These figures, plots of $\log_{10}W'_L$, $\log_{10}W'_D$, $\log_{10}W'_V$ against $\log_{10}(10.6x_D)$, show that W'_V can be represented by W'_D at some distance below the intersection point $(x_{D,I})$ of the W'_D and W'_L curves and by W'_L at some distance above $x_{D,I}$. For $a > 0.2$ there is no intersection point but a merging of the two curves into one; in this case the Lorentz and Voigt curves are identical over the entire range of optical depths.

When the Lorentz and Doppler curves do intersect, in the region about $x_{D,I}$, W'_V is slightly higher than either W'_L or W'_D. Note that the smaller the value of a, the larger the value of $x_{D,I}$. For example, for $a = 5\times10^{-4}$, $x_{D,I} \sim 10^4$ and for $a = 5\times10^{-2}$, $x_{D,I} \sim 5\times10^2$.

Above $x_{D,I}$ the curve for W'_V can be approximated by:

$$W'_V = W'_L\left[1+\frac{3}{8}\frac{\pi^{1/2}}{ax_D}+\frac{45}{128}\frac{\pi}{(ax_D)^2}+\frac{6615}{3072}\frac{\pi^{3/2}}{(ax_D)^3}\right]. \tag{7}$$

Equation (7) was obtained by using an approximate expression[6] for $k(\nu, a)$, valid for $a < 0.1$, and applying the conditions valid for strong lines. It is similar to an expression obtained by PLASS and FIVEL.[7] For $x_D \gg x_{D,I}$, $ax_D \gg 1$, and $\{\frac{3}{8}[\sqrt{(\pi)}/ax_D]\} \ll 1$ so that equation (7) reduces to $W'_V = W'_L$. Below $x_{D,I}$ an empirical fit to W'_V was made:

$$W'_V = W'_D\left(1+\frac{\sqrt{ax_D}}{2\pi^{n/2}}\right) \tag{8}$$

where n is specified by the order of magnitude of a: $a \sim 10^{-n+1}$. For example, for $a = 5\times10^{-4} \sim 10^{-3}$, $n = 4$. For $x_D \ll x_{D,I}$, $(\sqrt{ax_D}/2\pi^{n/2}) \ll 1$ so that equation (8) reduces to $W'_V = W'_D$.

Comparisons of values obtained with equations (7) and (8) and those obtained by VAN DER HELD[4] and read from the curves of PENNER and KAVANAGH[5] are given in Table 1. Equations (7) and (8) yield values which agree fairly well with those of van der Held. For $a = 5\times10^{-1}$ the values obtained with equation (4) were consistently smaller than those of van der Held. Except for one point, values of W'_V obtained with equation (4) agreed with values read off the graph[5] to within 10 per cent.

A more detailed presentation of these results is given in Ref. 8.

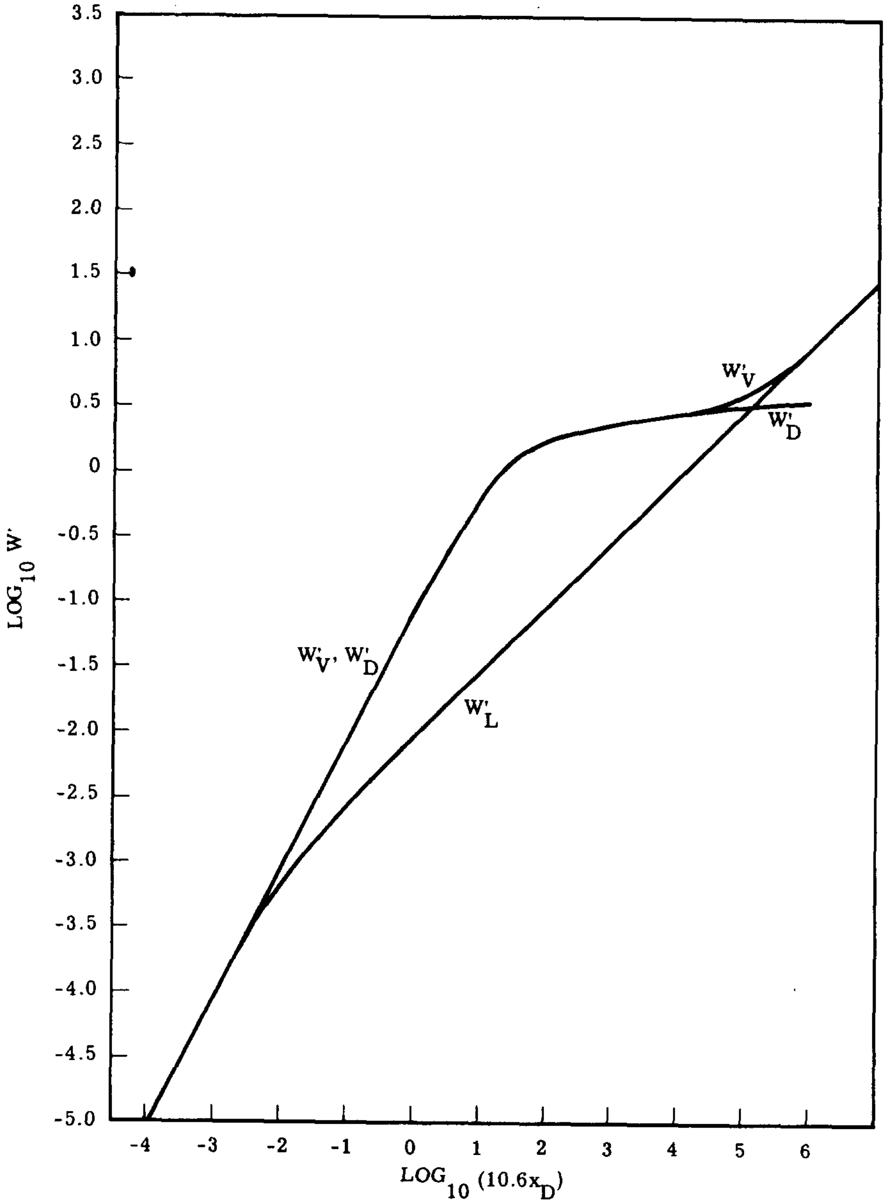

FIG. 1. Curves of growth for $a = 5 \times 10^{-4}$.

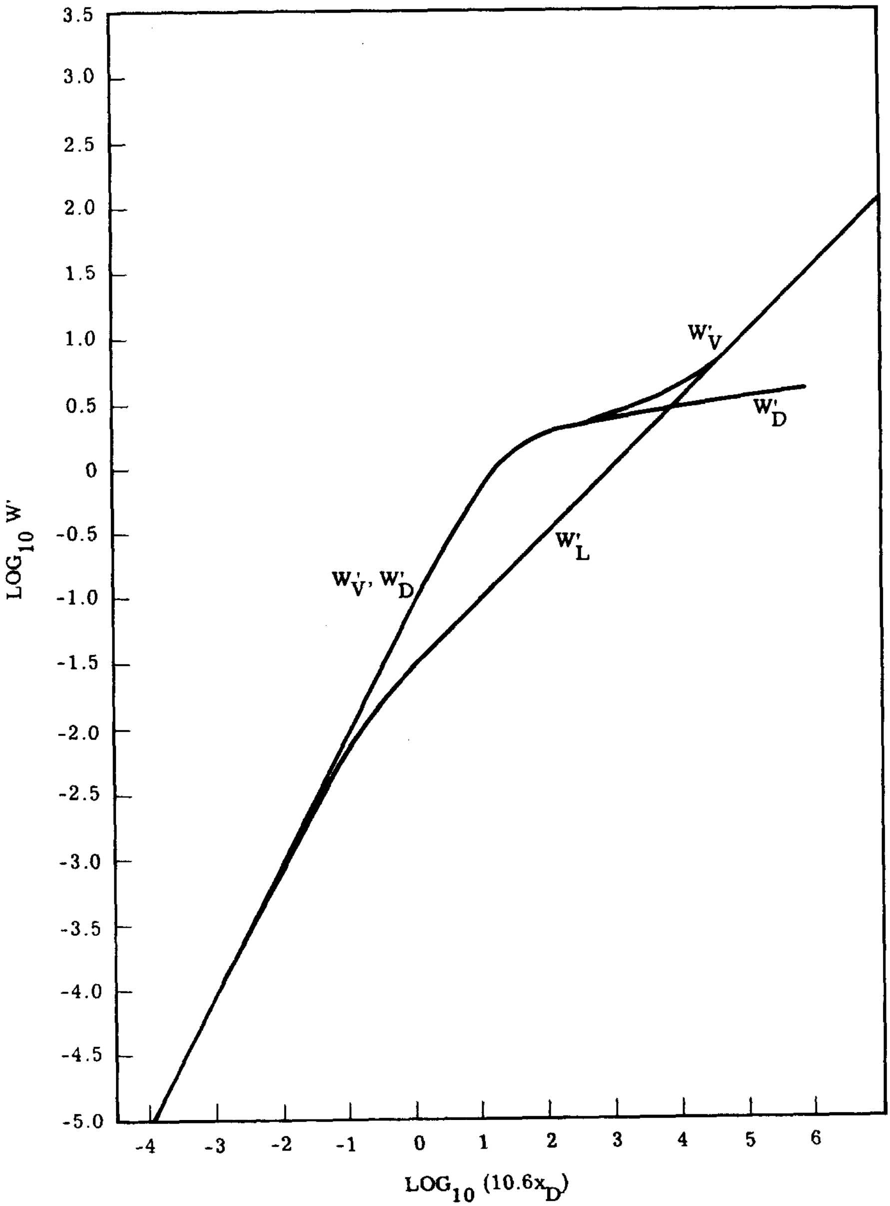

FIG. 2. Curves of growth for $a = 5 \times 10^{-3}$.

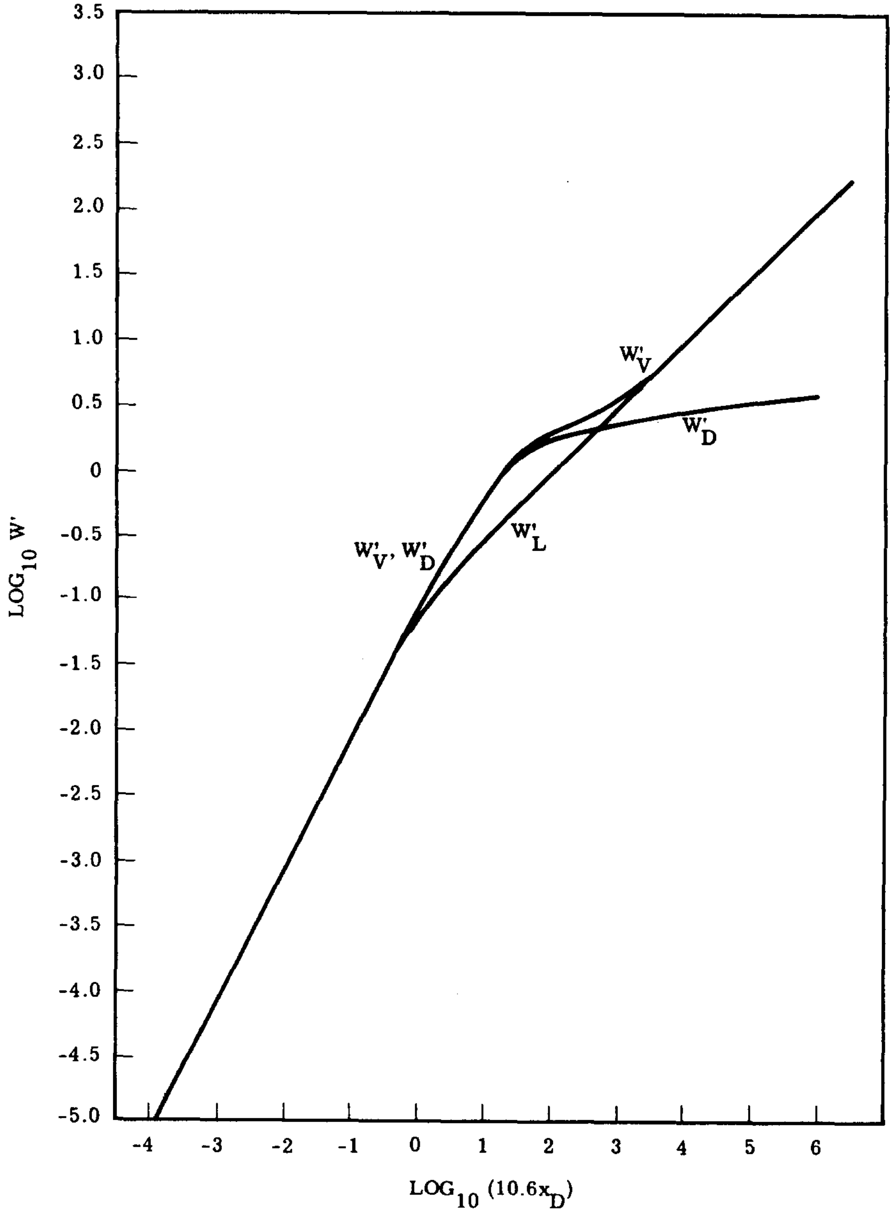

FIG. 3. Curves of growth for $a = 5 \times 10^{-2}$.

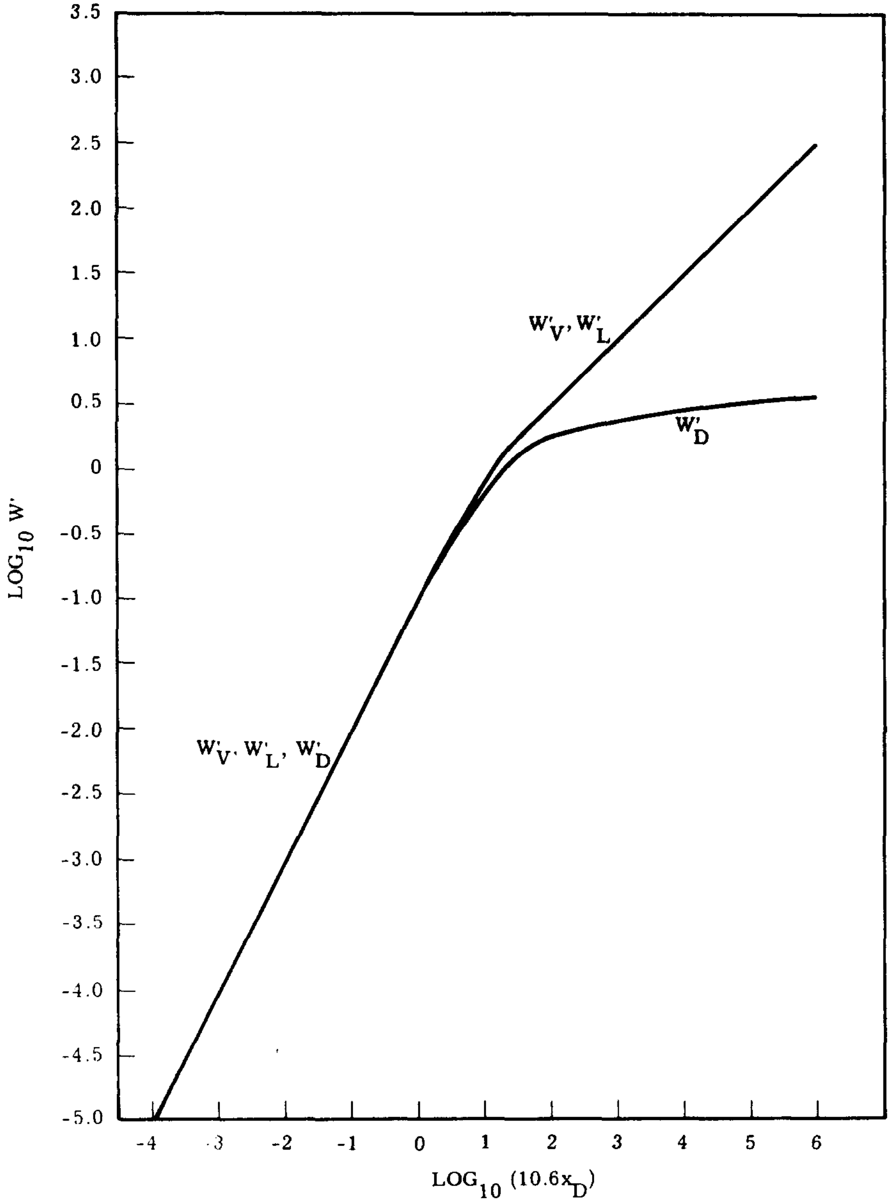

FIG. 4. Curves of growth for $a = 5 \times 10^{-1}$.

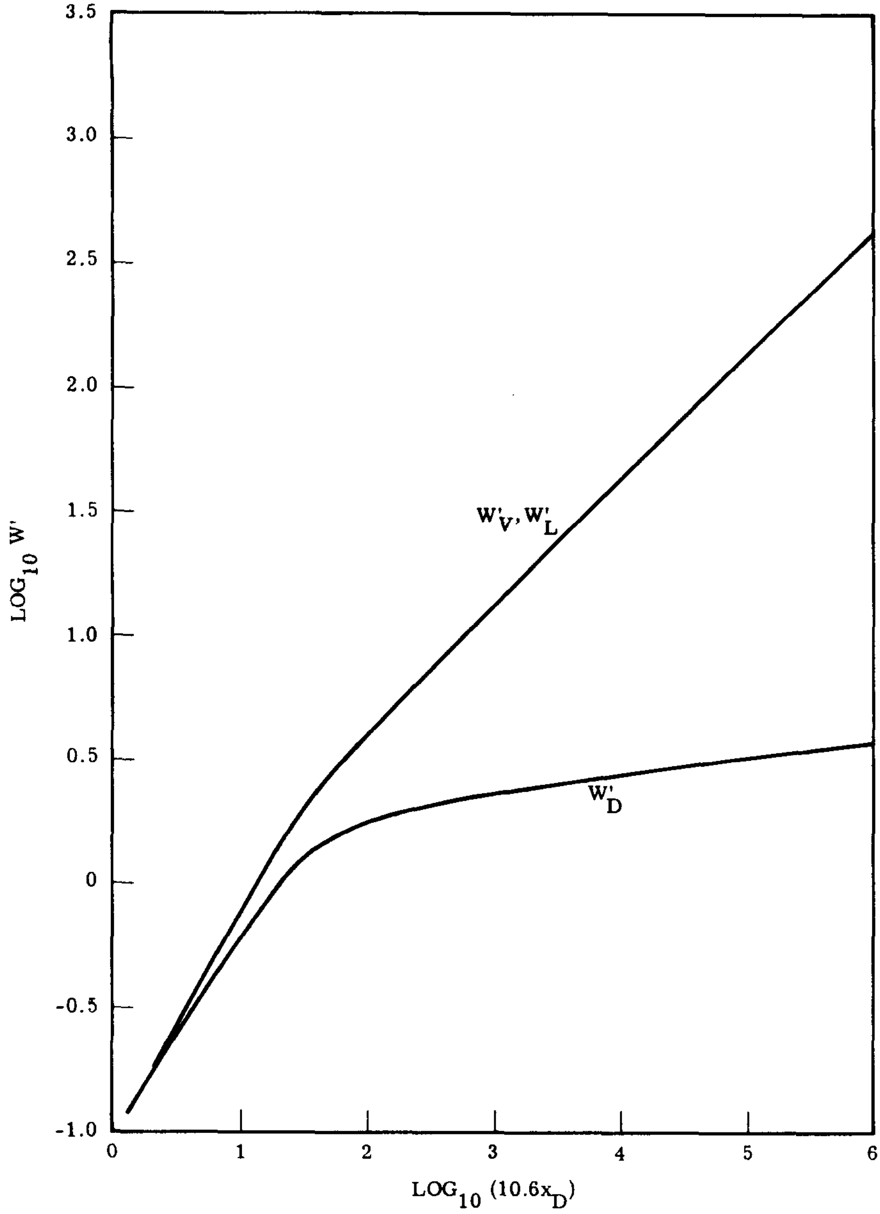

FIG. 5. Curves of growth for $a = 1$.

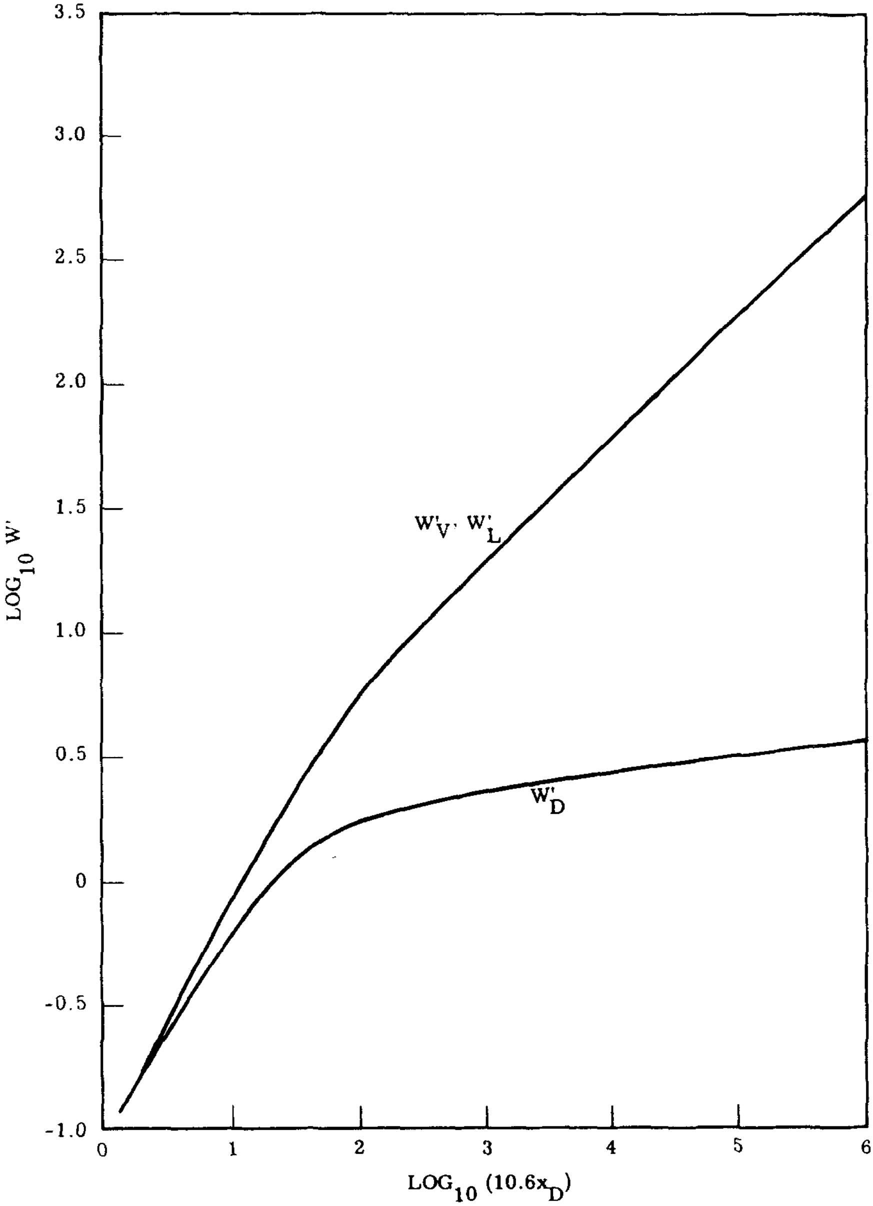

FIG. 6. Curves of growth for $a = 2$.

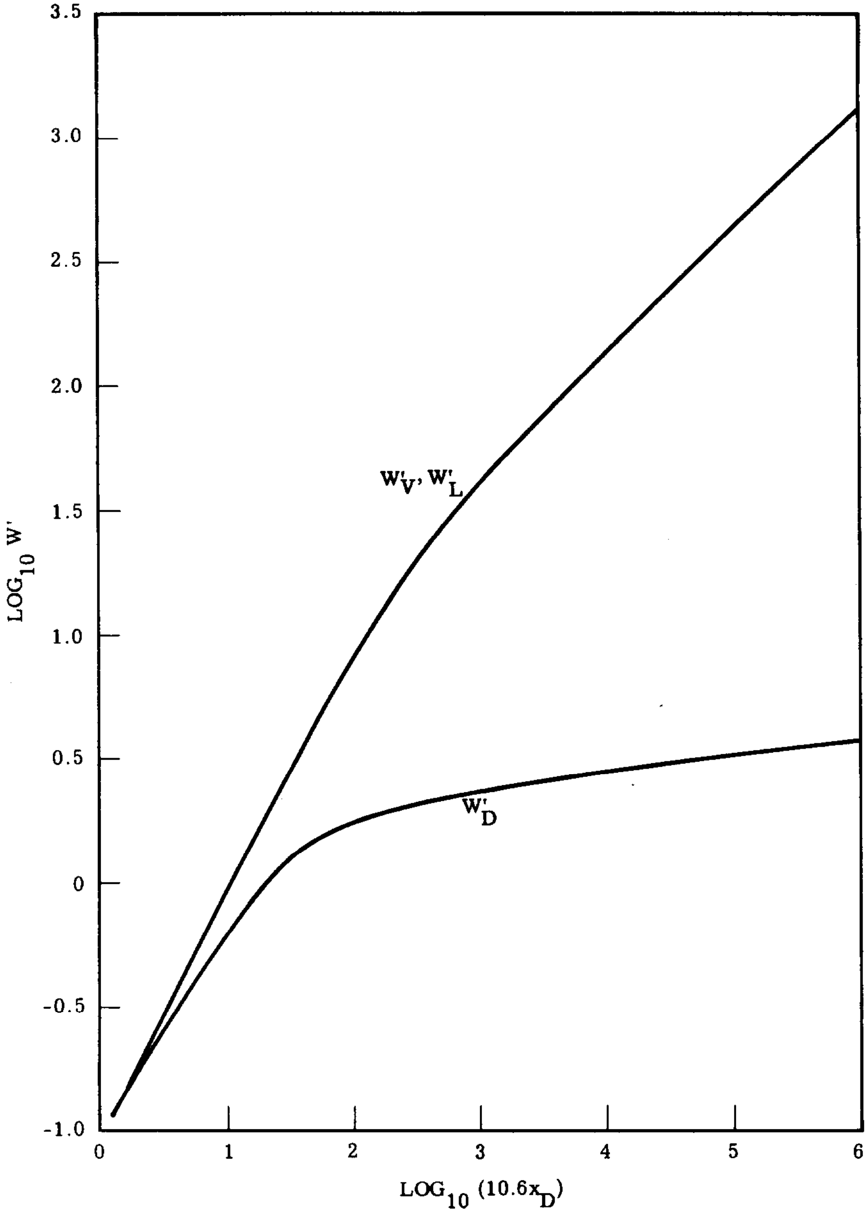

FIG. 7. Curves of growth for $a = 10$.

TABLE 1. CALCULATED VALUES OF W'_V

a	$\log_{10}(10.6x_D)$	W'_V calculated	W'_V	% Difference
5×10^{-4}	4.029	2.83 (E8)	2.86 (VDH)	1
	5.029	3.80 (E7)	4.02 (VDH)	6
	6.029	9.60 (E7)	9.67 (VDH)	0.7
5×10^{-3}	2.029	1.68 (E8)	1.67 (VDH)	−0.6
	3.029	2.42 (E8)	2.40 (VDH)	−0.8
	4.029	3.80 (E7)	3.76 (VDH)	−1
	5.029	9.60 (E7)	9.59 (VDH)	−0.1
	6.029	30.0 (E7)	29.8 (VDH)	−0.7
5×10^{-2}	1.029	0.670 (E8)	0.651 (VDH)	−3
	1.631	1.41 (E8)	1.41 (VDH)	0
	2.029	1.87 (E8)	1.86 (VDH)	−0.5
	3.029	3.79 (E7)	3.54 (VDH)	−7
	4.029	9.60 (E7)	9.51 (VDH)	−0.9
	5.029	30.0 (E7)	29.8 (VDH)	−0.7
5×10^{-1}	0.827	0.476 (E4)	0.499 (VDH)	5
	1.225	0.977 (E4)	1.066 (VDH)	8
	1.827	2.278 (E4)	2.572 (VDH)	11
	2.225	3.688 (E4)	3.959 (VDH)	7
	1.7	2.0 (E4)	2 (G)	0
	2.5	5.0 (E4)	5 (G)	0
	3.05	9.7 (E4)	10 (G)	3
	3.75	22 (E4)	20 (G)	−10
	4.55	54 (E4)	50 (G)	−8
	5.1	99 (E4)	100 (G)	1
	5.7	206 (E4)	200 (G)	−3
1.0	1.55	2 (E4)	2 (G)	0
	2.2	5 (E4)	5 (G)	0
	2.8	10 (E4)	10 (G)	0
	3.4	21 (E4)	20 (G)	−5
	4.2	51 (E4)	50 (G)	−2
	4.8	103 (E4)	100 (G)	−3
	5.4	207 (E4)	200 (G)	−4
	6.0	412 (E4)	400 (G)	−3
1.5	2.1	5 (E4)	5 (G)	0
	2.65	10 (E4)	10 (G)	0
	3.2	20 (E4)	20 (G)	0
	4.0	50 (E4)	50 (G)	0
	4.6	100 (E4)	100 (G)	0
	5.2	199 (E4)	200 (G)	0.5
	5.8	394 (E4)	400 (G)	2
	6.0	504 (E4)	500 (G)	−0.8
2	1.5	2 (E4)	2 (G)	0
	2.0	5 (E4)	5 (G)	0
	2.5	9.8 (E4)	10 (G)	2
	3.05	19 (E4)	20 (G)	5
	3.9	53 (E4)	50 (G)	−6
	4.5	101 (E4)	100 (G)	−1
	5.05	195 (E4)	200 (G)	2
	5.7	407 (E4)	400 (G)	−2
	6.0	571 (E4)	600 (G)	−5
10	1.8	5 (E4)	5 (G)	0
	2.2	11 (E4)	10 (G)	−10
	2.6	23 (E4)	20 (G)	−15
	3.2	50 (E4)	50 (G)	0
	3.8	103 (E4)	100 (G)	−3
	4.4	207 (E4)	200 (G)	−4
	5.0	412 (E4)	400 (G)	−3

E8 = calculated with equation (8).
E7 = calculated with equation (7).
E4 = calculated with equation (4).
VDH = value taken from van der Held's Table 1 (Ref. 4).
G = value read from graph (Ref. 5).

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