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On a Singular Boundary Value Problem for the Euler-Darboux Equation

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1.

The Euler–Darboux Equation

$$L_\mu[v] = \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial r^2} - \frac{2\mu}{r}\frac{\partial v}{\partial r} = 0, \tag{1.1}$$

where μ is a real parameter, is a classic example of a linear hyperbolic equation of the second order with a singular line. The presence of this line poses certain questions that are beyond the scope of the standard theory. This can be seen by examining its Riemann function. For this purpose it is more convenient to work with the self-adjoint equation

$$M_\mu[w] = r^\mu L_\mu[r^{-\mu}w] = \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial r^2} + \frac{\mu(\mu - 1)}{r^2} w = 0, \tag{1.2}$$

which is obtained from (1.1) by the substitution

$$w = r^\mu v. \tag{1.3}$$

Its Riemann function is a solution $R = R(r_0, t_0; r, t)$ which reduces to unity on the two characteristics $t - t_0 = \pm(r - r_0)$ which pass through the pole of the Riemann function, (r_0, t_0). This was determined by Darboux [3] who showed that

$$R = (1-\sigma)^\mu F(\mu, \mu; 1; \sigma), \qquad \sigma = \frac{(r - r_0)^2 - (t - t_0)^2}{(r + r_0)^2 - (t - t_0)^2}\,. \tag{1.4}$$

Now σ is infinite on the two characteristics $t - t_0 = \pm(r + r_0)$ which are the reflections of the characteristics through the pole in the singular line.

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The Riemann function is determined uniquely in any sufficiently small neighborhood of the pole, and so the fact that (1.4) satisfies (1.2) and reduces to unity on $t - t_0 = \pm(r - r_0)$, where $\sigma = 0$, guarantees that (1.4) is the Riemann function of (1.2) in $|t - t_0| < r + r_0$ if, for instance, $r_0 > 0$ and $r > 0$. But it cannot be expected that (1.4) holds when $|t - t_0| > r + r_0$; in fact, the Riemann function is then not defined at all.

This behavior of the Riemann function is immaterial when a solution of (1.2) is determined by Cauchy data on a line $r = c$ where, say, $c > 0$. For then (1.2) can be solved in $r > 0$ by Riemann's method. (In general, the solution becomes infinite on the singular line and cannot be continued beyond it). But there are situations in which a continuation of the Riemann function into $|t - t_0| > r + r_0$ is required. An instance of this is found in the theory of the diffraction of transient waves by a wedge [5]. Following Sommerfeld's original approach to the diffraction of harmonic wave trains, this diffraction problem can be reduced to the construction of many-valued solutions of the two-dimensional wave equation

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \tag{1.5}$$

that are, considered as functions of the polar coordinates r and $\theta(X = r\cos\theta$, $Y = r\sin\theta)$, periodic in θ with period 2α where α is the exterior angle of the wedge. When such a solution is expanded as a Fourier series in θ, a typical term is $U = u(r, t)\cos(\lambda\theta + \beta)$, where β is an arbitrary constant and λ is an integral multiple of π/α. Hence u satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{\lambda^2}{r^2}u \tag{1.6}$$

which reduces to (1.2) if one puts $u = r^{-1/2}w$, $\lambda = \mu - \frac{1}{2}$. The basic problem is then the initial value problem, where u and $\partial u/\partial t$ are given for $t = 0, r > 0$. These data must be supplemented by a regularity condition at $r = 0$, which corresponds to the edge condition in the diffraction problem. The problem must therefore be considered as a singular mixed boundary value problem. For $t < r$, the solution can be obtained by Riemann's method, using Darboux's Riemann function (1.4). This gives nothing new, as the same result can be deduced by the method of descent from the classical solution of the initial value problem for (1.5). The key to the solution is therefore precisely the continuation of the Riemann function into the domain $r > 0$, $|t - t_0| > r + r_0$. In [5], such a continuation was proposed, and was shown to lead to the known Green's function of the wedge. However, the argument by which it was derived is invalid, as will be explained below. The same continuation was obtained by Copson [1], by means of integral transforms, without explicit reference to the singular mixed boundary value problem.

In the present paper, we shall discuss the following singular mixed boundary value problem for the Euler–Darboux equation (1.1): Given v and v_t (partial derivatives will sometimes be denoted by subscripts) for $t = 0$, $r \geqslant 0$, find a solution of the equation in $r > 0$, $t > 0$ such that v_r and v_t remain bounded as $r \to 0$. It will be assumed that $\mu \geqslant \frac{1}{2}$. The case $\mu < \frac{1}{2}$ can be reduced to this by means of the well-known relation

$$L_\mu[r^{1-2\mu}v] = r^{1-2\mu}L_{1-\mu}[v],$$

but will not be considered here. The solution of the problem will be effected by a generalization of Riemann's method. Its relation to the classical Poisson representation of the solutions of the Euler–Darboux equation, and to another integral representation due to Volterra, has also been recently investigated by us, and will be published at a later date.

A useful guide to the general case is the case $\mu = \frac{1}{2}$. For v is then an axisymmetric solution of the two-dimensional wave equation (1.5), and can be derived by the method of descent from the solution of the initial value problem for this equation. Let us suppose, for simplicity, that the initial conditions are

$$v(r, 0) = 0, \qquad v_t(r, 0) = g(r). \tag{1.7}$$

The solution of the initial value problem for the wave equation is then (see e.g. [7], p. 50)

$$v(r_0\,, t_0) = \frac{1}{2\pi}\iint \frac{g(r)}{\{t_0^2 - (X - X_0)^2 - (Y - Y_0)^2\}^{1/2}}\, dX\, dY, \tag{1.8}$$

where $r_0 = (X_0^2 + Y_0^2)^{1/2}$, and the integral is taken over the circle $(X - X_0)^2 + (Y - Y_0)^2 \leqslant t_0^2$. It is known that (1.8) satisfies the wave equation and the initial conditions (1.7) if $g(r)$, as a function of X and Y, has first-order derivatives which satisfy Lipschitz conditions. This will be the case if and only if $g'(r)$ exists, satisfies a Lipschitz condition, and $g'(0) = 0$. Now (1.8) can be written as

$$v(r_0\,, t_0) = \int_0^\infty K(r_0\,, t_0\,; r)\, g(r)\, dr, \tag{1.9}$$

where

$$K(r_0\,, t_0\,; r) = \frac{r}{\pi}\int (t_0^2 - r_0^2 - r^2 + 2rr_0 \cos\theta)^{-1/2}\, d\theta \tag{1.10}$$

and the integral is taken over the subinterval of $(0, \pi)$ in which

$$t_0^2 - r_0^2 - r^2 + 2rr_0 \cos\theta > 0.$$

Let us put

$$\zeta = \frac{t_0^2 - (r - r_0)^2}{4rr_0}. \tag{1.11}$$

Then

$$K(r_0, t_0; r) = \frac{1}{2\pi}\left(\frac{r}{r_0}\right)^{1/2} \int \left(\zeta + \frac{1}{2}\cos\theta - \frac{1}{2}\right)^{-1/2} d\theta. \tag{1.12}$$

If $t_0 < |r - r_0|$, then $\zeta < 0$, so that the interval of integration is empty, and hence $K = 0$. If $|r - r_0| < t_0 < r + r_0$, then $0 < \zeta < 1$, so that one can put $\cos\theta = 1 - 2\zeta z$ where $0 < z < 1$. Then (1.12) becomes

$$\begin{aligned} K &= \frac{1}{2\pi}\left(\frac{r}{r_0}\right)^{1/2} \int_0^1 z^{-1/2}(1 - z)^{-1/2}(1 - \zeta z)^{-1/2}\, dz \\ &= \frac{1}{2}\left(\frac{r}{r_0}\right)^{1/2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \zeta\right). \end{aligned} \tag{1.13}$$

If, finally, $t_0 > r + r_0$, then $\zeta > 1$, and one can put $\cos\theta = 1 - 2z$ to obtain

$$\begin{aligned} K &= \frac{1}{2\pi}\left(\frac{r}{r_0}\right)^{1/2} \int_0^1 z^{-1/2}(1 - z)^{-1/2}(\zeta - z)^{-1/2}\, dz \\ &= \frac{1}{2}\left(\frac{r}{r_0}\right)^{1/2} \zeta^{-1/2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \zeta^{-1}\right). \end{aligned} \tag{1.14}$$

Hence

$$v(r_0, t_0) = \frac{1}{2}\int_{r_0 - t_0}^{r_0 + t_0} \left(\frac{r}{r_0}\right)^{1/2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \zeta\right) g(r)\, dr, \qquad 0 \leqslant t_0 \leqslant r_0, \tag{1.15}$$

and

$$\begin{aligned} v(r_0, t_0) &= \frac{1}{2}\int_0^{t_0 - r_0} \left(\frac{r}{r_0}\right)^{1/2} \zeta^{-1/2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \zeta^{-1}\right) g(r)\, dr \\ &\quad + \frac{1}{2}\int_{t_0 - r_0}^{t_0 + r_0} \left(\frac{r}{r_0}\right)^{1/2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \zeta\right) g(r)\, dr, \qquad t_0 \geqslant r_0. \end{aligned} \tag{1.16}$$

Now $w = r^{1/2}v$ satisfies $M_{1/2}[w] = 0$, and $w = 0$, $w_t = r^{1/2}g(r)$ when $t = 0$.

The Riemann representation of the solution of this initial value problem is

$$\begin{aligned} w(r_0, t_0) &= r_0^{1/2} v(r_0, t_0) \\ &= \frac{1}{2} \int_{r_0-t_0}^{r_0+t_0} R(r_0, t_0; r, 0)\, r^{1/2} g(r)\, dr, \qquad (0 \leqslant t_0 \leqslant r_0) \end{aligned} \tag{1.17}$$

where, by (1.4),

$$R(r_0, t_0; r, 0) = (1 - \sigma_0)^{1/2} F(\tfrac{1}{2}, \tfrac{1}{2}; 1; \sigma_0),$$

$$\sigma_0 = \frac{(r - r_0)^2 - t_0^2}{(r + r_0)^2 - t_0^2}.$$

This is, of course, identical with (1.15). For $\sigma_0 = \zeta/(\zeta - 1)$, and so, by a well-known property of the hypergeometric series,

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; \zeta\right) = (1 - \zeta)^{-1/2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\zeta}{\zeta - 1}\right) = R(r_0, t_0; r, 0).$$

We can therefore conclude from (1.16) that the continuation of the Riemann function into $| t - t_0 | > r + r_0$, which is appropriate for our singular mixed boundary value problem, is

$$\bar{R} = Z^{-1/2} F\left(\frac{1}{2}, \frac{1}{2}; 1; Z^{-1}\right), \qquad Z = \frac{(t - t_0)^2 - (r - r_0)^2}{4 r r_0}. \tag{1.18}$$

For $\zeta = 1$, both integrals (1.13) and (1.14) are divergent. This corresponds to the fact that both R and $\bar{R}$ become infinite as $Z \to 1$. But it can be shown that $\pi R - \log(1 - Z)$ and $\pi \bar{R} - \log(Z - 1)$ remain bounded. Thus the continuation (1.18) of the Riemann function has the following properties:

(i) It is a function of Z only

(ii) It vanishes like $r^{1/2}$ as $r \to 0$

(iii) $R/\log(1 - Z)$ and $\bar{R}/\log(Z - 1)$ tend to the same limit as $Z \to 1$.

We shall see that by postulating similar properties, an analogous continuation of the Riemann function can be constructed in the general case.

2.

We begin with a uniqueness theorem.

THEOREM 1. *Let v be a solution of $L_\mu[v] = 0$, $(\mu > 0)$, which is of class C^2 in $r > 0$. If $v = 0$, $v_t = 0$ for $t = 0$, $0 \leqslant r \leqslant c$, and v_r and v_t remain* *bounded as* $r \to 0$ *uniformly in* t *for* $0 \leqslant t \leqslant c$, *then* $v = 0$ *in* $r \geqslant 0$, $t \geqslant 0$, $r + t \leqslant c$.

It is well known that the uniqueness of the solution of the initial value problem can be proved by means of *a priori* integral estimates of energy type ([*2*], p. 441). This method can also be applied to the singular mixed boundary value problem. By (1.1), we have

$$2v_t r^{2\mu} L_\mu[v] = \frac{\partial}{\partial t}\{r^{2\mu}(v_r{}^2 + v_t{}^2)\} - 2\frac{\partial}{\partial r}(r^{2\mu} v_r v_t) = 0. \qquad (2.1)$$

Let us integrate this identity over the domain bounded by $t = 0$, $t + r = c$, $t = t_0$, and by $r = \epsilon$, where $0 < \epsilon < c$ and $0 < t_0 < c - \epsilon$ (Fig. 1). Since $v = 0$ and $v_t = 0$ on $t = 0$, and $dr = -dt$ on $t + r = c$, we obtain

$$\int_\epsilon^{c-t_0} r^{2\mu}[v_r{}^2 + v_t{}^2]_{t=t_0}\, dr + \int_{c-t_0}^{c} r^{2\mu}[v_r{}^2 + v_t{}^2]_{t=c-r}\, dr$$

$$- 2\int_0^{t_0} [r^{2\mu} v_r v_t]_{r=\epsilon}^{r=c-t}\, dt = 0.$$

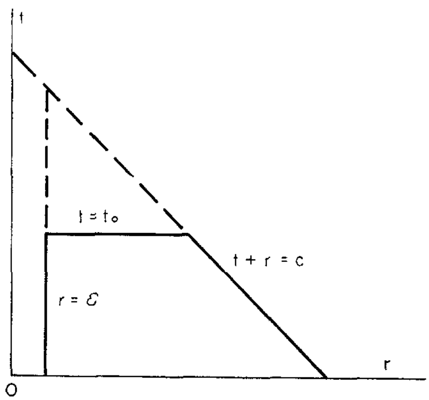

FIG. 1.

It follows from the assumption that v_r and v_t remain bounded uniformly in t as $r \to 0$, that we can make $\epsilon \to 0$, and so

$$\int_0^{c-t_0} r^{2\mu}[v_r{}^2 + v_t{}^2]_{t=t_0}\, dr = -\int_{c-t_0}^{c} r^{2\mu}[v_r - v_t]^2_{t=c-r}\, dr \leqslant 0, \qquad (2.2)$$

for $0 < t_0 < c$. Since $v \in C^2$ in $r > 0$, we therefore have $v_r = 0$, $v_t = 0$ in the triangle bounded by the axis and by the characteristic $t + r = c$.

Also, since $v = 0$ on $t = 0$, it follows that $v = 0$ in this triangle This. proves the theorem.

3.

. The elegant argument by which Darboux determined the Riemann function of (1.2) can be put into the following form: In terms of the characteristic variables

$$x = \tfrac{1}{2}(r + t), \qquad y = \tfrac{1}{2}(t - r)$$

the equation becomes

$$\frac{\partial^2 w}{\partial x\, \partial y} + \frac{\mu(\mu - 1)}{(x - y)^2} w = 0. \tag{3.2}$$

Since this equation is invariant under the transformation $x \to kx$, $y \to ky$, where k is a constant, it has solutions which are functions of y/x only. For our purpose, it is convenient to write these as

$$w = F(z), \qquad z = \frac{x}{x - y}.$$

A simple calculation then shows that $F(z)$ must satisfy the equation

$$z(1 - z) F''(z) + (1 - 2z) F'(z) + \mu(\mu - 1) F(z) = 0,$$

which is obtained from Gauss' equation

$$z(1 - z) F''(z) + [c - (a + b + 1) z] F'(z) - abF(z) = 0$$

by setting $a = \mu$, $b = 1 - \mu$, $c = 1$. But (3.2) is also invariant under the transformations

$$x \to \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad y \to \frac{\alpha y + \beta}{\gamma y + \delta},$$

where α, β, γ and δ are constants such that $\alpha\delta - \beta\gamma \neq 0$. Taking $\alpha = 1$, $\beta = -x_0$, $\gamma = 1$ and $\delta = -y_0$, where $x_0 \neq y_0$, it follows that (3.2) has the solutions

$$w = F(Z) \tag{3.3}$$

where

$$Z = \frac{\dfrac{x - x_0}{x - y_0}}{\dfrac{x - x_0}{x - y_0} - \dfrac{y - x_0}{y - y_0}} = \frac{(x - x_0)(y - y_0)}{(x_0 - y_0)(x - y)} = \frac{(t - t_0)^2 - (r - r_0)^2}{4rr_0}, \tag{3.4}$$

and $r_0 = x_0 - y_0$, $t_0 = x_0 + y_0$, provided that $F(Z)$ is a solution of

$$Z(1 - Z)F''(Z) + (1 - 2Z)F'(Z) + \mu(\mu - 1)F(Z) = 0. \tag{3.5}$$

Now the Riemann function of (3.2) with the pole (x_0, y_0) is the solution which reduces to unity when $x = x_0$ and when $y = y_0$, that is to say when $Z = 0$. Hence

$$R = F(\mu, 1 - \mu; 1; Z). \tag{3.6}$$

This is also the Riemann function of $M_\mu(w] = 0$, with the pole (r_0, t_0). It follows from the relation

$$F(a, b; c; z) = (1 - z)^{-a} F\left(a, c - b; c; \frac{z}{z - 1}\right), \tag{3.7}$$

and from $Z = \sigma/(\sigma - 1)$, that (3.6) is equivalent to Darboux' result (1.4).

When μ is a positive integer, R is a polynomial in Z of degree $\mu - 1$. It is, in fact, the Legendre polynomial of order $\mu - 1$ with argument $1 - 2Z$. In that case, R is constant on the characteristics $| t - t_0 | = r + r_0$, and we define its continuation $\bar{R}$ simply by setting $\bar{R} = 0$. We shall refer to this as the *exceptional case*.

When μ is not a positive integer, the behavior of R near $| t - t_0 | = r + r_0$, where $Z = 1$, can be inferred from the continuation formula ([*4*], p. 110)

$$F(a, b; a + b; z) = \frac{\Gamma(a + b)}{\Gamma(a)\,\Gamma(b)} \Big\{ F(a, b; 1; 1 - z) \log \frac{1}{1 - z}$$
$$+ \sum_{n=0}^{\infty} [2\psi(n + 1) - \psi(a + n) - \psi(b + n)] \frac{(a)_n (b)_n}{(n!)^2} (1 - z)^n \Big\}. \tag{3.8}$$

Here ψ denotes the logarithmic derivative of the Gamma function, and $(a)_n = a(a + 1) \ldots (a + n - 1)$. Applied to (3.6), this gives

$$R = \frac{\sin \pi\mu}{\pi} \Big\{ F(\mu, 1 - \mu; 1; 1 - Z) \log \frac{1}{1 - Z} + F_1(Z) \Big\}, \tag{3.9}$$

where

$$F_1(Z) = \sum_{n=0}^{\infty} [2\psi(n + 1) - \psi(\mu + n) - \psi(1 - \mu + n)] \frac{(\mu)_n (1 - \mu)_n}{(n!)^2} (1 - Z)^n \tag{3.10}$$

is a regular function of Z in $| 1 - Z | < 1$.

In order to obtain the continuation of R into $| t - t_0 | > r + r_0$, where

$Z > 1$, we observe that (3.5) has, in the neighborhood of $Z = \infty$, the solutions

$$w_1 = Z^{-\mu}F(\mu, \mu; 2\mu; Z^{-1}), \tag{3.11a}$$

$$w_2 = \begin{cases} Z^{\mu-1}F(1-\mu, 1-\mu; 2-2\mu; Z^{-1}), & \mu > \tfrac{1}{2}, \\ Z^{-1/2}[F(\tfrac{1}{2}, \tfrac{1}{2}; 1; Z^{-1}) \log Z + F^*(Z)], & \mu = \tfrac{1}{2}, \end{cases} \tag{3.11b}$$

where F^* is regular at $Z = \infty$. Now $r^{-\mu}w_1$ is a solution of $L_\mu(v] = 0$ that remains bounded as $r \to 0$, except when $|t - t_0| = |r - r_0|$. But $r^{-\mu}w_2$ behaves like $r^{1-2\mu}$ as $r \to 0$ (or like $\log r$ when $\mu = \frac{1}{2}$), and since we are assuming that $\mu \geqslant \frac{1}{2}$, this solution must be rejected. Hence we are led to assume a continuation of the Riemann function that is of the form

$$\bar{R} = CZ^{-\mu}F(\mu, \mu; 2\mu; Z^{-1}), \tag{3.12}$$

where C is a constant. By (3.8), we then have

$$\bar{R} = C\frac{\Gamma(2\mu)}{[\Gamma(\mu)]^2} Z^{-\mu} \Big\{ F(\mu, \mu; 1; 1 - Z^{-1}) \log \frac{Z}{Z-1} + 2\sum_{r=0}^{\infty} [\psi(n+1) - \psi(\mu+n)] \Big[\frac{(\mu)_n}{n!}\Big]^2 \Big(1 - \frac{1}{Z}\Big)^n \Big\}. \tag{3.13}$$

Following the case $\mu = \frac{1}{2}$, we now choose C so that $R/\log(1 - Z)$ and $\bar{R}/\log(Z - 1)$ tend to the same limit as $Z \to 1$. This gives

$$C = \frac{[\Gamma(\mu)]^2}{\Gamma(2\mu)} \frac{\sin \pi\mu}{\pi} = \frac{\pi^{1/2}}{2^{2\mu-1}\Gamma(\mu + 1/2)\,\Gamma(1-\mu)}, \tag{3.14}$$

where the second form follows from the duplication formula for the Gamma function. (This value of C also gives $\bar{R} = 0$ in the exceptional case). By (3.7),

$$F(\mu, \mu; 1; 1 - Z^{-1}) = Z^{\mu}F(\mu, 1 - \mu; 1; 1 - Z).$$

Hence it follows from (3.13) and (3.14) that

$$\bar{R} = \frac{\sin \pi\mu}{\pi} \Big\{ F(\mu, 1 - \mu; 1; 1 - Z) \log \frac{1}{Z-1} + F_2(Z) \Big\}, \tag{3.15}$$

where F_2 is C^∞, provided that $|Z - 1| < 1$. It is evident from (3.9) that $F_1(Z) - F(\mu, 1 - \mu; 1; 1 - Z) \log(Z - 1)$ is a solution of the hypergeometric equation (3.5). One can therefore express $\bar{R}$ as a linear combination of this function and of $F(\mu, 1 - \mu; 1; 1 - Z)$, which is a solution since the equation is invariant under $Z \to 1 - Z$. The constant factors can be determined by means of (3.13), and one finds that

$$F_2 - F_1 = \pi \cot \pi\mu F(\mu, 1 - \mu; 1; 1 - Z)].$$

The value (3.14) of the constant C was used by one of the authors in the wedge diffraction problem ([5], p. 114, equation 5.3.12), but the argument given there to justify this choice is irrelevant. The function equal to

$$r^{-1/2}F(\mu, 1 - \mu; 1; Z)\cos(\mu - \tfrac{1}{2})\,\theta \tag{3.16}$$

for $|r - r_0| < t - t_0 < r + r_0$, and equal to zero for $t - t_0 > r + r_0$, is a solution of the wave equation (1.5) that is discontinuous at the converging circular wave front $t - t_0 = r_0 - r$. This discontinuity is focused at $r = 0$, and emerges as a logarithmic singularity on $t - t_0 = r + r_0$. It was argued that the coefficient of $\log |1 - Z|$ must be continuous at this front, because of certain results on focusing [5], p. 67). But these do not apply to the focusing of many-valued solutions; they do not, for instance, account for the factor $\sin \pi\mu/\pi$. The correct argument is as follows: Suppose that (3.16) is continued into $t - t_0 > r + r_0$ as

$$Cr^{-1/2}Z^{-\mu}F(\mu, \mu; 2\mu; Z^{-1})\cos(\mu - \tfrac{1}{2})\,\theta. \tag{3.17}$$

One then has, in the neighborhood of the singularity front, a solution of the wave equation which is of the form

$$r^{-1/2}\{A(r, t)\log|t - t_0 - r - r_0| + B(r, t)\}\cos(\mu - \tfrac{1}{2})\,\theta,$$

where both A and B are discontinuous at $t - t_0 = r + r_0$. Such a function is admissible, say in the sense of the theory of distributions, provided that the discontinuities of both $r^{-1/2}A\cos(\mu - \tfrac{1}{2})\,\theta$ and $r^{-1/2}B\cos(\mu - \tfrac{1}{2})\,\theta$ satisfy the appropriate transport equations. For a circular front, these imply that the discontinuities of A and B must be constant. Now it follows from (3.9) and (3.15) that this is the case for the coefficient of the logarithm, A, but that B jumps by an amount

$$\left\{C\frac{\Gamma(2\mu)}{[\Gamma(\mu)]^2} - \frac{\sin \pi\mu}{\pi}\right\}\log\frac{r}{r + r_0} + \text{const.}$$

Hence the condition that the continuation of the Riemann function should correspond to a weak solution of the wave equation at once leads to the value (3.14) of C. The same argument can be applied to $M_\mu[w] = 0$, and is in fact the essential content of the lemma which will be proved in the next section.

4.

An elementary solution of $M_\mu[w] = 0$ is a distribution G which satisfies the equation

$$M_\mu[G] = \delta(r - r_0)\,\delta(t - t_0) \tag{4.1}$$

where δ denotes the Dirac δ-function. Since we are working in $r > 0$, G must be in the space dual to the space of test functions which are indefinitely differentiable, and have compact supports which are contained in $r > 0$. This will be denoted by $\mathscr{D}^+$.

In the nonsingular case, the elementary solutions of linear hyperbolic second-order equations with two independent variables are functions. We will therefore assume that G is a function. Then (4.1) is equivalent to the identity

$$\iint_{r>0} GM_\mu[\phi]\,dr\,dt = \phi(r_0, t_0), \qquad \phi \in \mathscr{D}^+, \tag{4.2}$$

since M_μ is self-adjoint. It is evident that the behavior of G on the singular line $r = 0$ is irrelevant to the validity of (4.2). However, we may heuristically define a Green's function of our singular mixed boundary value problem to be an elementary solution G such that $r^{-\mu}G$ remains bounded almost everywhere as $r \to 0$. We shall now prove the following:

LEMMA. *The two functions*

$$G(r_0, t_0; r, t) = \begin{cases} 0, & t > t_0 - |r - r_0|, \\ \tfrac{1}{2}R, & t_0 - |r - r_0| > t > t_0 - r_0 - r, \\ \tfrac{1}{2}\bar{R}, & t_0 - r_0 - r > t, \end{cases} \tag{4.3}$$

and

$$G^*(r_0, t_0; r, t) = G(r, t; r_0, t_0) \tag{4.4}$$

are Green's functions of M_μ in $r > 0$.

We have already determined $\bar{R}$ by the condition that $r^{-\mu}\bar{R}$ remains bounded almost everywhere as $r \to 0$. It therefore only remains to prove that (4.2) holds for both G and G^*. This can be done by a variant of Riemann's method. If ϕ and ψ are two functions of class C^2, then

$$\psi M_\mu[\phi] - \phi M_\mu[\psi] = \frac{\partial}{\partial t}(\psi\phi_t - \phi\psi_t) - \frac{\partial}{\partial r}(\psi\phi_r - \phi\psi_r).$$

Integrating this over a bounded domain ω with piecewise smooth boundary

$\partial\omega$, oriented so that ω is on the left of $\partial\omega$, we obtain the usual integral identity

$$\iint_\omega \{\psi M_\mu[\phi] - \phi M_\mu[\psi]\}\, dr\, dt$$
$$= \int_{\partial\omega} \{(\phi\psi_t - \psi\phi_t)\, dr + (\phi\psi_r - \psi\phi_r)\, dt\}. \tag{4.5}$$

We shall take $\phi \in \mathscr{D}^+$ and $\psi = 2G$. The exceptional case $\mu = k$, $k = 1, 2, \ldots$, is easily disposed of. Let ω be the rectangle bounded by the characteristics $t + r = t_0 + r_0$, $t - r = t_0 - r_0$, $t + r = t_0 - r_0$, and by a characteristic $r - t = c$, where c is a positive constant. The vertices of this rectangle are denoted by A, B, C, and P in Fig. 2. Since ϕ has compact support, $r - t > c$

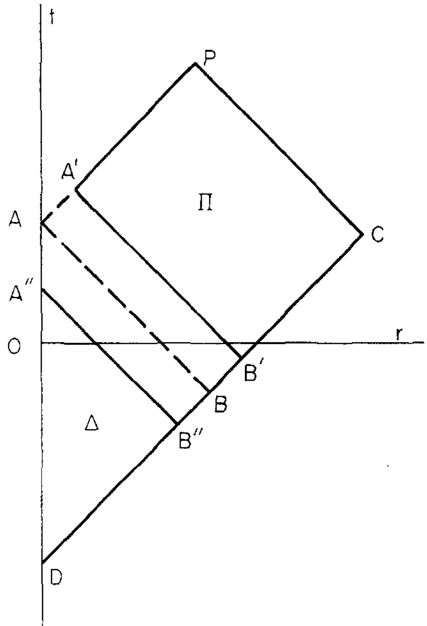

FIG. 2.

will be in the complement of the support of ϕ if c is sufficiently large, and we shall have $\phi = 0$ on BC. As the support of ϕ is in $r > 0$, there exists an $a > 0$ such that $\phi = 0$ in $0 \leqslant r \leqslant a$, so that there are no convergence difficulties at $r = 0$. Then (4.5) becomes

$$\iint_\omega RM_\mu[\phi]\, dr\, dt = \int_C^P (Rd\phi - \phi dR) + \int_P^A (\phi dR - Rd\phi)$$
$$+ \int_A^B (Rd\phi - \phi dR),$$

because $2G = R$ in ω, $dr = dt$ on AP, and $dr = -dt$ on CP, AB. Furthermore, $R = 1$ on CP and on PA, and $R = F(k, 1-k; 1; 1) =$ const. on AB. Hence

$$\iint_\omega RM_\mu[\phi]\, dr\, dt = 2\phi(r_0, t_0),$$

taking into account that $\phi = 0$ at A, B and C. But this is (4.2) for the exceptional case, when $\bar{R} = 0$, by definition.

When μ is not a positive integer, we must exclude the singular line $t + r = t_0 - r_0$. We therefore take ω to consist of a rectangle Π bounded by the characteristics through $P(r_0, t_0)$, by $t + r = t_0 - r_0 + 2\epsilon$ where $\epsilon > 0$, by $r - t = c$, of a triangle Δ bounded by $r - t = c$, $t + r = t_0 - r_0 - 2\epsilon$, and by $r = 0$. The vertices of Π are denoted by P, A', B' and C in Fig. 2, and those of Δ by A'', B'', and D. We can again take c large enough to ensure that $\phi = 0$ on $r - t = c$, and we shall have $\phi = 0$ at A' when ϵ is sufficiently small, since the support of ϕ is in $r > 0$. Since G is defined by (4.3), we have $2G = R$ in and $2G = \bar{R}$ in Δ, so that (4.5) becomes

$$\begin{aligned}\iint_\Pi RM_\mu[\phi]\, dr\, dt + \iint_\Delta \bar{R}M_\mu[\phi]\, dr\, dt \\ = \int_C^P (Rd\phi - \phi dR) + \int_P^{A'} (\phi dR - Rd\phi) \\ + \int_{A'}^{B'} (Rd\phi - \phi dR) - \int_{A''}^{B''} (\bar{R}d\phi - \phi d\bar{R}),\end{aligned} \tag{4.6}$$

in view of the fact that $dr = dt$ on $A'P$, and $dr = -dt$ on PC, $A'B'$, and $A''B''$; there is no contribution from $A''D$, since ϕ vanishes in a neighborhood of the singular line. As $R = 1$ on CP, PA', and $\phi = 0$ both at C and at A' (provided that ϵ is small enough),

$$\int_C^P (Rd\phi - \phi dR) + \int_P^{A'} (Rd\phi - \phi dR) = 2\phi(r_0, t_0). \tag{4.7}$$

The other two integrals in the second member of (4.6) can be transformed by partial integration into

$$2\int_{A'}^{B'} Rd\phi - 2\int_{A''}^{B''} \bar{R}d\phi. \tag{4.8}$$

Now on $A'B'$ one has

$$Z = Z' = 1 - \frac{(r + r_0 - \epsilon)\epsilon}{rr_0} \tag{4.9a}$$

and on $A''B''$,

$$Z = Z'' = 1 + \frac{(r + r_0 + \epsilon)\epsilon}{rr_0}. \tag{4.9b}$$

As ϕ vanishes in a neighborhood of $r = 0$, r will be bounded away from zero in both of the integrals (4.8), and so $Z' \to 1 -$ in the first integral, $Z'' \to 1 +$ in the second one, both uniformly in r. We can therefore substitute the second members of (3.9) and (3.15) in (4.8), which then takes the form

$$\begin{aligned} &\frac{2 \sin \pi\mu}{\pi} \int_{A'}^{B'} \left\{\log \frac{1}{1 - Z'} + H_1(Z')\right\} d\phi \\ &\frac{-2 \sin \pi\mu}{\pi} \int_{A''}^{B''} \left\{\log \frac{1}{Z'' - 1} + H_2(Z'')\right\} d\phi, \end{aligned} \tag{4.10}$$

where H_1 is continuous in $Z' \leqslant 1$, and H_2 is continuous in $Z'' \geqslant 1$. As $\epsilon \to 0$, the contributions from H_1 and H_2 tend to

$$\frac{2 \sin \pi\mu}{\pi} \{H_1(1) - H_2(1)\} \int_A^B d\phi = 0.$$

The logarithmic terms are, by (4.9a) and (4.9b),

$$\begin{aligned} \frac{2 \sin \pi\mu}{\pi} \int_0^\infty \Big\{&\log \frac{rr_0}{(r + r_0 - \epsilon)\epsilon}\, d\phi(r, t_0 - r_0 - r + 2\epsilon) \\ &- \log \frac{rr_0}{(r + r_0 + \epsilon)\epsilon}\, d\phi(r, t_0 - r_0 - r - 2\epsilon)\Big\}, \end{aligned}$$

where the integration is in effect over a finite interval $a \leqslant r \leqslant b$, $a > 0$, because of the properties of ϕ. Since $\phi \in C^\infty$, it is evident that this also tends to zero as $\epsilon \to 0$. We can therefore make $\epsilon \to 0$ in (4.6), and find by (4.7) that the limit of the second member is $2\phi(r_0, t_0)$. The limit of the first member is

$$2 \iint_{r>0} G(r_0, t_0; r, t)\, M_\mu[\phi(r, t)]\, dr\, dt,$$

since the support of ϕ is contained in $r - t < c$ and that of G is $t \leqslant t_0 - |r - r_0|$. Thus G is an elementary solution of M_μ.

The second assertion, (4.4), is the reciprocity property of the Green's function. It can be deduced from (4.3). For as Z remains unchanged if t_0 is replaced by $-t_0$ and t by $-t$, while the supports of G and G^* are interchanged, we can also write (4.4) as

$$G^*(r_0, t_0; r, t) = G(r_0, -t_0; r, -t). \tag{4.11}$$

If $\phi \in \mathscr{D}^+$, and $\psi = \phi(r, -t)$, then $\psi \in \mathscr{D}^+$; hence

$$\begin{aligned}\phi(r_0, t_0) = \psi(r_0, -t_0) &= \iint_{r>0} G(r_0, -t_0; r, t)\, M_\mu[\psi(r, t)]\, dr\, dt \\ &= \iint_{r>0} G^*(r_0, t_0; r, -t)\, M_\mu[\psi(r, t)]\, dr\, dt \\ &= \iint_{r>0} G^*(r_0, t_0; r, t)\, M_\mu[\phi(r, t)]\, dr\, dt.\end{aligned}$$

5.

In terms of distribution, the solution of the initial value problem

$$M_\mu[w] = 0, \qquad w(r, 0) = w_0(r), \qquad w_t(r, 0) = w_1(r), \tag{5.1}$$

is a distribution with support in $t \geqslant 0$ which satisfies

$$M_\mu[w] = w_0(r)\, \delta'(t) + w_1(r)\, \delta(t). \tag{5.2}$$

This equation shows that it is sufficient to consider the case where $w_0 = 0$, $w_1 \neq 0$, since the solution for $w_0 \neq 0$, $w_1 = 0$, can then be obtained by replacing w_1 by w_0, and differentiating with respect to t. The same remark applies to the solution of the singular mixed boundary value problem, and to the solution in the classical sense. We shall therefore consider this case only.

If w is a function, with support in $t \geqslant 0$, then (5.2) means that this function is a weak solution of the initial value problem, in the sense that

$$\int_0^\infty \int_0^\infty w M_\mu[\phi]\, dr\, dt = \int_0^\infty w_1(r)\, \phi(r, 0)\, dr, \tag{5.3}$$

for all $\phi \in \mathscr{D}^+$. Since $M_\mu[r^\mu v] = r^\mu L_\mu[v]$, one can go from (5.1) and (5.2) to the Euler–Darboux equation (1.1) by putting $w = r^\mu v$, $w_1 = r^\mu g$, to obtain the initial value problem

$$L_\mu[v] = 0, \qquad v(r, 0) = 0, \qquad v_t(r, 0) = g(r). \tag{5.4}$$

By (5.3), the weak solutions of this problem satisfy the identity

$$\int_0^\infty \int_0^\infty r^{2\mu} v L_\mu[\phi]\, dr\, dt = \int_0^\infty r^{2\mu} g(r)\, \phi(r, 0)\, dr, \qquad \phi \in \mathscr{D}^+. \tag{5.5}$$

Such a solution can be written down at once. We have

THEOREM 2. *If $g(r)$ is continuous, then*

$$v(r, t) = \int_0^{r+t} G(r, t; s, 0) \left(\frac{s}{r}\right)^{\mu} g(s)\, ds \tag{5.6}$$

is a weak solution of the initial value problem (5.4). If $v(r, t)$ is considered as a distribution in t that depends on r as a parameter, then it tends to a limit, for $t > 0$, as $r \to 0$,

$$\lim_{r\to 0} v(r, t) = \frac{\pi^{1/2}}{\Gamma(\mu + \frac{1}{2})\, \Gamma(n + 1 - \mu)} \left(\frac{\partial}{2t\, \partial t}\right)^n \int_0^t (t^2 - s^2)^{n-\mu} s^{2\mu} g(s)\, ds, \tag{5.7}$$

where n is any integer such that $n - \mu + 1 > 0$; the derivatives on the right-hand side are evaluated in the sense of the theory of distributions.

To prove that (5.6) is a weak solution of (5.4), we first show that

$$\begin{aligned} w(r, t) &= \int_0^{\infty} G(r, t; s, 0)\, w_1(s)\, ds \\ &= \int_0^{r+t} G(r, t; s, 0)\, w_1(s)\, ds \end{aligned} \tag{5.8}$$

satisfies (5.3). The support of $G(r, t; s, 0)$, considered as a function of s, is determined by the inequality $| r - s | < t$. It is empty for $t < 0$, and the second member of (5.8) is then zero. For $t > 0$, it is in $0 \leqslant s \leqslant r + t$, so that the integral (5.8) exists. Now if $\phi \in \mathscr{D}^+$, then

$$\begin{aligned} &\int_0^{\infty} \int_0^{\infty} w(r, t)\, M_\mu[\phi(r, t)]\, dr\, dt \\ &= \iint_{r>0} M_\mu[\phi(r, t)]\, dr\, dt \int_0^{\infty} G(r, t; s, 0)\, w_1(s)\, ds \\ &= \int_0^{\infty} w_1(s)\, ds \iint_{r>0} G(r, t; s, 0)\, M_\mu[\phi(r, t)]\, dr\, dt \\ &= \int_0^{\infty} w_1(s)\, ds \iint_{r>0} G^*(s, 0; r, t)\, M_\mu[\phi(r, t)]\, dr\, dt, \end{aligned}$$

where the interchange of the order of integration is justified by Fubini's theorem. By (4.4), this gives (5.3). Since $v = r^{-\mu} w$ and $g = r^{-\mu} w_1$, it follows that (5.6) is a weak solution of the initial value problem (5.4).

It has already been observed that for generalized solutions which are based on the class of test functions $\mathscr{D}^+$, the behavior as $r \to 0$ is irrelevant since the supports of the members of $\mathscr{D}^+$ are in $r > 0$. If one wants to consider the behavior of the weak solution (5.6) as $r \to 0$, still in terms of generalized solutions of the Euler–Darboux equation, one must adopt a different point of view. It can be shown that a consistent theory can be constructed by treating v as a distribution in t which depends on r as a parameter. We do not propose to develop this point of view in detail here, but will only show that in this sense, v tends to a limit as $r \to 0$.

In $0 \leqslant s \leqslant t + r$, $G(r, t; s, 0)$ is a function of

$$\zeta = \frac{t^2 - (r - s)^2}{4rs} \tag{5.9}$$

only,

$$G(r, t; s, 0) = \begin{cases} \frac{1}{2}F(\mu, 1 - \mu; 1; \zeta), & 0 \leqslant \zeta < 1 \\ \frac{1}{2}C\zeta^{-\mu}F(\mu, \mu; 2\mu; \zeta^{-1}), & \zeta > 1. \end{cases}$$

This can be written in the form

$$G(r, t; s, 0) = \left(\frac{d}{d\zeta}\right)^n G_n(\zeta), \tag{5.10}$$

where $n = 1, 2, \ldots$, and

$$G_n(\zeta) = \begin{cases} \dfrac{\zeta^n}{2\Gamma(n + 1)} F(\mu, 1 - \mu; n + 1; \zeta), & 0 \leqslant \zeta \leqslant 1, \\ \dfrac{\Gamma(\mu)\, \zeta^{n-\mu}}{2\Gamma(2\mu)\, \Gamma(n + 1 - \mu)} F(\mu - n, \mu; 2\mu; \zeta^{-1}), & \zeta \geqslant 1. \end{cases} \tag{5.11}$$

For $\zeta \neq 1$, this can be verified by term-by-term differentiation of the hypergeometric series. Moreover, we have

$$F(a, b; c; 1) = \frac{\Gamma(c)\, \Gamma(c - a - b)}{\Gamma(c - b)\, \Gamma(c - a)}$$

when $c - a - b > 0$, and this formula shows that $G_n(\zeta)$ is continuous at $\zeta = 1$. Since

$$\frac{d}{d\zeta} = \frac{2rs}{t} \frac{\partial}{\partial t},$$

and G_n vanishes for $s = r + t$, $(c \neq 0, -1, -2, \cdots)$ which corresponds to $\zeta = 0$, when $n \geqslant 1$, we can therefore write (5.6) as

$$v(r, t) = \left(\frac{\partial}{2t\, \partial t}\right)^n \int_0^{r+t} G_n(\zeta)(2r)^{n-\mu}(2s)^{n+\mu} g(s)\, ds, \tag{5.12}$$

when g is continuous. Let us multiply this equation by an indefinitely differentiable function $\phi(t)$ with compact support contained in $t > 0$, and integrate with respect to t. Then we obtain

$$\int_{-\infty}^{\infty} v(r, t)\,\phi(t)\,dt = \iint_D G_n(\zeta)\,\phi_n(t)\,g(s)(2r)^{n-\mu}(2s)^{n+\mu}\,ds\,dt, \tag{5.13}$$

where we have put

$$\phi_n(t) = (-1)^n \left(\frac{\partial}{2t\,\partial t}\right)^n \phi(t), \tag{5.14}$$

and D is the domain in which $|\,r - s\,| < t$ and $\phi \neq 0$. By hypothesis there exist positive constants a and b such that $\phi = 0$ for $t \leqslant a$ and $t \geqslant b$. If we suppose (as we may) that $r < a$, then D can be taken as $a < t < b$, $0 < s < t - r$, and (5.13) becomes

$$\int_{-\infty}^{\infty} v(r, t)\,\phi(r)\,dt = J_1 + J_2, \tag{5.15}$$

where, by (5.11),

$$J_1 = \frac{1}{2\Gamma(n+1)} \int_a^b dt \int_{t-r}^{t+r} F(\mu, 1-\mu; n+1; \zeta) \times g(s)\,\phi_n(t)[t^2 - (r-s)^2]^n \left(\frac{s}{r}\right)^{\mu} ds, \tag{5.16}$$

$$J_2 = \frac{\Gamma(\mu)}{2\Gamma(2\mu)\,\Gamma(n+1-\mu)} \int_a^b dt \int_0^{t-r} F(\mu - n, \mu; 2\mu; \zeta^{-1}) \times g(s)\,\phi_n(t)[t^2 - (r-s)^2]^{n-\mu}(2s)^{2\mu}\,ds. \tag{5.17}$$

For $n \geqslant 1$, the hypergeometric functions in the integrands are continuous. Hence

$$|\,J_1\,| \leqslant K \int_a^b dt \int_{t-r}^{t+r} (t + r - s)^n\, r^{-\mu}\,ds = O(r^{n-\mu+1}) \tag{5.18}$$

where K is constant. If also $n \geqslant \mu$, then the integrand of J_2 is continuous, and bounded uniformly in r. Hence we can make $r \to 0$ under the integral sign in J_2. By (5.18), J_1 tends to zero, and so

$$\lim_{r\to 0} \int_{-\infty}^{\infty} v(r, t)\,\phi(t)\,dt = \frac{2^{2\mu-1}\Gamma(\mu)}{\Gamma(2\mu)\,\Gamma(n+1-\mu)} \times \int_{-\infty}^{\infty} \phi_n(t)\,dt \int_0^t (t^2 - s^2)^{n-\mu} s^{2\mu} g(s)\,ds, \tag{5.19}$$

where we have restored the nominal limits of integration $(-\infty, \infty)$. By the duplication formula for the Gamma function,

$$\frac{2^{2\mu-1}\Gamma(\mu)}{\Gamma(2\mu)} = \frac{\pi^{1/2}}{\Gamma(\mu + \frac{1}{2})},$$

and in view of (5.14), (5.19) is equivalent to (5.7), with $n \geqslant 1$, $n \geqslant \mu$. But as distributional derivatives are identical with ordinary derivatives when they exist, we have, for any integer m,

$$\frac{1}{\Gamma(m+1-\mu)}\left(\frac{\partial}{2t\,\partial t}\right)^{m}\int_0^t (t^2 - s^2)^{m-\mu} s^{2\mu} g(s)\,ds$$
$$= \frac{1}{\Gamma(m-\mu)}\left(\frac{\partial}{2t\,\partial t}\right)^{m-1}\int_0^t (t^2 - s^2)^{m-\mu-1} s^{2\mu} g(s)\,ds,$$

if $m > \mu$. Hence (5.7) is valid for $n - \mu + 1 > 0$. In particular, if $\frac{1}{2} \leqslant \mu < 1$, Hence (5.7) is valid for $n - \mu + 1 > 0$. In particular, if $\frac{1}{2} \leqslant \mu < 1$, we can take $n = 0$, and the second member of (5.7) is then a continuous function.

In the exceptional case, when μ is a positive integer k, one can take $n = k$ in (5.7), and carry out one differentiation with respect to t. This gives (in terms of distributions)

$$\lim_{r\to 0} v(r, t) = \frac{1}{1.3 \cdots (2k-1)}\left(\frac{\partial}{t\,\partial t}\right)^{k-1}\{t^{2k-1}g(t)\}. \tag{5.20}$$

This result could also have been obtained by the method of descent, since a spherically symmetrical solution of the wave equation in $2k + 1$ spatial dimensions satisfies $L_k[v] = 0$. Conversely, one can recover a well-known form of the solution of the initial value of the first-order derivative, with respect to the time, of the dependent variable (see [*2*], p. 682). A similar remark applies to (5.7) when $\mu = k + \frac{1}{2}$.

6.

Since distributional derivatives are equal to derivatives in the usual sense when these exist, we can now pass from weak solutions to solutions in the usual sense by considering the differentiability properties of (5.6). In view of the application to the initial value problem in which $v(r, 0) \neq 0$, $v_t(r, 0) = 0$, we shall discuss what conditions must be imposed on $g(r)$ to ensure that $v \in C^3$. For $t < r$, this is a very simple matter. For (5.6) then reduces to the

Riemann representation of the solution of the initial value problem,

$$v(r, t) = \frac{1}{2}\int_{r-t}^{r+t} F(\mu, 1-\mu; 1; \zeta)\left(\frac{s}{r}\right)^{\mu} g(s)\, ds, \tag{6.1}$$

where ζ is again defined by (5.9). Since

$$0 \leqslant \zeta = \frac{1}{2} - \frac{r^2 - t^2}{4rs} - \frac{s}{4r} < \frac{1}{2},$$

$F(\mu, 1-\mu; 1; \zeta)$ is of class C^∞ in all three variables r, s and t, and $v \in C^m$ if $g \in C^{m-1}$. For $m \geqslant 2$, (6.1) is then the unique solution of the initial value problem (5.4), and this can, of course, be verified easily by direct differentiation.

We must now consider the case $t > r$. We shall assume for the present that μ is not a positive integer; the exceptional case will be treated separately. We then have

$$v(r, t) = \int_0^{r+t} G(r, t; s, 0)\left(\frac{s}{r}\right)^{\mu} g(s)\, ds, \tag{6.2}$$

and G has a logarithmic singularity at $s = t - r$. Now it follows from (3.9) and (3.15) that, if a and b are two constants such that $0 < a < t - r < b < t + r$, and that $|\zeta - 1| < 1$ in $a \leqslant s \leqslant b$, then

$$G(r, t; s, 0) + \frac{\sin \pi\mu}{2\pi} F(\mu, 1-\mu; 1; 1-\zeta)\log|t - r - s|$$
$$= \begin{cases} G_1(r, t; s), & a \leqslant s \leqslant t - r \\ G_2(r, t; s), & t - r \leqslant s \leqslant b, \end{cases} \tag{6.3}$$

where both G_1 and G_2 are functions of class C^∞. Hence (6.2) can be written as

$$\begin{aligned} v(r, t) = {} & \frac{1}{2} C \int_0^a \zeta^{-\mu} F(\mu, \mu; 2\mu; \zeta^{-1})\left(\frac{s}{r}\right)^{\mu} g(s)\, ds \\ & + \frac{1}{2}\int_b^{r+t} F(\mu, 1-\mu; 1; \zeta)\left(\frac{s}{r}\right)^{\mu} g(s)\, ds \\ & + \int_a^{t-r} G_1(r, t; s)\left(\frac{s}{r}\right)^{\mu} g(s)\, ds + \int_{t-r}^{b} G_2(r, t; s)\left(\frac{s}{r}\right)^{\mu} g(s)\, ds \\ & - \frac{\sin \pi\mu}{2\pi}\int_a^b F(\mu, 1-\mu; 1; 1-\zeta)\left(\frac{s}{r}\right)^{\mu} g(s)\log|t - r - s|\, ds, \end{aligned} \tag{6.4}$$

where C is the constant (3.14).

The derivatives of the last term in (6.4), which is the only one that presents any difficulty, can be obtained by a well-known device. Suppose that a function $\phi(s)$ satisfies a Lipschitz condition $|\phi(s_2) - \phi(s_1)| \leqslant K \mid s_2 - s_1 \mid$ in $a \leqslant s \leqslant b$, and put

$$\psi(x) = \int_a^b \phi(s) \log \mid s - x \mid ds = \int_{a-x}^{b-x} \phi(s + x) \log \mid s \mid ds. \qquad (6.5)$$

By partial integration, this can be transformed into

$$\psi(x) = \left[\log \mid s \mid \int_x^{x+s} \phi(\sigma)\, d\sigma\right]_{a-x}^{b-x} - \int_{a-x}^{b-x} \frac{ds}{s} \int_x^{x+s} \phi(\sigma)\, d\sigma. \qquad (6.6)$$

Now

$$\left| \frac{\partial}{\partial x} \int_x^{x+s} \phi(\sigma)\, d\sigma \right| = |\phi(x + s) - \phi(x)| \leqslant Ks,$$

so that differentiation with respect to x of the last integral in (6.6) can be carried out under the integral sign, and $\psi'(x)$ exists for $a < x < b$. One finds in this way that

$$\psi'(x) = -\int_a^b \frac{\phi(s)}{s - x}\, ds, \qquad (6.7)$$

where the integral is a principal value. The derivatives of higher order can be obtained in the same way. For if $\phi \in C^{m-1}$ and $\phi^{(m-1)}$ satisfies a Lipschitz condition, then we can differentiate the last integral in (6.5) $m - 1$ times, and then use the same device to establish the existence of $\psi^{(m)}$.

We can therefore conclude from (6.4) that if $g \in C^{m-1}$ and $g^{(m-1)}$ satisfies a Lipschitz condition, then $v \in C^m$ in $r > 0, t > r$. If these conditions are fulfilled in $0 < r < c$, then the conclusion holds in $0 < r < t$, $t + r < c$. But it still remains to consider the behavior of v and of its derivatives at $t = r$. It is obvious that v itself is continuous there, and that

$$v(r, r) = \frac{1}{2} \int_0^{2r} F\left(\mu, 1 - \mu; 1; \frac{2r - s}{4r}\right)\left(\frac{s}{r}\right)^\mu g(s)\, ds. \qquad (6.8)$$

Hence $v(r, r) \in C^m$ if $g \in C^{m-1}$. We need therefore only discuss the derivatives of v in a direction that is not tangential to $t = r$, say those with respect to t. For $t < r$, we have

$$v_t(r, t) = \frac{1}{2} g(r + t) \left(\frac{r + t}{r}\right)^\mu + \frac{1}{2} g(r - t) \left(\frac{r - t}{r}\right)^\mu$$
$$+ \frac{t}{4} \int_{r-t}^{r+t} F'(\mu, 1 - \mu; 1; \zeta) \frac{s^{\mu-1}}{r^{\mu+1}} g(s)\, ds,$$

and since $\mu > 0$, it follows that

$$v_t(r, r-) = 2^{\mu-1}g(2r) + \frac{1}{4}\int_0^{2r} F'\left(\mu, 1-\mu; 1; \frac{2r-s}{4r}\right) \frac{s^{\mu-1}}{r^{\mu}} g(s)\, ds. \quad (6.9)$$

For $t > r$, we can again use (6.4). But we must now make a a function of r and t, in order to ensure that $a < t - r$. It is sufficient to take $a = \frac{1}{2}(t-r)$. For ζ is a decreasing function of s when $t > r$, and we have, when $s = \frac{1}{2}(t-r)$,

$$\zeta - 1 = \frac{t^2 - (r+s)^2}{4rs} = \frac{1}{2} + \frac{3(t-r)}{8r},$$

so that $\zeta < 2$ in $\frac{1}{2}(t-r) \leqslant s \leqslant t-r$ for $t - r < \frac{4}{3}r$, which we may assume to be the case. We must also take $b < 2r$, so that $b < t + r$ for all sufficiently small values of $t - r$. The limiting value of the derivative of the last term on the right-hand side of (6.4) can again be computed by means of (6.7). We must now take $a = \frac{1}{2}x$ in (6.5); it is easily seen that (6.7) then becomes

$$\frac{d}{dx}\int_{x/2}^{b} \phi(s) \log|x-s|\, ds = -\frac{1}{2}\phi\left(\frac{x}{2}\right)\log\frac{x}{2} - \phi(x)\log\frac{2(b-x)}{x} - \int_{x/2}^{b} \frac{\phi(s)-\phi(x)}{s-x}\, ds. \quad (6.10)$$

In the application of this equation to (6.4), we shall have $\phi = s^{\mu}\phi_1$, where ϕ_1 satisfies a Lipschitz condition in $0 \leqslant s \leqslant b$. Then

$$\begin{aligned}\left|\frac{\phi(s)-\phi(x)}{s-x}\right| &= \left|\frac{s^{\mu}\phi_1(s) - x^{\mu}\phi_1(x)}{s-x}\right| \\ &\leqslant s^{\mu}\left|\frac{\phi_1(s)-\phi_1(x)}{s-x}\right| + \phi_1(x)\left|\frac{s^{\mu}-x^{\mu}}{s-x}\right| \quad (6.11) \\ &\leqslant Ks^{\mu} + M\left|\frac{s^{\mu}-x^{\mu}}{s-x}\right|,\end{aligned}$$

where K is the Lipschitz constant of ϕ_1, and M is an upper bound for $|\phi_1|$ in $0 \leqslant s \leqslant b$. Also

$$\left|\frac{s^{\mu}-x^{\mu}}{s-x}\right| \leqslant \begin{matrix}\mu b^{\mu-1}, & \mu \geqslant 1 \\ \mu s^{\mu-1}, & 0 < \mu < 1.\end{matrix}$$

Hence the integrand of the integral in the second member of (6.10) is dominated by an integrable function, and we can make $x \to 0$ under the integral sign, to obtain

$$\lim_{x\to 0} \frac{d}{dx} \int_{x/2}^{b} s^{\mu}\phi_1(s) \log | s - x | \, ds = - \int_0^b s^{\mu-1}\phi_1(s) \, ds. \qquad (6.12)$$

This is exactly what one would obtain if the integral were differentiated formally, and then put $x = 0$. We can therefore compute the limiting value of v_t as $t \to r +$ from (6.4) by the same rule. Since the nonsingular contributions from the variable limits $a = \frac{1}{2}(t - r)$ and $t - r$, and the derivative of the integral from *0* to a, vanish in the limit, this again gives (6.9). Hence v_t is continuous at $t = r$.

The derivatives of higher order can be treated in the same manner. But as $\partial\zeta/\partial t = t/2rs$, convergence difficulties may arise at $s = 0$. In $t < r$, the mth derivative of (6.1) will consist of an integral whose integrand contains a term

$$(\tfrac{1}{2}t)^m \, r^{-m-\mu} \, s^{\mu-m} g(s) \, F^{(m)}(\mu, 1 - \mu; 1; \zeta),$$

and of contributions arising from the differentiation of the variable limits that include terms which are proportional to $(r - t)^{\mu-m+1} g(r - t)$, $(r - t)^{\mu-m+2} g'(r - t),\ldots$. As $t \to r -$, the limit will certainly exist if either $m < \mu + 1$, or if $g^{(j)}(0) = 0$ for $0 \leqslant j \leqslant [m - \mu - 1]$, where $[z]$ denotes, as usual, the greatest integer less than z. We shall then have $g(s) = 0(s^{(m-\mu)})$ as $s \to 0$, so that $s^{\mu-m}g$ will be integrable at $s = 0$, and the terms arising from differentiation of the lower limit will tend to zero as $r - t \to 0$. When $\partial^m v/\partial t^m$ is computed in $t > r$ by the same method as that which was used above to evaluate $\partial v/\partial t$, it is found that the same conditions ensure that the limit exists as $t \to r +$, and is equal to that found as $t \to r -$.

It would therefore appear at first sight that if v is to be of class C^m in $r > 0$, and $m > \mu + 1$, then the subsidiary conditions $g^{(j)}(0) = 0$, $0 \leqslant j \leqslant [m - \mu - 1]$ must be added to the basic requirement that $g \in C^{m-1}$ and that $g^{(m-1)}$ should satisfy a Lipschitz condition. But it can be shown that only the derivatives of odd order need be assumed to vanish at the origin. For $t < r$, this can be proved as follows: If $g = r^{2l}$, $l = 0, 1, 2,\ldots$, then the solution of the initial value problem (5.4) is a polynomial $p_l(r, t)$,

$$p_0 = t, \qquad p_1 = r^2 t + \tfrac{2}{3}(\mu + \tfrac{1}{2})t^3, \qquad (6.13)$$

$$p_l(r, t) = \sum_{j=0}^{l} \frac{2^{2j}\Gamma(l + 1)\,\Gamma(l + \mu + \frac{1}{2})}{\Gamma(l - j + 1)\,\Gamma(l - j + \mu + \frac{1}{2})} \frac{r^{2l-2j}t^{2j+1}}{(2j + 1)!} .$$

But the solution of the initial value problem in $t < r$ is unique; hence

$$p_l(r, t) = \frac{1}{2} \int_{r-t}^{r+t} F(\mu, 1 - \mu; 1; \zeta) \frac{s^{\mu+2l}}{r^{\mu}} \, ds, \qquad (t < r, l = 0, 1,\ldots). \quad (6.14)$$

We can therefore write (6.1) as

$$v(r, t) = \sum_{l=0}^{k} g^{(2l)}(0) \frac{p_l(r, t)}{(2l)!} + \frac{1}{2} \int_{r-t}^{r+t} F(\mu, 1 - \mu; 1; \zeta) \left(\frac{s}{r}\right)^{\mu} g_k(s)\, ds,$$

$$g_k(s) = g(s) - \sum_{l=0}^{k} g^{(2l)}(0) \frac{s^{2l}}{(2l)!}\,.$$

Clearly, $g_k^{(2j)}(0) = 0$ for $0 \leqslant j \leqslant k$. Since we can work with this identity instead of (6.1), it tells us that only the requisite number of derivatives of odd order of g must vanish at the origin in order to ensure the existence of the limit of $\partial^m v/\partial t^m$ as $t \to r\,-$.

We cannot establish the analog of (6.14) for $t > r$ by an appeal to the uniqueness theorem, since we are in fact trying to prove that (5.6) is a solution of the singular mixed boundary value problem which satisfies, for suitable g, the conditions of the uniqueness theorem. To overcome this difficulty, the argument can be divided into two steps. For $l = 0$, the identity

$$\int_{0}^{r+t} G(r, t; s, 0) \left(\frac{s}{r}\right)^{\mu} ds = t \tag{6.15}$$

can be proved directly. The proof will be given at the end of the next section. We can therefore write (5.6) as

$$v(r, t) = g(0)t + \int_{0}^{r+t} G(r, t, s, 0) \left(\frac{s}{r}\right)^{\mu} [g(s) - g(0)]\, ds. \tag{6.16}$$

Since $g(s) - g(0) = 0(s)$ and $(\mu + 1) - 2 > -1$, we can conclude that if $g \in C'$, and g' satisfies a Lipschitz condition, then (5.6) is a solution of class C^2 of the initial value problem (5.4) in $r > 0$. It then follows from Theorem 4, which will be proved below, that for $g = r^{2l}$, $l = 1, 2,\ldots$, v_r and v_t remain bounded as $r \to 0$. Hence Theorem 1 shows that the identities

$$p_l(r, t) = \int_{0}^{r+t} G(r, t; s, 0) \frac{s^{\mu+2l}}{r^{\mu}}\, ds, \qquad l = 1, 2,\ldots, \tag{6.17}$$

are valid, and we can argue as in $t < r$. We can therefore state the following theorem on the differentiability properties of (5.6):

THEOREM 3. *If $g(r)$ is of class C^{m-1}, $m \geqslant 1$, in $0 \leqslant r \leqslant c$, and $g^{(m-1)}(r)$ satisfies a Lipschitz condition, then* (5.6) *is of class C^m in $r > 0$, $t \geqslant 0$, $r + t \leqslant c$, provided that either $m < \mu + 1$ or that, if $m > \mu + 1$, $g^{(2j+1)}(0) = 0$ for $j = 0, 1, 2,\ldots$, and $2j + 1 < m - \mu - 1$. If $m \geqslant 2$, then* (5.6) *is a solution of class C^m of the initial value problem* (5.4) *in $r > 0$.*

In the exceptional case $\mu = k$, $k = 1, 2,\ldots$, (6.16) remains valid, and takes the form

$$v(r, t) = g(0)t + \frac{1}{2}\int_{|t-r|}^{t+r} F(k, 1-k; 1; \zeta)\left(\frac{s}{r}\right)^k [g(s) - g(0)]\, ds. \quad (6.18)$$

Now $F(k, 1-k; 1; \zeta)$ is a polynomial in ζ. Hence there is no need for a Lipschitz condition in $t > r$. Then $g \in C^{m-1}$ implies that $v \in C^m$ for $r > 0$, $t \neq r$. It follows from (6.18) that the derivatives of v of the first and second order are continuous at $t = r$ provided only that $g \in C'$. The continuity of the derivatives of higher order can be discussed by the same method as in the case of nonintegral μ. One finds that the subsidiary conditions at the origin, which are required when the order m of the differentiability class exceeds $k + 1$, are $g^{(2j+1)}(0) = 0, j = 0, 1, 2,\ldots$, for $2j + 1 \leqslant m - k - 1$.

7.

We can now complete the discussion of the singular mixed boundary value problem by considering the behavior of (5.6) as $r \to 0$. We shall again first assume that μ is not a positive integer. Now if $k < \mu < k + 1$, $k = 0, 1, 2,\ldots$, and $g(r) \in C^k$, then the distribution (5.7) is a continuous function. The fact that v tends to this function in the sense of the theory of distributions does not imply that it does so in the ordinary sense. However, we can prove directly that this is indeed the case, and that there is a corresponding result for v_t when $g \in C^{k+1}$. The proof can be effected by means of an integral representation which is of some intrinsic interest. It should be noted that although we are assuming that $\mu \geqslant \frac{1}{2}$ in this paper, the proof of Theorem 4 is valid for $\mu > 0$.

THEOREM 4. *If* $k < \mu < k + 1$, $k = 0, 1, 2,\ldots$, *and* $g(r) \in C^{k+1}$ *in* $0 \leqslant r \leqslant c$, *then* $v(r, t)$, *given by* (5.6), tends to

$$\frac{\pi^{1/2}}{\Gamma(\mu + \frac{1}{2})\,\Gamma(1 - \mu)}\int_0^t (t^2 - s^2)^{-\mu} s^{2\mu} g(s)\, ds \quad (7.1)$$

as $r \to 0$ *in* $0 < t + r \leqslant c$; *for* $\mu > 1$, *this integral must be interpreted as a finite part in the sense of Hadamard. Also,* $v_r \to 0$, *and* v_t *tends to the derivative of* (7.1).

Let us first suppose that $0 < \mu < 1$. We can then derive an integral representation for $G(r, t; s, 0)$ that is similar to (1.10). It follows from the well-known relation

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 \sigma^{b-1}(1-\sigma)^{c-b-1}(1-\sigma z)^{-a}\, d\sigma, \quad (7.2)$$

which is valid for $\operatorname{Re} b > 0$, $\operatorname{Re}(c - b) > 0$ and $|z| < 1$, that for $\zeta > 1$,

$$\frac{1}{2} C\zeta^{-\mu} F(\mu, \mu; 2\mu; \zeta^{-1}) = \frac{\sin \pi\mu}{2\pi} \int_0^1 \sigma^{\mu-1}(1-\sigma)^{\mu-1}(\zeta-\sigma)^{-\mu}\, d\sigma \quad (7.3)$$

where C is the constant defined by (3.14). Again, for $0 \leqslant \zeta < 1$,

$$\frac{1}{2} F(\mu, 1-\mu; 1; \zeta) = \frac{\sin \pi\mu}{2\pi} \int_0^1 \sigma^{\mu-1}(1-\sigma)^{-\mu}(1-\zeta\sigma)^{\mu-1}\, d\sigma,$$

and if we replace σ by σ/ζ, then this becomes

$$\frac{1}{2} F(\mu, 1-\mu; 1; \zeta) = \frac{\sin \pi\mu}{2\pi} \int_0^\zeta \sigma^{\mu-1}(1-\sigma)^{\mu-1}(\zeta-\sigma)^{-\mu}\, d\sigma. \quad (7.4)$$

Now when ζ is the quantity defined by (5.9), then the first members of (7.3) and (7.4) are the values of $G(r, t; s, 0)$ in $0 \leqslant s < t - r$ and in $t - r < s \leqslant t + r$, respectively. We can combine the two integrals by putting $\sigma = \frac{1}{2}(1 - \cos\theta)$, to obtain

$$G(r, t; s, 0) = \frac{\sin \pi\mu}{\pi} \int \frac{\sin^{2\mu-1}}{(4\zeta - 2 + 2\cos\theta)^\mu}\, d\theta$$

where the integral is taken over $0 < \theta < \pi$ if $\zeta > 1$, and over $0 < \theta < \cos^{-1}(1 - 2\zeta)$ if $\zeta < 1$. For $\zeta = 1$ it is, of course, not defined. Substituting $4rs\,\zeta = t^2 - (r - s)^2$, we find then that

$$G(r, t; s, 0) = \frac{\sin \pi\mu}{\pi} \int_0^\pi \frac{(rs)^\mu \sin^{2\mu-1}\theta}{(t^2 - r^2 - s^2 + 2rs\cos\theta)^\mu}\, d\theta \quad (7.5)$$

where the integrand is defined to be zero when $t^2 - r^2 - s^2 + 2rs\cos\theta < 0$.

We can now multiply (7.5) by $(s/r)^\mu g(s)$, integrate with respect to s from 0 to $t + r$, and put $s\cos\theta = \xi$, $s\sin\theta = \eta$. Then, by (5.6),

$$v(r, t) = \frac{\sin \pi\mu}{\pi} \iint_D \frac{\eta^{2\mu-1} g(s)}{[t^2 - (\xi - r)^2 - \eta^2]^\mu}\, d\xi\, d\eta, \quad (7.6)$$

where $s = (\xi^2 + \eta^2)^{1/2}$, and D is the domain in which $(\xi - r)^2 + \eta^2 < t^2$, $\eta > 0$. For continuous g this procedure is justified by absolute convergence. We can now put

$$\xi = r + \xi_1 t, \qquad \eta = \eta_1 t, \qquad \rho_1 = (\xi_1^2 + \eta_1^2)^{1/2}, \quad (7.7)$$

and write (7.6) as

$$v(r, t) = \frac{t \sin \pi\mu}{\pi} \iint_{D_0} \frac{\eta_1^{2\mu-1} g(s)}{(1 - \rho_1^2)^\mu} \, d\xi_1 \, d\eta_1 , \tag{7.8}$$

where D_0 is the domain $0 \leqslant \rho_1 < 1$, $\eta > 0$, and

$$s = [(r + \xi_1 t)^2 + \eta_1^2 t^2]^{1/2}. \tag{7.8a}$$

If $0 \leqslant r \leqslant c$ and $0 \leqslant r + t \leqslant c$, then $0 \leqslant s \leqslant c$ in D_0, so that $g(s)$ is continuous, and bounded uniformly in r. We can therefore make $r \to 0$ under the integral sign in (7.8). The result will be the same as that which is obtained by making $r \to 0$ in (7.6),

$$\begin{aligned} v(0, t) &= \frac{\sin \pi\mu}{\pi} \iint_D \frac{\eta^{2\mu+1} g(s)}{(t^2 - s^2)^\mu} \, d\xi \, d\eta \\ &= \frac{\sin \pi\mu}{\pi} \int_0^\pi \sin^{2\mu-1}\theta \, d\theta \int_0^t (t^2 - s^2)^{-\mu} s^{2\mu} g(s) \, ds \end{aligned}$$

and since

$$\int_0^\pi \sin^{2\mu-1}\theta \, d\theta = \frac{\Gamma(\mu)\,\Gamma(\frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} ,$$

this gives (7.1).

Also, if $g \in C^1$ for $0 \leqslant r \leqslant c$, then we can differentiate under the integral sign in (7.8), and find that

$$v_r(r, t) = \frac{t \sin \pi\mu}{\pi} \iint_{D_0} \frac{\eta_1^{2\mu-1} g'(s)}{(1 - \rho_1^2)^\mu} \frac{r + \xi_1 t}{s} \, d\xi_1 \, d\eta_1 , \tag{7.9a}$$

$$v_t(r, t) = \frac{\sin \pi\mu}{\pi} \iint_{D_0} \left\{ g(s) + \left[\frac{\xi_1(r + \xi_1 t)}{s} + \frac{\eta_1^2 t}{s} \right] t g'(s) \right\} \frac{\eta_1^{2\mu-1}}{(1 - \rho_1^2)^\mu} \, d\xi_1 d\eta_1 . \tag{7.9b}$$

Since $|(r + \xi_1 t)/s \mid \leqslant \mid$ and $0 \leqslant \eta_1 t/s \leqslant 1$, we again let r tend to zero under the integral signs. Thus

$$v_r(0, t) = \frac{t \sin \pi\mu}{\pi} \iint_{D_0} \frac{\eta_1^{2\mu-1} \xi_1 g'(\rho_1 t)}{\rho_1 (1 - \rho_1^2)^\mu} \, d\xi_1 \, d\eta_1 = 0,$$

as the integrand is an odd function of ξ_1. Again,

$$\begin{aligned} v_t(0, t) &= \frac{\sin \pi\mu}{\pi} \iint_{D_0} \{ g(\rho_1 t) + \rho_1 t g'(\rho_1 t) \} \frac{\eta_1^{2\mu-1}}{(1 - \rho_1^2)^\mu} \, d\xi_1 \, d\eta_1 \\ &= \frac{\pi^{1/2}}{\Gamma(\mu + \frac{1}{2})\,\Gamma(1 - \mu)} \int_0^1 (1 - \rho_1^2)^{-\mu} \rho_1^{2\mu} \{ g(\rho_1 t) + \rho_1 t g'(\rho_1 t\} \, d\rho_1 \\ &= \frac{\pi^{1/2}}{\Gamma(\mu + \frac{1}{2})\,\Gamma(1 - \mu)} \frac{d}{dt} \int_0^1 (1 - \rho_1^2)^{-\mu} \rho_1^{2\mu} t g(\rho_1 t) \, d\rho_1 , \end{aligned}$$

and the substitution $s = \rho_1 t$ shows at once that $v_t(0, t)$ is obtained from (7.1) by differentiation with respect to t. We have thus proved Theorem 4 when $k = 0$.

To extend this argument to $k \geqslant 1$, we observe first that (7.2) is valid for all b and c, except $c = 0, -1, -2, \ldots$, if the integral is regularized by the method of Gelfand and Shilov ([6], p. 118). This regularization is uniquely defined by analytic continuation. When neither b nor $c - b$ is a non-positive integer, it is given by the finite part of the divergent integral in the sense of Hadamard, since this provides the analytic continuation of an integral with an algebraic infinity ([8], p. 40). Thus (7.5) also remains valid for nonintegral $\mu > 1$, if the integral is interpreted as a finite part.

On the other hand, we can also interpret the integral in the second member of (7.8) as a finite part, provided that $\eta_1^{2\mu-1} g(s)$ is of class C^k in a neighborhood of $\rho_1 = 1, \eta_1 > 0$, where $k = [\mu]$. It is easy to see that this is the case. For, in the context of the proof of Theorem 4, we can assume that $r/t < 1$. Then it follows from $g(r) \in C^k$ that $g(s)$ is of class C^k in $b < \rho_1 < 1, \eta_1 > 0$, where b is a constant such that $b > r/t$, since s is bounded away from zero there. Also, $2\mu - 1 > 2k - 1$ and $k \geqslant 1$, so that $\eta_1^{2\mu-1} \in C^k$. Now we can compute the finite part of (7.8) by reversing our previous argument: We first make the transformation (7.7), then introduce polar coordinates, and finally calculate the integral by integrating first with respect to θ and then with respect to s, the inner integral being evaluated as a finite part. By (7.5) we therefore find again that the second member of (7.8) is equal to the second member of (5.6).

Furthermore, $\eta_1^{2\mu-1} g(s)$ and its derivatives of all orders up to k are continuous functions of ξ_1, η_1 and r in $b < \rho_1 < 1$, $\eta > 0, 0 \leqslant r < bt$. It therefore follows from the properties of the finite parts of divergent integrals ([7], p. 149) that (7.8) depends continuously on r as $r \to 0$. The calculation of $v(0, t)$ is formally the same as before; the only difference is that the end result must be interpreted as finite part. Also, if $g(r) \in C^{k+1}$, then the same reasoning can be applied to (7.9a) and (7.9b), and thus Theorem 3 is proved for all k.

We can now establish the identity (6.15) which was used in the proof of Theorem 3. We need only put $g = 1$ in (7.8). This gives, since (7.8) is equivalent to (5.6),

$$\int_0^{r+t} G(r, t; s, 0) \left(\frac{s}{r}\right)^{\mu} ds = \frac{t \sin \pi\mu}{\pi} \iint_{D_0} \frac{\eta_1^{2\mu-1}}{(1 - \rho_1^2)^{\mu}} \, d\xi_1 \, d\eta_1 .$$

For $\frac{1}{2} \leqslant \mu < 1$, the integral can be evaluated at once by transforming to polar coordinates:

$$\frac{t \sin \pi\mu}{\pi} \iint_{D_0} \frac{\eta_1^{2\mu-1}}{(1-\rho_1{}^2)^\mu} \, d\xi_1 \, d\eta_1 = \frac{t \sin \pi\mu}{\pi} \int_0^\pi \sin^{2\mu-1}\theta \, d\theta \int_0^1 \frac{\rho_1^{2\mu}}{(1-\rho_1{}^2)^\mu} \, d\rho_1$$

$$= \frac{\pi^{1/2} t}{\Gamma(\mu+\frac{1}{2}) \, \Gamma(1-\mu)} \int_0^1 \frac{\rho_1^{2\mu}}{(1-\rho_1{}^2)^\mu} \, d\rho_1 = t.$$

It now follows by analytic continuation that this equation remains valid for $\mu > 1$ if the integral is evaluated as a finite part, and this proves (6.15).

The exceptional case can be treated directly by means of (5.12). When $\mu = k$, $k = 1, 2, \ldots$, then (5.11) gives

$$2G_{k-1}(\zeta) = \begin{cases} \dfrac{1}{(k-1)!} \zeta^{k-1}(1-\zeta)^{k-1}, & 0 \leqslant \zeta \leqslant 1, \\ 0 & \zeta > 1 \end{cases}$$

which is simply a form of Rodrigues' formula, since $F(k, 1-k; 1; \zeta) = P_{k-1}(1-2\zeta)$, where P_{k-1} denotes the Legendre polynomial of order $k-1$. We can therefore take $n = k-1$ in (5.12):

$$v(r, t) = \frac{2^{k-2}}{(k-1)!} \left(\frac{\partial}{t \, \partial t}\right)^{k-1} \int_{|t-r|}^{t+r} \zeta^{k-1}(1-\zeta)^{k-1} \frac{s^{2k-1}}{r} g(s) \, ds. \tag{7.10}$$

Since $4rs \, \zeta = t^2 - (r-s)^2$, we have

$$\zeta(1-\zeta) = \frac{[(r+t)^2 - s^2][s^2 - (t-r)^2]}{(4rs)^2},$$

and can therefore put

$$s^2 = (r-t)^2 + 4rt \, \sigma \tag{7.11}$$

in (7.10), to obtain

$$v(r, t) = \frac{2^{k-1}}{(k-1)!} \left(\frac{\partial}{t \, \partial t}\right)^{k-1} \int_0^1 \sigma^{k-1}(1-\sigma)^{k-1} t^{2k-1} g(s) \, d\sigma. \tag{7.12}$$

For $g = 1$, this again gives the identity (6.15). Now if $g(r) \in C^{k-1}$ and $r < t$, then $g(s)$ is of class C^{k-1} in all three variables r, t and σ. It is then legitimate to take the differentiations under the integral sign in (7.12), and to make $r \to 0$. In the limit, $s = t$, and so

$$\begin{aligned} v(0, t) &= \frac{2^{k-1}[(k-1)!]^2}{(k-1)! \, (2k-1)!} \left(\frac{\partial}{t \, \partial t}\right)^{k-1} [t^{2k-1} g(t)] \\ &= \frac{1}{1.3 \cdots (2k-1)} \left(\frac{\partial}{t \, \partial t}\right)^{k-1} [t^{2k-1} g(t)]. \end{aligned} \tag{7.13}$$

This is the same as (5.20), but whereas the derivatives in (5.20) were distributional ones, (7.13) holds in the usual sense.

8.

When 2μ is an integer, the solutions of the Euler–Darboux equation are also axisymmetric solutions of the wave equation in $2\mu + 1$ spatial dimensions. The solution of the singular mixed boundary value problem which has been constructed is one for which there are no sources on the axis $r = 0$. The proof of Theorem 1 remains valid if one assumes only that $r^{2\mu}(v_r^2 + v_t^2)$ is integrable at $r = 0$ and that $r^{2\mu}v_r v_t$ tends to zero uniformly in t, as $r \to 0$. One can interpret these by saying that the total energy of the disturbance represented by v must be finite at any instant, and that the rate of energy flux across a cylinder $r = \epsilon$ must tend to zero as $\epsilon \to 0$. The counterpart to this solution of the wave equation is one that represents a line source on the axis. A similar solution of the Euler–Darboux equation can be constructed for general μ.

The function

$$H = (rr_0)^{-\mu}\, G^*(r_0, t_0; r, t), \tag{8.1}$$

where G^* is defined by (4.4), satisfies the equation

$$L_\mu[H] = r_0^{-2\mu}\delta(r - r_0)\,\delta(t - t_0), \tag{8.2}$$

and has the support $t \geqslant t_0 + |r - r_0|$. If one thinks of r as a spatial variable, and of t as the time, then H can be interpreted as a disturbance due to a point source at $r = r_0$, $t = t_0$. Now when $r_0 \to 0$, one has

$$H \to \frac{\pi^{1/2}}{\Gamma(\mu + \frac{1}{2})\,\Gamma(1 - \mu)}\,[(t - t_0)^2 - r^2]^{-\mu}, \qquad t > t_0 + r, \tag{8.3}$$

as the field due to a point source on the singular line, if one ignores the singularity on $t = r$. By superposition, one can now form the solution due to a line source on the singular line,

$$v(r, t) = \frac{\pi^{1/2}}{\Gamma(\mu + \frac{1}{2})\,\Gamma(1 - \mu)} \int_{-\infty}^{t-r} \frac{\sigma(t')}{[(t - t')^2 - r^2]^\mu}\, dt'. \tag{8.4}$$

This representation of solutions of the Euler–Darboux equation seems to have been first given by Volterra [9].

Let us suppose that $k < \mu < k + 1$, $k = 0, 1, 2, \ldots$, that $\sigma(t)$ vanishes for sufficiently large negative t, and that $\sigma \in C^{k+1}$, with $\sigma^{(k+1)}$ satisfying a Lipschitz condition in any finite t-interval. Then the integral (8.4) exists, in the usual sense for $k = 0$, and as a finite part for $k > 0$. Its derivatives of the first and second order can be obtained by formal differentiation under the integral sign, provided that the resulting integrals are read as finite parts.

Since $L_\mu[(t^2 - r^2)^{-\mu}] = 0$, (8.4) is a solution [of class C^2] of the Euler–Darboux equation in $r > 0$. Its support is $t \geqslant r + t_0$, where t_0 is the greatest lower bound of the support of $\sigma(t)$. We can show that

$$\lim_{r\to 0} r^{2\mu} \frac{\partial v}{\partial r} = -\sigma(t). \tag{8.5}$$

To prove (8.5), we differentiate (8.4) with respect to r and then put $t' = t - r - rs$. This gives

$$r^{2\mu} \frac{\partial v}{\partial r} = - \frac{2\pi^{1/2}}{\Gamma(\mu + \frac{1}{2})\,\Gamma(-\mu)} \int_0^\infty \frac{\sigma(t - r - rs)}{[s(2 + s)]^{\mu+1}}\, ds, \tag{8.6}$$

where the second member is a finite part. It is easy to see that one can make $r \to 0$ under the integral sign in this equation. One can divide the interval of integration into $(0, a)$ and (a, ∞), where $a > 0$. In (a, ∞), $\sigma(t - r - rs)$ is bounded uniformly in r for fixed t, and $[s(2 + s)]^{-\mu-1}$ is integrable. Hence the passage to the limit is justified by Lebesgue's theorem. In $(0, a)$, $\sigma(t - r - rs)$ and its derivatives with respect to s up to order $k + 1$ depend continuously on r for $r \geqslant 0$; it is known that this implies the continuity of the finite part of the integral with respect to r. Hence

$$\lim_{r\to 0} r^{2\mu} \frac{\partial v}{\partial r} = - \frac{2\pi^{1/2}\,\sigma(t)}{\Gamma(\mu + \frac{1}{2})\,\Gamma(1 - \mu)} \int_0^\infty \frac{ds}{[s(s + 2)]^{\mu+1}}. \tag{8.7}$$

To evaluate the finite part of this integral, we note that it converges for $-\frac{1}{2} < \operatorname{Re} \mu < 0$, and is then an analytic function of μ. It can be computed by setting $s + 1 = z^{-1/2}$, which gives

$$\int_0^\infty \frac{ds}{[s(s + 1)]^{\mu+1}} = \frac{1}{2} \int_0^1 (1 - z)^{-\mu-1} z^{\mu-1/2}\, dz = \frac{\Gamma(-\mu)\,\Gamma(\mu + \frac{1}{2})}{2\Gamma(\frac{1}{2})}. \tag{8.8}$$

By analytic continuation, the finite part of the integral in the first member of (8.8) is given by the second member of (8.8) except when $\mu = 0, 1, 2, \ldots$. Thus (8.5) follows from (8.7) and (8.8).

The exceptional case $\mu = k$, $k = 1, 2, \ldots$, is elementary. Clearly, one has, for $k = 1$,

$$v = \frac{\sigma(t - r)}{r}.$$

Now it is well known that

$$\frac{1}{r}\frac{\partial}{\partial r} L_\mu[v] = L_{\mu+1}\left[\frac{1}{r}\frac{\partial v}{\partial r}\right].$$

Hence

$$v(r, t) = \frac{(-1)^{k-1}}{1.3 \cdots (2k - 1)} \left(\frac{\partial}{r\,\partial r}\right)^{k-1} \frac{\sigma(t - r)}{r} \tag{8.9}$$

is the appropriate solution of $L_k[v] = 0$, if $\sigma \in C^{k+1}$; it is easily verified that (8.6) remains valid.

The Volterra representation (8.4) can also be treated as a distribution in t which depends on r as a parameter. Let

$$E(r; t) = \begin{cases} \dfrac{\pi^{1/2}}{\Gamma(\mu + \frac{1}{2})\,\Gamma(1 - \mu)} (t^2 - r^2)^{-\mu}, & t > r \\ 0, & t < r \end{cases} \tag{8.10}$$

For Re $\mu < 1$, the linear form determined by this distribution is

$$\int_{-\infty}^{\infty} E(r; t)\,\phi(t)\,dt, \tag{8.11}$$

where $\phi(t)$ is an indefinitely differentiable function with compact support. For Re $\mu \geqslant 1$, it is defined by analytic continuation, and is the finite part of the integral (8.11) when μ is not a positive integer. Let $\sigma(t)$ be a distribution whose support is contained in $t \geqslant a$, for some finite a. Then the convolution

$$v = E(r; t)^* \sigma(t) \tag{8.12}$$

exists. It is an elementary exercise in distribution theory to prove that $L_\mu[E] = 0$ for $r > 0$, and that

$$\lim \left(r^{2\mu} \frac{\partial E}{\partial r}\right) = -\delta(t). \tag{8.13}$$

Hence v, considered as a distribution in t that depends on r as a parameter, satisfies $L_\mu[v] = 0$ in $r > 0$, and $r^{2\mu} v_r$ tends to $-\sigma(t)$ as $r \to 0$.

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