Brief Communications

*On the Existence of Upper Bounds on the Performance Index of Nonlinear Systems**

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ABSTRACT: *Several interesting results are presented in determining the value of an integral performance index for certain types of finite changes in the characteristics of the system. In particular, we consider the assumed model of the system to be linear, while an accurate representation is given to a system containing a single nonlinearity. These assumptions, in many physical situations are quite reasonable. The performance index is specified as the integral of a quadratic function of the state of the system. Two theorems concerning the existence of upper-bounds for the values of the performance index are proved. Consideration of these theorems may lead to a meaningful insight into the relationship between changes in the system characteristics and the corresponding changes in the performance index.*

Introduction

In many control situations there is a meaningful performance criteria which can be established. For various reasons, changes will occur in the characteristics of the plant. In order to determine an upper bound on the corresponding values of the performance criteria, it is desirable to consider the most general type of change for which the existence of an upper bound can be guaranteed. We present such conditions on the changes in the system characteristics where the changes can be represented by a single nonlinearity.

Several authors have recently been concerned with determining the effects of various system characteristics on the performance of an optimally designed system. This problem is of considerable importance, since in any physical system uncertainties of various kinds inevitably occur. In particular, uncertainty may occur in the state measurement process if the control is of a feedback form, and uncertainty may occur in the controlled part of the system due to environmental and aging effects. Also, there are always inherent uncertainties in the choice of a mathematical model both for the controlled system and the controller. Thus, for several reasons there are discrepancies between any physical process and the mathematical model chosen as its representation.

There have been various attempts to develop theoretical methods with which to study this problem. The so-called *performance index sensitivity vector*, as pro-

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[1] A prime denotes the transpose of a vector.

posed by Dorato (1), has been used by Pagurek (2, 3) and Witenhausen (4), to investigate the change in a performance index due to changes in certain system parameters. However, this approach has certain inherent difficulties due to the fact that Dorato's sensitivity vector is defined as the gradient of the performance index with respect to a parameter vector, hence, it can yield information only of a purely local nature. Several other approaches based on the consideration of essentially finite changes in the system characteristics have been considered. Howard and Rekasius (5) discuss the worst possible changes in the sense that the performance index is maximized. Rissanen (6) and McClamroch (7) consider the related problem of specifying an upper bound on the change in the performance index and of determining admissable changes in the system characteristics. Similar results are given by Rissanen and Durbeck (8).

Assume that the mathematical model for a system is given by the linear differential equations

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0 \tag{1}$$

where x is an n-vector and u is an m-vector. The minimization of the integral criteria

$$J = \int_0^\infty (x'Qx + u'Ru)\, dt^1 \tag{2}$$

where Q and R are positive definite matrices is achieved by the control law

$$u = -R^{-1}B'Px \tag{3}$$

where P is a positive definite symmetric matrix which satisfies, (9) $A'P + PA - PBR^{-1}B'P + Q = 0$. Using the control law Eqs. 3, 1 and 2 become

$$\dot{x} = Fx, \qquad x(0) = x_0 \tag{4}$$

$$J = \int_0^\infty x'Gx\, dt \tag{5}$$

where $F = A - BR^{-1}B'P$ and $G = Q + PBR^{-1}B'P$. The value of J in (5) for any matrices F and G can easily be determined by classical linear techniques (10). The only restriction is that F have eigenvalues with negative real parts and that G be a positive definite matrix. However, for reasons discussed previously, a more accurate representation of the system might be given by

$$\begin{aligned} \dot{x} &= Fx + by, \qquad x(0) = x_0 \\ y &= \phi(\sigma) \\ \sigma &= a'x \end{aligned} \tag{6}$$

where b and a are constant n-vectors, and the continuous nonlinearity $\phi(\sigma)$ satisfies

$$0 \leq \phi(\sigma)\sigma \leq k\sigma^2, \qquad k > 0 \tag{7}$$

and $\phi(0) = 0$. The problem is now to determine, if possible, the value of the integral given in (5), denoted by J, for the plant given by Eqs. 6 and 7. The problem, as it is posed, is definitely not amenable to an analytical treatment. Therefore, we take the following approach. Denote by $\hat{\Phi}$ a set of admissable nonlinearities. Denote by S a subset of the state space with the property that if $x(0) \in S$, then $x(t) \in S$, for all $t \geq 0$. Let $\hat{J}(\hat{\Phi}, \hat{S})$ denote the set of real numbers which correspond to the values of (5) for any $\phi(\sigma) \in \hat{\Phi}$ and $x_0 \in \hat{S} \subseteq S$. Instead of trying to obtain the value of J for some $\phi(\sigma) \in \hat{\Phi}$ and some $x_0 \in \hat{S}$, the problem becomes one of trying to determine an upper-bound for the set $\hat{J}(\hat{\Phi}, \hat{S})$. In fact, this is really the more important problem since we usually do not know the exact specification for the nonlinearity, except possibly that it satisfies the conditions in (7). Also, we might be interested in some set of initial conditions, rather than a single initial condition. Thus, specification of the sets $\hat{\Phi}$ and $\hat{S}$ makes sense from a purely physical reasoning.

The following lemma is used in the proof of Theorem II and is stated below for completeness. The proof of the lemma easily follows from similar results in (**7**, **8**).

Lemma: Consider the differential equation $\dot{x} = f(x)$, $x(0) \in \hat{S}$ with the performance criteria

$$\int_0^\infty L(x)\,dt, \qquad L(x) > 0, \qquad x \neq 0, \qquad L(0) = 0.$$

Let $W(x)$ denote a scalar valued function which is continuously differentiable in S, such that for some real $\rho > 0$,

$$W_x'f(x) \leq -\rho L(x), \qquad \text{for all } x \in S,$$

then

$$\int_0^\infty L(x)\,dt \leq (1/\rho)W[x(0)].$$

Instead of solving the problem directly, we obtain sufficient conditions on the sets $\hat{\Phi}$ and $\hat{S}$ to guarantee that the set $\hat{J}(\hat{\Phi}, \hat{S})$ has an upperbound. As will be shown, the sets $\hat{\Phi}$ and $\hat{S}$ cannot be arbitrarily chosen.

Theorem I. It is not sufficient to choose $\hat{\Phi}_H$ as the set of characteristics lying in the Hurwitz sector in order to guarantee that for any bounded set $\hat{S}$, the set $\hat{J}(\hat{\Phi}_H, \hat{S})$ has an upper-bound.

Proof: This theorem gives a negative result. Thus, all that is needed is a counterexample, which is easily obtained by referring to a paper presenting counterexamples to Aizerman's conjecture (**11**). This paper presents examples for which there is a $\phi(\sigma) \in \hat{\Phi}_H$ such that the solution to (6) is unstable. Therefore, the corresponding value of J does not exist, and $\hat{J}(\hat{\Phi}_H, \hat{S})$ does not have an upperbound.

Theorem II. It is sufficient to choose $\hat{\Phi}_P$ as the set of characteristics lying in the Popov sector in order to guarantee that in any bounded set $\hat{S}$, the set $\hat{J}(\hat{\Phi}_P, \hat{S})$ has an upperbound. The Popov sector is defined as the set of nonlinearities satisfying the following conditions: there exists finite real positive numbers q and k such that for all $\omega \geq 0$, Re $(1 + iq\omega)W(i\omega) + (1/k) > 0$. [See (**12**) for the notation.]

Proof: Assuming that the Popov conditions are satisfied, Aizerman and Gantmacher (**12**) proved that there exists a Liapunov function for (6) of the form

$$V(x) = x'Px + \beta \int_0^\sigma \phi(\sigma)\, d\sigma.$$

Thus, $dV(x)/dt$ is negative definite along the trajectories of (6), that is,

$$x'(PF + F'P)x + \phi(\sigma)(2b'P + \beta a'F)x + \beta a'b\phi^2(\sigma) < 0, \qquad x \neq 0$$

for all $\phi(\sigma) \in \hat{\Phi}_P$. If we now consider the following continuous function of the scalar ρ, and $G > 0$

$$\psi(\rho) = x'(PF + F'P + \rho G)x + \phi(\sigma)(2b'P + \beta a'F)x + \beta a'b\phi^2(\sigma),$$

since $\psi(\rho)$ is uniformly continuous at $\rho = 0, x \in S$, there exists a $\rho > 0$ such that $\psi(\rho) < 0, x \in S - \{0\}$. Therefore,

$$V_x'(x)[Fx + b\phi(\sigma)] < -\rho x'Gx, \qquad \rho > 0, x \in S - \{0\}.$$

If we use the lemma presented earlier and choose $W(x) = V(x)$, then we have

$$\int_0^\infty x'Gx\, dt \leq (1/\rho)V[x(0)], x(0) \in \hat{S}.$$

Thus, we have exhibited an upper bound for the set $\hat{J}(\hat{\Phi}_P, \hat{S})$, and the theorem is proved.

If we are interested in the problem of specifying the sets $\hat{\Phi}$ and $\hat{S}$, and in determining an upper bound or the set $\hat{J}(\hat{\Phi}, \hat{S})$, then it is sufficient to require that $\hat{\Phi} \in \hat{\Phi}_P$ and $\hat{S}$ are bounded in order to guarantee that the set $\hat{J}(\hat{\Phi}, \hat{S})$ has an upper bound.

Since Theorem II gives only sufficient conditions for the existence of an upper bound for $\hat{J}(\hat{\Phi}, \hat{S})$, it may be possible to obtain slightly more general conditions, say, conditions which are necessary as well as sufficient. However, at present this seems unlikely.

We have obtained conditions which guarantee that the set $\hat{J}(\hat{\Phi}, \hat{S})$ has an upper bound. However, we have not presented a constructive procedure by which this upper bound can be determined. In some cases it is not difficult to construct an upper bound using the lemma. As an example, consider

$$\dot{x} = -ax + a\phi(x), \qquad x(0) = x_0$$

where $0 \leq x\phi(x) \leq hx^2$, $h < 1$, and the performance index is given by

$$J = \int_0^\infty x^2 dt.$$

We assume that $W(x) = x^2$ and determine ρ so that the hypothesis of the lemma is satisfied. Thus,

$$\begin{aligned} W_x'f(x) &= -2ax^2 + 2ax\phi(x) \\ &\leq -2ax^2(1 - h). \end{aligned}$$

Thus, if $\rho = 2a(1 - h)$ the hypothesis is satisfied and we obtain

$$J \leq [x_0^2/2a(1 - h)].$$

This example is quite simple; in fact the procedure used for determining ρ is a slight modification of a procedure proposed by Rissanen and Durbeck (8). The point we emphasize is that even though it may not be possible to obtain a simple expression for an upper bound, it can be asserted that one exists if the conditions of Theorem II are satisfied.

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