Flows on the Solid Torus Asymptotic to the Boundary*

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Received October 21, 1966

INTRODUCTION

If $X$ is a topological space and $T$ denotes the real numbers, then by a flow we mean a continuous map $\phi : X \times T \to X$ such that $\phi(x, 0) = x$ and $\phi(x, s + t) = \phi(\phi(x, s), t)$. We shall denote $\phi(x, t)$ by $x_t$. If $X$ is a differentiable manifold and $V$ is a vector field on $X$, then $V$ is said to generate $\phi$ where $V_x(f) = d/dt[f(x_t)]_{t=0}$ for every differentiable function, $f$.

In [A], Seifert raised the question: Does there exist a flow on $S^3$ which contains no closed, that is, periodic orbit? He showed that if $V_o$ is a vector field on $S^3$ which generates a flow whose orbits are the fibers of the Hopf fibration and $V$ is sufficiently close to $V_o$, in the $C^0$ sense, then the flow generated by $V$ must contain at least one closed orbit. Since $S^3$ is the union of two solid tori whose intersection is a two-dimensional torus, it is of interest to study flows on a solid torus, $K = D^2 \times S^1$, where $D^2 = \{z \mid z$ complex, $|z| \leq 1\}$. If it is possible to construct such a flow on $K$ with no closed orbit, then it is possible to construct such a flow on $S^3$.

Considering $K$, one might think that if the flow were such that the restriction to the boundary was the irrational flow, then a closed orbit, encircling the "hole" (that is, a closed orbit not contractible to a point) would exist in the interior of $K$. However, in [I] Fuller has constructed a flow on $K$ whose only closed orbits are null homotopic.

In this paper, we shall approach the problem from a somewhat different standpoint. We consider a flow on $K$ such that every interior orbit approaches the boundary as $t \to \infty$, and show that the boundary of $K$ must contain a closed orbit. Thus if the boundary of $K$ contains no closed orbit, the interior of $K$ is not completely unstable.

We shall not assume that $\phi$ is generated by a vector field. We will make

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* This research was supported by the National Science Foundation, N.S.F. Grant No. 06962.
considerable use of the covering of $K$ by $\hat{K}$ a simply connected, noncompact cylinder.

1. Preliminary Definitions and Propositions

1.1. Definition. Let $p : \hat{X} \to X$ be a covering of $X$ by $\hat{X}$ with projection $p$. If $\phi : \hat{X} \times T \to \hat{X}$ and $\phi : X \times T \to X$ are flows such that $p(\phi_t) = [p(x)]_t$, $\phi$ is said to cover $\psi$.

As a consequence of the covering homotopy property we have the following proposition:

1.2. Proposition. If $p : \hat{X} \to X$ is a covering of $X$ by $\hat{X}$ and $\phi : X \times T \to X$ is a flow, there exists a unique flow $\hat{\phi} : \hat{X} \times T \to \hat{X}$ which covers $\psi$.

1.3. Notation. $D^2$ will denote the unit disc, $\{z \mid z$ complex, $|z| \leq 1\}$, $S^1$ the unit circle, $\{z \mid |z| = 1\}$, and $T^+$ the positive real numbers. We also use the notation $K = D^2 \times S^1$ and $\hat{K} = D^2 \times T$.

The covering $p : \hat{K} \to K$ is defined by $p(d, t) = (d, e^{2\pi it})$. We denote $I = S^1 \times S^1 = \text{boundary of } K$ and $\hat{I} = S^1 \times T = \text{boundary of } \hat{K}$.

We shall also use $p$ to denote the restriction of $p$ to $\hat{I}$.

We shall consider a flow $\phi : K \times T \to K$ and its covering $\hat{\phi} : \hat{K} \times T \to \hat{K}$. Finally, if $(d, t)$ is in $\hat{K}$ and $l$ is an integer we denote

$$(d, t) + l = (d, t + l).$$

Since

$$p((x + l)\iota) = [p(x + l)]\iota = [p(x)]\iota = p(x) = p(x + l)$$

and $(x + l)\iota = x\iota + l$ for $t = 0$, it follows from the uniqueness of covering paths that

1.4. $(x + l)\iota = (x\iota + l)$ for $x$ in $\hat{K}$, $t$ real, and $l$ an integer.

It follows from the continuity of $\hat{\phi}$ and the compactness of $\hat{K} = D^2 \times [-1, 1]$ that there exists a function $\Delta' : T^+ \times T^+ \to T^+$ such that if
x and y are in $\mathbb{R}^n$, $\epsilon > 0$, $t > 0$, and $\text{dist}(x, y) < \Delta(\epsilon, t)$, then $\text{dist}(x_s, y_s) < \epsilon$ for $|s| \leq t$. Applying 1.4, we have

1.5. There exists a function $\Delta : T^+ \times T^+ \to T^+$ such that if $x$ and $y$ are in $\mathbb{R}^n$, $\epsilon > 0$, $t > 0$, and $\text{dist}(x, y) < \Delta(\epsilon, t)$, then $\text{dist}(x_s, y_s) < \epsilon$ for $|s| \leq t$.

(Note: we assume that $K = D^2 \times S^1$ and $\mathbb{R} = D^2 \times T$ are equipped with their product metrics.)

We now introduce some concepts from topological dynamics.

1.5. **Definition.** If $x$ is a point of $K$ or $\mathbb{R}$, by the omega limit set of $x$, $\Omega_x$, we mean $\cap_{t \in T} \overline{\{x_s \mid s \geq t\}}$; where $\overline{\{\ldots\}}$ denotes the closure of $\{\ldots\}$.

It is easy to verify that

$$\Omega_x = \{y \in K \mid x_{t(k)} \to y \text{ for some } t(k) \to \infty\}. \quad (1.6)$$

1.7. **Definition.** A compact set $M \subset K$ is called minimal where $M$ is nonempty invariant (i.e., $M_t = M$ for all $t$ in $T$), and contains no such proper subset.

By an application of Zorn's lemma one sees that

1.8. Every compact invariant set contains a minimal set.

1.9. **Definition.** $x_T$ is called a closed orbit where $x_T$ is compact. Thus

1.10. $x_T$ is a closed orbit if and only if $x_h = x$ for some $h \neq 0$.

Note that a fixed point, $x_T = \{x\}$ is a closed orbit.

1.11. **Definition.** Consider a flow in $L$ or $\overline{L}$. We say a closed orbit is bounding in $L$ or $\overline{L}$ if its complement has two components, at least one of which is bounded.

As a consequence of the Brouwer fixed-point theorem we have

1.11. Every bounding orbit in $\overline{L}$ contains a fixed point in the bounded component of its complement.

Note that by invariance of domain, $L$ or $\overline{L}$ must be an invariant set in $K$ or $\overline{K}$.

We shall make use of a somewhat generalized version of the Poincaré-Bendixson theory. As a rule, Poincaré-Bendixson theorems are proved for flows in the plane or the two-dimensional sphere, $S^2$, generated by continuous vector fields (see, for example, [2]). However, the only use made of the vector field is in the construction of transversal line segments. Whitney has shown, [6], that transversal line segments may be constructed at any regular point (i.e., nonfixed point) of a two-dimensional flow. Thus we have
1.12. Theorem (Poincaré-Bendixson). Given a flow on $S^2$, and a point $x$ in $S^2$ then

(a) $\Omega_x = x_T$ if $x_T$ is closed. On the other hand, if $x_T$ is not closed we have

(b) $\Omega_x = \gamma = \text{boundary of } C$, where $C$ is an open two cell containing $x$ and a fixed point. Moreover, if $S^2$ contains finitely fixed points $c_1, \ldots, c_n$ then

(c) $\Omega_x$ is either a closed orbit or $\Omega_x$ is the union of some of the fixed points and orbits $y_1^T, \ldots, y_m^T$ satisfying $y_i^T \to c_k$ as $t \to -\infty$ and $y_i^T \to c_l$ as $t \to +\infty$, for some $k$ and $l$.

Now, consider a flow on $L$ without fixed points. We may embed $\bar{L}$ in $S^2$ so that $S^2 - \bar{L} = \{(0, 0, -1), (0, 0, +1)\}$. We state that $(0, 0, -1)$ and $(0, 0, +1)$ are fixed points and thereby extend the flow to $S^2$. If $x$ is in $\bar{L}$ and $\Omega_x$ contains no fixed point, then $\Omega_x$ is a closed orbit, nonbounding in $L$. On the other hand, if $\Omega_x$ contains one fixed point, $(0, 0, 1)$, $S^2 - \Omega_x$ must contain the other. If, in addition, $\Omega_x$ were to contain a regular orbit $y_T$, then $y_T \cup \{(0, 0, 1)\}$ would separate $S^2$ into two regions, each containing a fixed point, which is impossible. Thus we have

1.13. If $L$ contains no fixed point, then every omega limit set is either a nonbounding orbit or empty.

We may reformulate this by introducing the following definitions:

1.14. Notation. For $(d, t)$ in $\bar{K}$, denote $\pi(d, t) = t$.

1.15. Definition. Let $x$ be in $K$, $p(\tilde{x}) = x$. If $\pi(\tilde{x}_t) \to +\infty (-\infty)$ as $t \to +\infty$ we say $x_t \to \infty(-\infty)$ as well as $\tilde{x}_t \to \infty(-\infty)$. Thus, we have as a corollary to 1.13,

1.16. If $L$ contains no closed orbit, and $x$ is in $L$ then either $x_t \to +\infty$ or $x_t \to -\infty$.

It will take a good deal more effort to show that all orbits tend, in some sense uniformly, to the same limit.

2. The Behavior of the Flow on L

In [5], Siegel showed that if $L$ contains no compact orbit, it must contain a cross-section, $\Gamma$, that is, a simple closed curve, nowhere tangent to the field generating $\phi$, which intersects every orbit.

If $\Gamma$ were covered by a closed curve $\tilde{\Gamma}$ in $\bar{L}$. It would be easy to show that every orbit tended to $+\infty$ or every orbit tended to $-\infty$. Although we may construct a covering $p^* : L^* \to L$ so that $\Gamma$ is covered by a closed curve $\Gamma^*$, it may not be possible to extend $p^*$ to a covering of $K$. 
The difficulty to be avoided is exemplified by the following system in the plane:

\[
\frac{dx}{dt} = \cos 2\pi y,
\]

\[
\frac{dy}{dt} = \sin 2\pi y.
\]

Here all orbits tend to $-\infty$ except $y = 2\pi k$, $k = 0, +1, +2,...$, which tend to $+\infty$. The orbits $y = 2\pi k$ serve as examples of the following concept:

2.1. DEFINITION. $x_T$ is called a \textit{separatrix} where $x_t \to +\infty$ and there exists $y(k) \to x$ such that for each $k$, $y(k)_t \to -\infty(+\infty)$.

Our immediate aim is to show that separatrices in $L$ are closed orbits.

As a consequence of 1.13 we have the following lemma:

2.2. LEMMA. If $\tilde{L}$ contains at least one orbit $\tilde{x}_T$, such that $\tilde{x}_t \to \infty$ or $\tilde{x}_t \to -\infty$, but no fixed point, then $\tilde{L}$ contains no closed orbit.

Proof. If $\tilde{L}$ contains a closed orbit, $y$, it must be nonbounding. If we embed $\tilde{L}$ in $S^2$ as before, $y$ separates $(0, 0, 1)$ and $(0, 0, -1)$. We may select $\tilde{x}$ in $p^{-1}(x)$ so that $\tilde{x}$ is in the same component of $S^2 - y$ as $(0, 0, -1)$. Thus $\tilde{x}_t \to (0, 0, 1)$, which is to say, $x_t \to \infty$. Similarly $x_t \to -\infty$, which proves the lemma.

Our next lemma limits the amount of time an orbit may remain in a compact portion of $\tilde{L}$.

2.3. LEMMA. If $\tilde{L}$ contains no closed orbit, there exists a function $M : T^+ \to T^+$ such that $\text{diam}(y(t), (t)) > t$ for all $y$ in $\tilde{L}$ and $t > 0$.

Proof. Suppose, on the contrary, for some $t_0 > 0$ and $\{y(k) | k = 1, 2,...\} \subset \tilde{L}$ we have $\text{diam}(y(k)) \leq t_0$. According to 1.4 we may assume $\{y(k)\} \subset S^1 \times [0, 1]$ and by choosing a subsequence, if necessary, we may assume $y(k) \to \tilde{y}$.

If $\text{diam} \tilde{y} \leq 2t_0$, then $\Omega_2$ must be a closed orbit, contrary to hypothesis. If $\text{diam} \tilde{y} \geq 2t_0$, then $\text{dist}(\tilde{y}, \tilde{y}_h) > t_0$ for some $h$, $\text{dist}(y(k), y(h)) > t_0$ for sufficiently large $k$, and for $k > h$ we have $\text{diam}(y(k), y(h)) > t_0$, contradicting the supposition. The lemma is proved.

The next lemma, in a sense, limits the “speed” of any orbit.

2.4. LEMMA. There exists a positive number $F$ such that for any $x$ in $\tilde{L}$,

(i) $|\pi(x_s) - \pi(x)| \leq 1$ if $s$ in $[0, F)$, and

(ii) $|\pi(x_t) - \pi(x)| \leq \frac{t}{F} + 1$ for $t \geq 0$. 

Proof. Let

\[ F_x = \inf \{ t \geq 0 \mid |\pi(x_t) - \pi(x)| = 1\} \]

and

\[ F = \inf \{ F_x \mid x \text{ in } \bar{L}\}. \]

Clearly, \( F \) satisfies (i). That \( F > 0 \) follows from the continuity of \( \check{\phi} \), the compactness of \( \bar{K}' = D^2 \times [-1, 1] \), and 1.4. Now, if \( N \) is an integer such that

\[ 0 \leq (N - 1)F \leq t \leq NF, \]

we have

\[ |\pi(x_t) - \pi(x)| \leq \sum_{k=0}^{N-2} |\pi(x_{kF}) - \pi(x_{(k+1)F})| + |\pi(x_t) - \pi(x_{(N-1)F})| \]

\[ \leq N \leq \frac{t}{F} + 1. \]

We now come to the key theorem of this section.

2.5. THEOREM. If \( L \) contains no fixed point, every separatrix is a closed orbit in \( L \).

Proof. (See Figure 1). Let \( x_T \) be a separatrix. Let us say \( x_t \to +\infty \). Suppose \( x_T \) is not closed in \( L \). Let \( x = p(\check{x}) \). We may assume \( \pi(\check{x}) = 0 \). (See 1.14).

Let

\[ t(k) = \sup \{ t \mid \pi(\check{x}_t) = k \} \]

and

\[ y(k) = \check{x}_{t(k)} - k. \] (See 1.3).

Thus \( \{y(k)\mid k = 1, 2, \ldots\} \) is an infinite subset of \( \Pi_0 = \{\xi \in \bar{L} \mid \pi(\xi) = 0\} \subset \bar{L} \). Moreover,

\[ y(k)_t > 0 \quad \text{for} \quad t > 0, \]

\[ y(k)_t \to +\infty, \]

\[ p(y(k)_t) = x_T \quad \text{for all} \quad k. \]

We choose

\[ r(k) = \inf \{ t \geq 0 \mid \pi(y(k)_t) = 1\} \]

thus

\[ 0 < \pi(y(k)_t) < 1 \quad \text{for} \quad 0 < t < r(k) \]
so that

\[ 0 < F \leq r(k) \leq M(1), \]

(2.6)

where \( F \) is defined in 2.4 and \( M(1) \) is defined in 2.3. Note \( x_t \to +\infty \) implies \( T \) contains no closed orbit according to 2.2.

Now by taking a subsequence, if necessary, we may assume that \( y(k) \to \bar{y} \).
Moreover, if we suitably coordinatize a neighborhood \( U \), of \( \Pi_o \) near \( \bar{y} \) by \( \theta : U \to T \), we may assume (again taking a subsequence if necessary)

\[ \theta(y(1)) > \theta(y(2)) > \cdots > \theta(y(k)) > \cdots. \]

Now, \( y(k)_{[0,\infty)} \cup y(k + 2)_{[0,\infty)} \) separates \( S^1 \times [0, \infty) \) into two components, \( C_k \) and \( D_k \), with \( y(k + 1) \) in \( C_k \).

By supposition, we may assume there is a point \( w(k) \) in \( \Pi_o \) satisfying

(i) \( w(k) \to -\infty \),

(ii) \( w(k) \) is in \( C_k \),

(iii) for some \( s > 0 \), \( \pi(s w(k)_s) > 1 \), and \( w(k)_s \) is in \( C_k \).

We may satisfy (iii) by choosing \( s > 0 \) such that \( \pi(y(k + 1)_s) > 1 \) and choosing \( w(k) \) sufficiently close to \( y(k + 1) \).
Let
\[ s(k) = \sup \{ s \mid \pi(w(k)_s) = 1, w(k)_s \in C_k \} \]
and
\[ u(k) = w(k)_{s(k)}. \]

Since \( u(k) \) is in \( w(k)_F \), \( u(k)_F \rightarrow -\infty \). We may set
\[ v(k) = \inf \{ v > 0 \mid \pi(u(k)) = 0 \}. \]

Now \( u(k)_{(0,v(k))} \) cannot cross \( y(k)_{(0,\infty)} \) or \( y(k+2)_{(0,\infty)} \), thus,
\[ u(k)_{(0,v(k))} \subset C_k \]
and
\[ \theta(y(k)) > \theta(u(k)_{v(k)}) > \theta(y(k+2)). \]

According to 2.4 and 2.3,
\[ 0 < F \leq v(k) \leq M(1). \]

Next we suitably coordinatize \( V = \Pi_1 - \{ y(1)_{r(\Omega)} \} \), where \( \Pi_1 = \{ \xi \in \tilde{L} \mid \pi(\xi) = 1 \} \), by \( \psi : V \rightarrow T \). We have
\[ \psi(y(k)_{r(\Omega)}) > \psi(u(k)) > \psi(y(k+2)_{r(\Omega+\omega)}) \]
so that, according to (2.9) and (2.6),
\[ \lim_{k \to \infty} y(k)_{r(\Omega)} = \lim_{k \to \infty} u(k) = \tilde{y}_F, \]
where
\[ \tilde{r} = \lim_{k \to \infty} r(k) \geq F. \]

(Note that the uniqueness of the limit of \( \{ y(k)_{r(\Omega)} \} \) implies the uniqueness of the limit of \( \{ r(k) \} \).) Furthermore, from (2.7) and (2.8) it follows that
\[ u(k)_{v(k)} \rightarrow \tilde{y}_{\tilde{r} + \tilde{v}} = \tilde{y}, \]
where
\[ \tilde{v} = \lim_{k \to \infty} v(k) \geq F. \]

Thus \( \tilde{y}_F \) is a closed orbit in \( \tilde{L} \). But, according 2.2, this contradicts the hypothesis. The theorem is proved.

As a corollary to 2.5 we have

2.10. COROLLARY. If \( L \) contains no closed orbit, all orbits tend to \( +\infty \) or all orbits tend to \( -\infty \).
Proof. If $L$ contains no closed orbit, a fortiori $\mathcal{L}$ contains no closed orbit. Thus, according to 1.16, $\mathcal{L} = A \cup B$ where $A = \{x \mid x_i \to \infty\}$ and $B = \{x \mid x_i \to -\infty\}$. But according to 2.5, $A$ and $B$ are closed. Since $A \cap B$ is empty and $\mathcal{L}$ is connected, $\mathcal{L} = A$ or $\mathcal{L} = B$, which was to be shown.

Having established that all orbits tend to the same limit, we now show that they tend to this limit "uniformly".

2.11. Theorem. If $L$ contains no closed orbit, there exists a function $\rho : T^+ \to T^+$ such that for any $x$ in $\mathcal{L}$,

$$|\pi(x_t) - \pi(x)| \geq t \quad \text{if} \quad x \geq \rho(t). \quad (2.12)$$

Proof. Let us assume $x_i \to \infty$ for all $x$ in $L$. For each $x$ in $L$ and $r > 0$, let

$$A_x(r) = \inf\{t \geq 0 \mid \pi(x_t) - \pi(x) = r\}$$

and let

$$A(r) = \sup\{A_x(r) \mid x \in \mathcal{L}\}.$$ 

The finiteness of $A(r)$ follows from the continuity of $\phi$, and 1.4. Recall that, according to 2.4 (i), if

$$F_x = \inf\{t > 0 \mid |\pi(x_t) - \pi(x)| = 1\},$$

then

$$F = \inf\{F_x \mid x \in \mathcal{L}\} > 0.$$ 

Now set $C = A(2 + A(1)/F)$. We assert that

$$\pi(x_t) - \pi(x) \geq 1 \quad \text{if} \quad t \geq C. \quad (2.13)$$

Suppose $\pi(x_t) - \pi(x) < 1$ for some $t_0 \geq C$. Then for some $u$ in $[0, C] \subset [0, t_0]$ we have

$$\pi(x_u) - \pi(x) = 2 + \frac{A(1)}{F},$$

and for some $v > t_0$ we have

$$\pi(x_v) - \pi(x) = 1.$$

Now let

$$\bar{u} = \sup\left\{u \in [0, v] \mid \pi(x_u) - \pi(x) = 2 + \frac{A(1)}{F}\right\}$$

so that

$$\pi(x_s) - \pi(x) < 2 + \frac{A(1)}{F} \quad \text{if} \quad \bar{u} < s \leq v, \quad (2.14)$$
and
\[ \pi(x_\delta) - \pi(x_\beta) = 1 + \frac{A(1)}{F}. \]

Thus, according to 2.4 (ii), \( v - \delta \geq A(1) \). But by the definition of \( A \),
\[ \pi(x_\delta) \geq \pi(x_\beta) + 1 = \pi(x) + \frac{A(1)}{F} + 3 \]
for some \( s \) in \([\delta, v]\), which contradicts (2.14). Thus (2.13) is proved.
Applying (2.13) we have, for any positive integer \( N \) and \( t \geq NC \),
\[ \pi(x_t) - \pi(x) = \pi(x_t) - \pi(x_{(N-1)C}) + \sum_{k=1}^{N-1} \pi(x_{kc}) - \pi(x_{(k-1)c}) \geq N. \]
Thus \( p(t) = (t + 1) C \) satisfies (2.12).

Having established that every orbit tends to \( +\infty \) or \(-\infty\) uniformly, in
the sense of (2.12), on \( L \), we turn to a consideration of the flow on \( K \).

3. Proof of the Main Theorem

3.1. Theorem. Let \( \phi \) be a flow on \( K = D^2 \times S^1 \) such that \( \Omega_x \subset L = S^1 \times S^1 \) = boundary of \( K \) for each \( x \) in \( K \). Then \( L \) contains a closed orbit.

In order to prove the theorem, we establish a series of lemmas. The first
two lemmas extend the conclusions of 2.10 and 2.11 from \( L \) to \( \bar{K} \).

3.2. Lemma. If \( \Omega_x \subset L \) for each \( x \) in \( K \) and \( L \) contains no closed orbit, then
each orbit in \( K \) tends to \( +\infty \) or each orbit tends to \( -\infty \).

Proof. Let \( \Delta : T^1 \times T^1 \rightarrow T^1 \) be as in 1.5 and \( \delta = \Delta(1/10, \rho(1)) \).
Since \( \Omega_x \subset L \) for each \( x \) in \( K \), there exists a function \( \sigma : K \rightarrow T^+ \) such that
if \( x \) is in \( K \) and \( s \geq \sigma(x) \) then \( \text{dist}(x_s, L) < \delta \). Since the metrics on
\( K = D^2 \times S^1 \) and \( \bar{K} = D^2 \times T \) are product metrics we have for \( x \) in \( \bar{K} \),
and \( S(x) = \sigma(p(x)) \),
\[ \text{dist}(x_s, L) < \delta \quad \text{for} \quad s \geq S(x). \]

Now for any positive integer \( n \), and \( x \) in \( \bar{K} \) we have, setting \( S = S(x) \) and \( R = \rho(1) \) (where \( \rho \) is defined in 2.11),
\[ P_n = \pi(x_{S+nR}) - \pi(x_S) = \sum_{k=0}^{n-1} \pi(x_{S+(k+1)R}) - \pi(x_{S+kR}), \]
but
\[ Q_k = \pi(x_{S+ (k+1)R}) - \pi(x_{S+kR}) \]
\[ = \pi(x_{S+ (k+1)R}) - \pi(y^k_R) + \pi(y^k_R) - \pi(y^k) + \pi(y^k) - \pi(x_{S+kR}), \]
where \( y^k \) may be chosen in \( L \) so that
\[ \text{dist}(y^k, x_{S+kR}) < \delta < 1/10 \]
and thus
\[ \text{dist}(y^k_R, x_{S+(k+1)R}) < 1/10. \]
Since \( y^k \) is in \( L \), \( \pi(y^k_R) - \pi(y^k) \geq 1 \). Therefore \( Q_k \geq 8/10 \) and \( P_n \geq 8/10n \), and the conclusion of the lemma follows.

In proving 2.11, we used the fact that every orbit in \( L \) tended to the same limit, \( +\infty \) or \( -\infty \), that \( L \) was the product of a compact set, \( S^1 \), and \( T \), and that the flow on \( L \) covered a flow on \( L = S^1 \times S^1 \). As a factor of \( L \), we used no property of \( S^1 \) other than its compactness. Therefore the proof of 2.11 may be repeated to obtain

3.3. LEMMA. If \( L \) contains no closed orbit, and \( \Omega_x \subset L \) for each \( x \) in \( K \); then there exists a function, \( \omega : T^+ \rightarrow T^+ \), such that for each \( x \) in \( K \),
\[ |\pi(x_s) - \pi(x)| \geq t \quad \text{if} \quad s \geq \omega(t). \quad (3.4) \]
Assuming \( x_t \rightarrow +\infty \) for each \( x \) in \( K \), we have the following corollary:

3.5. COROLLARY. If \( L \) contains no closed orbit and \( \Omega_x \subset L \) for each \( x \) in \( K \) then
\[ \pi(x_{\omega(t)}) - \pi(x) \geq 1 \quad \text{and} \quad \pi(x_{-\omega(t)}) - \pi(x) \leq -1 \]
for each \( x \) in \( K \).

Our next aim is to construct a global cross section of \( K \). In [3], Montgomery and Zippin showed that under certain conditions, a flow in Euclidean space has such a cross section. They employed the existence of local cross sections, as proved by Whitney, [6], in their proof. Rather than show that the necessary conditions are present, we shall derive the existence of a global cross section directly. However, the spirit of the derivation owes much to the above-mentioned authors.

We first prove

3.6. LEMMA. There exists a continuous function \( W : K \rightarrow T \) such that for any \( x \) in \( K \) and \( h > 0 \), \( W(x_h) = W(x) \geq h \).
Proof. Let \( w = \omega(1) \). Define \( W : \mathbb{R} \rightarrow T \) as follows:

\[
W(x) = \int_0^w \pi(x_s) \, ds.
\]

Then for \( h > 0 \) we have

\[
W(x_A) - W(x) = \int_h^{w+h} \pi(x_s) \, ds - \int_0^w \pi(x_s) \, ds,
\]

\[
= \int_0^h (\pi(x_{w+s}) - \pi(x_s)) \, ds \geq h.
\]

The continuity of \( W \) follows from that of \( \phi \). The lemma is proved.

3.7. Notation. \( \Sigma = \{ x \mid W(x) = 0 \} \).

3.8. Lemma. (i) \( \Sigma \) is bounded in \( \mathcal{K} \). (ii) For each orbit, \( x_T, \Sigma \cap x_T \) consists of one point, \( a(x) \). (iii) \( a : \mathcal{K} \rightarrow \Sigma \) is continuous.

Proof. If \( R = \max \{ |\pi(\xi_1) - \pi(\xi)| \mid |t| \leq w \} \) then \( |\pi(x)| \leq R \) for \( x \) in \( \Sigma \) which implies (i).

From (3.4) it follows that for each \( x \), \( f(t) = W(x_t) \) is unbounded from above or below. Thus, since \( f \) is continuous, each \( f(t) = 0 \) for some \( t \) which implies \( \Sigma \cap x_T \) is not empty. On the other hand, since \( f \) is a strictly increasing function, \( \Sigma \cap x_T \) contains but one point. Thus (ii) is valid.

To prove the continuity of \( \sigma \), we first consider \( \psi : \mathcal{K} \rightarrow T \) defined by \( x_{\psi(x)} = a(x) \), which is equivalent to \( W(x_{\psi(x)}) = 0 \). Now, let \( \varepsilon > 0 \) and \( \bar{x} \) in \( \mathcal{K} \) be given. We then have

\[
W(x_{\psi(x)} - \varepsilon) < 0 < W(x_{\psi(x)} + \varepsilon).
\]

Thus, for \( y \) sufficiently close to \( \bar{x} \),

\[
W(y_{\psi(x) - \varepsilon}) < 0 < W(y_{\psi(x) + \varepsilon}),
\]

so that

\[
\psi(\bar{x}) - \varepsilon < \psi(y) < \psi(\bar{x}) + \varepsilon,
\]

which establishes the continuity of \( \psi \). Since \( \sigma(x) = x_{\psi(x)} \), \( \sigma \) must be continuous. This completes the proof of 3.8.

3.9. Corollary. \( \Sigma \) has the fixed point property, since \( \Sigma \) is a retract of \( \{ x \in \mathcal{K} \mid -R \leq \pi(x) \leq R \} \) under \( \sigma \).
We are now ready to complete the proof of 3.1. Suppose \( \Omega_x \subset L \) for each \( x \) in \( K \) and \( L \) contains no closed orbit. Let \( \Sigma \) be as above, let \( S \) be an integer greater than \( 2R + 1 \), and let \( \Sigma' = \{ x + S | x \in \Sigma \} \). Thus, for any \( x \) in \( \Sigma' \) and \( y \) in \( \Sigma \) we have
\[
\pi(x) > -R + 2R + 1 > \pi(y),
\]
so that \( \Sigma \cap \Sigma' \) is empty. Now let \( h : \Sigma' \to \Sigma' \) be defined by
\[
h(x) = \sigma(x) + S = x_{\psi(x)} + S,
\]
where we note that \( \psi(x) \not= 0 \). According to 3.9, for some \( \dot{x} \) in \( \Sigma' \) \( h(\dot{x}) = \dot{x} \). Thus, \( \dot{x}_{\psi(\dot{x})} = \dot{x} - S \), which implies \( [p(\dot{x})]_{\psi(\dot{x})} = p(\dot{x}) \), so that \( p(\dot{x})_T \) is closed. Since \( p(\dot{x})_T = \Omega_{p(\dot{x})} \subset L \), this contradicts our assumption and the theorem is proved.

REFERENCES