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EFFECT OF THERMAL RADIATION ON THE LAMINAR FREE CONVECTION FROM A HEATED VERTICAL PLATE

VEDAT S. ARPACI

University of Michigan, Ann Arbor, Michigan 48104, U.S.A.

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Abstract—An analytical attempt is made to understand the non-equilibrium interaction between thermal radiation and laminar free convection in terms of a heated vertical plate in a stagnant radiating gas. The effect of radiation is taken into account in the integral formulation of the problem as a one-dimensional heat flux, evaluated by including the absorption in thin gas approximation and the wall effect in thick gas approximation. The local Nusselt numbers thus obtained help to interpret the gas domains from transparent to opaque and from cold to hot. The present thick gas model approximates the radiant flux as

$$q^R = -\frac{16\sigma}{3\alpha}\left[1 - \left(1 - \frac{\epsilon_w}{2}\right)\exp\left(-\tfrac{3}{2}\alpha y\right)\right] T^3 \frac{\partial T}{\partial y}$$

whose limit for large α and small but non-zero y is the Rosseland gas, $q^R_{Rs} = -(16\sigma/3\alpha)\, T^3(\partial T/\partial y)$, and that for $y = 0$ and large but finite α is $q^R_w = -\epsilon_w(8\sigma/3\alpha)\, T^3_w(\partial T/\partial y)_w$.

NOMENCLATURE

a,	thermal diffusivity, $k/\rho c_p$;
B,	Bouguer number, $\alpha\delta$;
B_x,	local Bouguer number, αx;
c_p,	specific heat at constant pressure;
$E_n(y)$,	integro-exponential function of order n;
g,	gravitational acceleration;
G_x,	local Grashof number, $g\beta\rho^2(T_w - T_\infty)x^3/\mu^2$;
h,	heat-transfer coefficient;
k,	thermal conductivity;
N_x,	local Nusselt number, hx/k;
P,	Prandtl number, $\nu/\alpha = \mu c_p/k$;
$\mathscr{P}_\infty$,	ambient Planck number, $\alpha k T_\infty/4\sigma T_\infty{}^4$;
q,	heat flux;
T,	absolute temperature;
u,	x-component of velocity;
U,	maximum of u;
x,	variable along plate wall;
y,	variable normal to plate wall.

Greek symbols

α,	volumetric absorption of gas;
β,	coefficient of thermal expansion;
Γ_x,	local dimensionless number, $B_x/(G_x/4)^{\frac{1}{4}}$;
δ,	boundary-layer thickness;
ϵ_w,	diffuse emissivity of plate wall;
ζ,	dimensionless variable, $1 - \xi$;
η,	dimensionless variable, αy;
λ,	temperature ratio, $(T_w - T_\infty)/T_\infty$;
μ,	viscosity;
ν,	kinematic viscosity;
ξ,	dimensionless variable, η/B;
ρ,	density;
ρ_w,	diffuse reflectivity of plate wall;
σ,	Stefan–Boltzmann constant;
Φ,	φ_0; φ_1; φ_2; φ_3; φ_4; φ_5; functions defined for thick gas;
Ψ,	ψ_0; ψ_1; ψ_2; ψ_3; ψ_4; ψ_5; functions defined for thin gas.

Superscripts

C,	convection;
R,	radiation;
$-$,	mean value;
$'$,	dummy variable.

Subscripts

- 0, first approximation;
- 1, second approximation;
- w, plate wall;
- ∞, ambient;
- x, local;
- Rs, Rosseland gas.

INTRODUCTION

MOTIVATED by the technological demand and provided by the present level of applied science, the effect of thermal radiation on gas dynamics and/or heat-transfer problems has received increased attention in the last decade. Because of the size of the literature, no attempt will be made here to give a complete list (see, however, references cited in Viskanta [1] for heat-transfer problems and those in Cheng [2] for gas dynamics problems). Although recent works on gas dynamics consider multi-dimensional radiation effects and place no restriction on the absorption of gas, studies on boundary layers (which are mainly on forced convection), are restricted to one-dimensional radiation effects, evaluated on the basis of thick gas and thin gas approximations. In thin gas, the absorption of gas is neglected in the boundary layer; in thick (Rosseland) gas, the wall effect is excluded. Although the assumption of one-dimensional radiation can be justified on grounds of boundary-layer physics, the existing thin gas and thick gas approximations need improvement in order that the effect of radiation on boundary layers may be shown for all values of the absorption of gas. So far as the author is aware, no published work exists on free convection except a recent attempt by Blake [3] which rests on the approximations above and that by Cess [4] involving the cases of hot gas or slightly absorbing gas.

A preliminary study is made here to understand the non-equilibrium interaction between thermal radiation and laminar convection for all values of the absorption of gas, using a heated vertical plate in a stagnant gas as a vehicle. The main objective of the work is to consider a model for thin gas absorbing inside, as well as outside, of the boundary layer, and that for thick gas including the wall effect. Thus it becomes possible to represent the local Nusselt number evaluated on the basis of these approximations, as well as velocity and temperature profiles, in terms of common dimensionless numbers, and to interpret the gas domains from transparent to opaque and from cold to hot. Furthermore, the behavior of Rosseland gas at boundaries is clarified.

FORMULATION

Consider a heated, semi-infinite, vertical plate in an infinite expanse of stagnant, radiating gas. To simplify the problem and isolate the influence of radiation, the following assumptions are made: the gas is perfect and gray; radiation scattering, radiation pressure, and the contribution of radiation to internal energy are negligible; the effect of radiation is included to the energy equation as a one-dimensional heat flux; the plate diffusely radiates as a gray body; non-equilibrium effects other than diffusion and radiation are negligible.

On the foregoing basis, the usual integral formulation of the problem is modified to include the effect of radiation. This gives

$$\rho \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta u^2\,\mathrm{d}y = \rho g\beta\int_0^\delta (T - T_\infty)\,\mathrm{d}y - \mu\left(\frac{\partial u}{\partial y}\right)_w, \tag{1}$$

$$\rho c_p \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta u(T - T_\infty)\,\mathrm{d}y = -k\left(\frac{\partial T}{\partial y}\right)_w - q^R\Big|_w^\delta, \tag{2}$$

where the radiant flux (see, for example, Blake [3]) is

$$q^R = 2\sigma\Big[\epsilon_w T_w^4 E_3(\eta) + \int_0^\eta T^4(\eta')\,E_2(\eta - \eta')\,\mathrm{d}\eta' - \int_\eta^\infty T^4(\eta')\,E_2(\eta' - \eta)\,\mathrm{d}\eta' + 2\rho_w E_3(\eta)\int_0^\infty T^4(\eta')\,E_2(\eta')\,\mathrm{d}\eta'\Big], \tag{3}$$

σ being the Stefan–Boltzmann constant, $\eta = \alpha y$, $\eta' = \alpha y'$, α the absorption coefficient of gas, ϵ_w and ρ_w the diffuse emissivity and diffuse reflectivity of the plate walls, respectively, and E_2 and E_3 the special cases of the integro-exponential function of n, defined by

$$E_n(y) = \int_0^1 t^{n-2} \exp(-y/t)\,dt. \tag{4}$$

A full discussion of this function and its properties may be found in Kourganoff [5].

Equation (3) may be integrated by parts for later convenience (see Vincenti and Baldwin [6] for similar manipulations on the divergence of radiant heat flux). The result is

$$q^R = -8\sigma\Big[\int_0^\eta T^3(\partial T/\partial\eta')\,E_3(\eta - \eta')\,d\eta' + \int_\eta^\infty T^3(\partial T/\partial\eta')\,E_3(\eta' - \eta)\,d\eta' - 2\rho_w E_3(\eta)\int_0^\infty T^3(\partial T/\partial\eta')\,E_3(\eta')\,d\eta'\Big]. \tag{5}$$

The difference between the values of q^R evaluated at the boundary layer and the plate walls may readily be found from equation (5) to be

$$q^R\big|_w^\delta = 8\sigma\Big\{\int_0^{\alpha\delta} T^3(\partial T/\partial\eta')\big[E_3(\eta') - E_3(\alpha\delta - \eta')\big]\,d\eta' - \rho_w[1 - 2E_3(\alpha\delta)] \times \int_0^{\alpha\delta} T^3(\partial T/\partial\eta')\,E_3(\eta')\,d\eta'\Big\}. \tag{6}$$

Next the solution of the problem is considered.

SOLUTION

The solution of equations (1, 2, 6) presents, even in terms of simple profiles, considerable mathematical difficulties for the intended investigation. In view of this, the integro-exponential function is approximated in the usual manner by a simple exponential of the form

$$E_n(y) = a\exp(-by).$$

The appropriate choice of the constants a and b depending on the values of n is not unique, and a number of values have already been used for them on different grounds. Noting, however, that the present problem requires the approximation of $E_3(y)$ only, and that the use of the correct value of $E_3(0)$ is important for boundary-layer problems, the Lick approximation [7] is employed. This gives

$$E_3(y) = \exp(-3y/2)/2. \tag{7}$$

Inserting equation (7) into (6), introducing the Bouguer number $B = \alpha\delta$, and the dimensionless variables $\xi' = \eta'/B$ and $\zeta' = 1 - \xi'$, the radiant flux may be arranged to give

$$q^R\big|_w^\delta = 4\sigma\Big\{\int_0^1 T^3(\partial T/\partial\zeta')\exp(-3B\zeta'/2)\,d\zeta' - \{1 - \rho_w[1 - \exp(-3B/2)]\}\exp(-3B/2) \times \int_0^1 T^3(\partial T/\partial\zeta')\exp(+3B\zeta'/2)\,d\zeta'\Big\}. \tag{8}$$

Now the solution of equations (1, 2, 8) presents, at least in principle, no difficulties in terms of polynomial profiles. However, the integrals associated with the radiant flux yield rather lengthy expressions for any practical use. This difficulty will be circumvented by evaluating the radiant flux for small and large values of B. Hereafter the conditions $B \ll 1$ and $B \gg 1$ will be referred to as the definition of thin gas and that of thick gas, respectively.

(a) *Thin gas* ($B \ll 1$)

Replacement of the exponential terms in equation (8) by the first two terms of their Maclaurin expansions gives

$$q^R\big|_w^\delta = 4\sigma(3B/2)\Big[(1 + \rho_w)\int_0^1 T^3(\partial T/\partial\zeta')\,d\zeta' - 2\int_0^1 T^3(\partial T/\partial\zeta')\,\zeta'\,d\zeta' + O(B)\Big], \tag{9}$$

where O implies the order.

Recalling the preliminary nature of the present investigation, the selection of the velocity and temperature profiles may be confined to the first order profiles,

$$u = U(y/\delta)(1 - y/\delta)^2, \tag{10}$$

$$(T - T_\infty)/(T_w - T_\infty) = (1 - y/\delta)^2, \tag{11}$$

previously used by Squire for the same problem in the absence of radiation (see, for example, Howarth [8]).

Expressing equation (11) in terms of ζ and then introducing into equation (9) yields

$$q^R|_w^\delta = 4\lambda\sigma T_\infty^4 B[\psi_1 + O(B)], \tag{12}$$

where

$$\psi_1 = \tfrac{3}{2}[(1+\rho_w)(1 + 3\lambda/2 + \lambda^2 + \lambda^3/4) - 4(\tfrac{1}{3} + 3\lambda/5 + 3\lambda^2/7 + \lambda^3/9)],$$

and $\lambda = (T_w - T_\infty)/T_\infty$.

Inserting equations (10–12) into equations (1) and (2), and neglecting terms of order B^2 and higher results in

$$\frac{1}{105}\frac{\mathrm{d}}{\mathrm{d}x}(U^2\delta) = \tfrac{1}{3}g\beta(T_w - T_\infty)\,\delta - \nu\frac{U}{d}, \tag{13}$$

$$\frac{1}{30a}\frac{\mathrm{d}}{\mathrm{d}x}(U\delta) = \frac{2}{\delta} + \left(\frac{4\sigma T_\infty^4}{kT_\infty}\right)\psi_1\delta, \tag{14}$$

where $a = k/\rho c_p$ is the thermal diffusivity. Note that for the limiting case $\sigma T_\infty^4 \to 0$ (cold gas), equations (13) and (14) reduce to those obtained by Squire, as expected.

Clearly, $B = \alpha\delta \ll 1$ suggests the solution procedure for equations (13) and (14) as that of a regular perturbation on the Squire problem. Thus multiplying equations (13) and (14) by δ, introducing the mean Bouguer number $\bar{B} = \alpha\bar{\delta}$ (in terms of a mean boundary-layer thickness which will drop out later), and using the first two terms of the expansions

$$U = \sum_{n=0}^{N} (\bar{B}^2/\mathscr{P}_\infty)^n\, U_n, \qquad \delta = \sum_{n=0}^{N} (\bar{B}^2/\mathscr{P}_\infty)^n\, \delta_n, \tag{15}$$

where $\mathscr{P}_\infty = \alpha k T_\infty/4\sigma T_\infty^4$ is the ambient Planck number, yields for $(\bar{B}^2/\mathscr{P}_\infty)^0$

$$\frac{\delta_0}{105}\frac{\mathrm{d}}{\mathrm{d}x}(U_0^2\delta_0) = \tfrac{1}{3}g\beta(T_w - T_\infty)\,\delta_0^2 - \nu U_0, \tag{16}$$

$$\frac{\delta_0}{30\alpha}\frac{\mathrm{d}}{\mathrm{d}x}(U_0\delta_0) = 2, \tag{17}$$

and for $(\bar{B}^2/\mathscr{P}_\infty)^1$

$$\frac{\delta_0}{105}\frac{\mathrm{d}}{\mathrm{d}x}(U_0^2\delta_1 + 2U_0U_1\delta_0) + \frac{\delta_1}{105}\frac{\mathrm{d}}{\mathrm{d}x}(U_0^2\delta_0) = \tfrac{2}{3}g\beta(T_w - T_\infty)\,\delta_0\delta_1 - \nu U_1, \tag{18}$$

$$\frac{\delta_0}{30a}\frac{\mathrm{d}}{\mathrm{d}x}(U_0\delta_1 + U_1\delta_0) + \frac{\delta_1}{30a}\frac{\mathrm{d}}{\mathrm{d}x}(U_0\delta_0) = \psi_1\frac{\delta_0^2}{\bar{\delta}^2}. \tag{19}$$

Squire obtained the solution of equations (16) and (17) by assuming the velocity and the boundary-layer thickness of the form

$$U_0 = C_0 x^{m_0}, \qquad \delta_0 = D_n x^{n_0}. \tag{20}$$

The result is

$$U_0 = 40[g\beta(T_w - T_\infty)]^{\frac{1}{2}} x^{\frac{1}{2}}/\psi_2^2,$$
$$\delta_0 = (2a)^{\frac{1}{2}}\psi_2 x^{\frac{1}{4}}/[g\beta(T_w - T_\infty)]^{\frac{1}{4}}, \tag{21}$$

where $\psi_2 = 2^{\frac{1}{2}}(100/7 + 15P)^{\frac{1}{4}}$ and P is the Prandtl number.

In general, the form of first approximations does not necessarily imply the form of second approximations. On the contrary, second approximations often remain in differential form when they are expressed in terms of first approximations. However, the inherent nature of the present problem allows the solutions of similar nature,

$$U_1 = C_1 x^{m_1}, \qquad \delta_1 = D_1 x^{n_1}, \tag{22}$$

for equations (18) and (19). This gives

$$U_1 = a\psi_1\psi_2^4\psi_3\psi_4 x/\bar{\delta}^2, \tag{23}$$

$$\delta_1 = (2a)^{\frac{3}{2}}\psi_1\psi_2^3\psi_3 x^{\frac{3}{4}}/[g\beta(T_w - T_\infty)]^{\frac{1}{4}}\bar{\delta}^2, \tag{24}$$

where

$$\psi_3 = (8/3 + P)/4(68/21 + 3P),$$
$$\psi_4 = 8(P - 4/21)/(8 + 3P)(20/21 + P),$$

and other ψ's were defined following equations (11) and (21).

A parametric study of the velocity and temperature profiles would be rather space consuming due to the number of parameters involved. Because of this fact the study will be confined to the local heat flux only. The local heat-transfer coefficient may now be defined, including the radiant as well as convective effects, as

$$q_w = h(T_w - T_\infty) = q_w^R + q_w^C, \tag{25}$$

where q_w^C being the convective flux,

$$q_w^C = -k(\partial T/\partial y)_w = 2k(T_w - T_\infty)/\delta, \tag{26}$$

evaluated by equation (11). In terms of the exponential approximation given by equation (7), the radiant flux is obtained from equation (5) to be

$$q_w^R = 4\sigma\epsilon_w \exp(-3B/2)\int_0^1 T^3(\partial T/\partial \xi') \times \exp(+3B\xi'/2)\,\mathrm{d}\xi'. \tag{27}$$

For small values of B, retaining again the first two terms in the Maclaurin expansion of the exponentials, equation (27) may be reduced to

$$q_w^R = 4\sigma\epsilon_w\Big[\int_0^1 T^3(\partial T/\partial\xi')\,\mathrm{d}\xi' - (3B/2) \times \int_0^1 T^3(\partial T/\partial\xi')(1-\xi')\,\mathrm{d}\xi'\Big]. \tag{28}$$

Insertion of the temperature profile given by equation (11) into equation (28) yields

$$q_w^R = 4\lambda\epsilon_w\sigma T_\infty^4[\psi_0 - B\psi_5 + O(B^2)], \tag{29}$$

where

$$\psi_0 = 1 + 3\lambda/2 + \lambda^2 + \lambda^3/4$$

and

$$\psi_5 = \tfrac{1}{2} + 9\lambda/20 + 3\lambda^2/14 + \lambda^3/24.$$

Introducing equations (26) and (29) into equation (25), and neglecting terms of order B^2 and higher, the local Nusselt number is found to be

$$N_x = hx/k = 2(x/\delta) + \epsilon_w(\alpha x)(\psi_0 - \psi_5\alpha\delta)/\mathscr{P}_\infty, \tag{30}$$

where x/δ and $\alpha\delta$ may be obtained from the second expansion of (15), the second term of (21) and equation (24) as

$$\delta/x = \psi_2\Psi/P^{\frac{1}{2}}(G_x/4)^{\frac{1}{4}}, \tag{31}$$

$$\alpha\delta = \psi_2\Psi\Gamma_x/P^{\frac{1}{2}}, \tag{32}$$

where

$$\Psi = 1 + \psi_1\psi_2^2\psi_3(\Gamma_x/\mathscr{P}_\infty)^2\mathscr{P}_\infty/P,$$

$$\Gamma_x = B_x/(G_x/4)^{\frac{1}{4}},$$

$B_x = \alpha x$ (the local Bouguer number) and $G_x = g\beta\rho^2(T_w - T_\infty)x^3/\mu^2$ (the local Grashof number). Finally, rearranging equation (30) in terms of equation (31) and (32) results in

$$N_x/(G_x/4)^{\frac{1}{4}} = 2P^{\frac{1}{2}}/\psi_2\Psi + \epsilon_w(\Gamma_x/\mathscr{P}_\infty)[\psi_0 - \psi_2\psi_5\Psi(\Gamma_x/\mathscr{P}_\infty)\mathscr{P}_\infty/P^{\frac{1}{2}}]. \tag{33}$$

The use of $G_x/4$ in equations (31) and (33) rather than G_x is for customary reasons only. In the limit as $\Gamma_x/\mathscr{P}_\infty \to 0$, $\Psi \to 1$ and equation (33) approaches the Squire solution,

$$N_x/(G_x/4)^{\frac{1}{4}} = (2P)^{\frac{1}{2}}/(100/7 + 15P)^{\frac{1}{2}}, \tag{34}$$

provided $0 \leqslant \mathscr{P}_\infty < \infty$. The physical implications of $\Gamma_x/\mathscr{P}_\infty \to 0$ and $\mathscr{P}_\infty \to 0$ are identical to those discussed following equation (14). Although the limit $\mathscr{P}_\infty \to \infty$ is not permissible, by keeping $\Gamma_x/\mathscr{P}_\infty \to 0$, $\mathscr{P}_\infty$ may be increased to a reasonably large value. This corresponds to a reasonably opaque, cold gas. Equation (33) will be plotted together with the similar expression to be obtained from the thick gas approximation that we consider next.

(b) *Thick gas* ($B \gg 1$)

The details of the thick gas solution are left out because of their similarity to those of the thin gas solution.

Noting that $\exp(-3B/2) \ll 1$ for $B \gg 1$, the radiant flux given by equation (8) may be rearranged as

$$q_w^R\Big|_w^\delta = 4\sigma\Big[\int_0^1 T^3(\partial T/\partial\zeta')\exp(-3B\zeta'/2)\,\mathrm{d}\zeta' - \epsilon_w\exp(-3B/2)\int_0^1 T^3(\partial T/\partial\zeta')\exp(+3B\zeta'/2)\,\mathrm{d}\zeta'\Big]. \tag{35}$$

In terms of the temperature profile selected, and by the successive use of integration by parts, equation (35) becomes

$$q^R\Big|_w^\delta = -8\lambda\sigma T_\infty{}^4(1/B)\left\{\tfrac{2}{3}\epsilon_w(1+\lambda)^3 - (1/B)\varphi_1 + O[(1/B)^2]\right\}, \quad (36)$$

where

$$\varphi_1 = \tfrac{4}{9}[1 + \epsilon_w(1+7\lambda)(1+\lambda)^2].$$

The momentum equation resulting in equation (13) remains valid for the thick gas; equation (14) is modified, however, in terms of equation (36). This gives, after neglecting terms of order $(1/B)^2$ and higher,

$$\frac{1}{30a}\frac{\mathrm{d}}{\mathrm{d}x}(U\delta) = \frac{2}{\delta}\left(\varphi_0 - \frac{\varphi_1}{\alpha\delta\mathscr{P}_\infty}\right) \quad (37)$$

where $\varphi_0 = 1 + 2\epsilon_w(1+\lambda)^3/3\mathscr{P}_\infty$, and φ_1 was defined following equation (36). The limiting case $\mathscr{P}_\infty \to \infty$, which corresponds to $\alpha \to \infty$ (opaque gas) or to $\sigma T_\infty^4 \to 0$ (cold gas), is the Squire problem, as expected.

Inspection of equations (13) and (37) by the consideration of $B = \alpha\delta \gg 1$ suggests again a perturbation procedure for the solution. Thus multiplying equation (13) by δ and equation (37) by δ^2, and using the first two terms of the expansions

$$U = \sum_{n=0}^{N} U_n/\bar{B}^n, \qquad \delta = \sum_{n=0}^{N} \delta_n/\bar{B}^n \quad (38)$$

gives for $1/\bar{B}^0$

$$\frac{\delta_0}{105}\frac{\mathrm{d}}{\mathrm{d}x}(U_0^2\delta_0) = \tfrac{1}{3}g\beta(T_w - T_\infty)\,\delta_0^2 - \nu U_0, \quad (39)$$

$$\frac{\delta_0^2}{60a}\frac{\mathrm{d}}{\mathrm{d}x}(U_0\delta_0) = \varphi_0\delta_0, \quad (40)$$

and for $1/\bar{B}^1$

$$\frac{\delta_0}{105}\frac{\mathrm{d}}{\mathrm{d}x}(U_0^2\delta_1 + 2U_0U_1\delta_0) + \frac{\delta_1}{105}\frac{\mathrm{d}}{\mathrm{d}x}(U_0^2\delta_0) = \tfrac{2}{3}g\beta(T_w - T_\infty)\,\delta_0\delta_1 - \nu U_1, \quad (41)$$

$$\frac{\mathrm{d}_0^2}{60a}\frac{\mathrm{d}}{\mathrm{d}x}(U_0\delta_1 + U_1\delta_0) + \frac{\delta_0\delta_1}{30a}\frac{\mathrm{d}}{\mathrm{d}x}(U_0\delta_0) = \varphi_0\delta_1 - \frac{\varphi_1\bar{\delta}}{\mathscr{P}_\infty}. \quad (42)$$

Since equations (16) and (39) are identical, and equations (17) and (40) differ by a parameter only, the particular forms similar to those of equation (20),

$$U_0 = C_0^* x^{m_0}, \qquad \delta_0 = D_0^* x^{n_0}, \quad (43)$$

may be used for the solution of equations (39) and (40). (Clearly, the effect of radiation is included in the coefficients C_0^* and D_0^*, the exponents m_0 and n_0 remaining identical to those found in the absence of radiation.) The result is

$$U_0 = 40[g\beta(T_w - T_\infty)]^{\frac{1}{2}}x^{\frac{1}{2}}/\varphi_2^2,$$
$$\delta_0 = (2a\varphi_0)^{\frac{1}{2}}x^{\frac{1}{4}}/[g\beta(T_w - T_\infty)]^{\frac{1}{4}}, \quad (44)$$

where $\varphi_2 = 2^{\frac{1}{4}}(100/7 + 15P/\varphi_0)^{\frac{1}{4}}$, and φ_0 was defined following equation (37). Note that $\varphi_2 \to \psi_2$ as $\varphi_0 \to 1$ for $\mathscr{P}_\infty \to \infty$.

Inspection reveals that equations (41) and (42), in a manner similar to the solution of first approximations, assume solutions of the form

$$U_1 = C_1^* x^{m_1^*}, \qquad \delta_1 = D_1^* x^{n_1^*}. \quad (45)$$

This gives

$$U_1 = -(2/a\varphi_0)^{\frac{1}{2}}\varphi_1\varphi_2\varphi_3\varphi_4\bar{\delta}\,[g\beta(T_w - T_\infty)]^{\frac{3}{4}}x^{\frac{1}{4}}/\mathscr{P}_\infty, \quad (46)$$

$$\delta_1 = -\varphi_1\varphi_3\bar{\delta}/\varphi_0\mathscr{P}_\infty, \quad (47)$$

where

$$\varphi_3 = (32\varphi_0/21 + P)/(24\varphi_0/7 + 11P),$$

$$\varphi_4 = (\tfrac{4}{3} + 14P/\varphi_0)/(20 + 21P/\varphi_0)(32\varphi_0/21 + P),$$

and other φ's were defined following equations (36, 37, 44).

The wall radiant flux given by equation (27) is also valid for the present case. This equation yields, in terms of the temperature profile assumed,

$$q_w^R = 8\lambda\sigma T_\infty^4(1/B)\{\tfrac{2}{3}\epsilon_w(1+\lambda)^3 - \tfrac{1}{5}\varphi_5 + O[(1/B)^2]\}, \quad (48)$$

where $\varphi_5 = 4(1 + 7\lambda)(1 + \lambda)^2/9$.

Finally, combining equations (25, 26, 48), and neglecting terms of order $(1/B)^3$ and higher, yields the local Nusselt number,

$$N_x = hx/k = 2(x/\delta)(\varphi_0 - \epsilon_w\varphi_5/\alpha\delta\mathscr{P}_\infty), \quad (49)$$

for which x/δ and $\alpha\delta$ may be evaluated from the second expansion of (38), the second term of (44) and equation (48) as

$$\delta/x = \Phi/(G_x/4)^{\frac{1}{4}}, \qquad \alpha\delta = \Phi\Gamma_x, \quad (50)$$

where $\Phi = (\varphi_0/P)^{\frac{1}{2}}\varphi_2 - \varphi_1\varphi_3/\varphi_0\mathscr{P}_\infty\Gamma_x$. Insertion of equation (50) into equation (49) results in

$$N_x/(G_x/4)^{\frac{1}{4}} = (2/\Phi)(\varphi_0 - \epsilon_w\varphi_5/\Phi\mathscr{P}_\infty\Gamma_x). \quad (51)$$

Note that the limit $\mathscr{P}_\infty\Gamma_x \to \infty$ gives the asymptote of equation (51) depending on $0 < \mathscr{P} \leqslant \infty$. As $\mathscr{P}_\infty\Gamma_x \to \infty$, $\Phi \to (\varphi_0/P)^{\frac{1}{2}}\varphi_2$ and equation (51) becomes

$$N_x/(G_x/4)^{\frac{1}{4}} = 2(\varphi_0 P)^{\frac{1}{2}}/\varphi_2, \qquad 0 < \mathscr{P}_\infty \leqslant \infty. \quad (52)$$

Furthermore, as $\mathscr{P}_\infty \to \infty$, $\varphi_0 \to 1$ and $\varphi_2 \to \psi_2$, and equation (52) reduces to equation (34), the result previously obtained by Squire. The physics associated with the limit $\mathscr{P}_\infty \to \infty$ was discussed following equation (37). The other limit $\mathscr{P}_\infty \to 0$ cannot be materialized because of the restriction $\mathscr{P}_\infty\Gamma_x \to \infty$. However, by keeping $\mathscr{P}_\infty\Gamma_x$ large, $\mathscr{P}_\infty$ may be reduced to a reasonably small value. This corresponds to a small α (nearly transparent gas) or to a large σT_∞^4 (reasonably hot gas).

Note that the local Nusselt numbers evaluated on the basis of thin gas and thick gas approximations, and given by equations (33) and (51), are in terms of common dimensionless numbers, and are expressible in the form

$$N_x/(G_x/4)^{\frac{1}{4}} = f\,[\lambda, \epsilon_w, P, \mathscr{P}_\infty, B_x/(G_x/4)^{\frac{1}{4}}].$$

Figure 1 shows $N_x/(G_x/4)^{\frac{1}{4}}$ vs. $B_x/(G_x/4)^{\frac{1}{4}}$ for values $0 < \mathscr{P}_\infty \leqslant \infty$; the three dimensionless numbers remaining are held constant, assuming a typical value of Prandtl number for gases, $P = 0{\cdot}733$, black walls for the plate, $\epsilon_w = 1$, and $\lambda = 0{\cdot}1$ which may be interpreted as the walls being 54 degF above the ambient gas at 77°F.

An important problem of gas radiation, the behavior of thick gas near boundaries, may be discussed now by comparing the present model

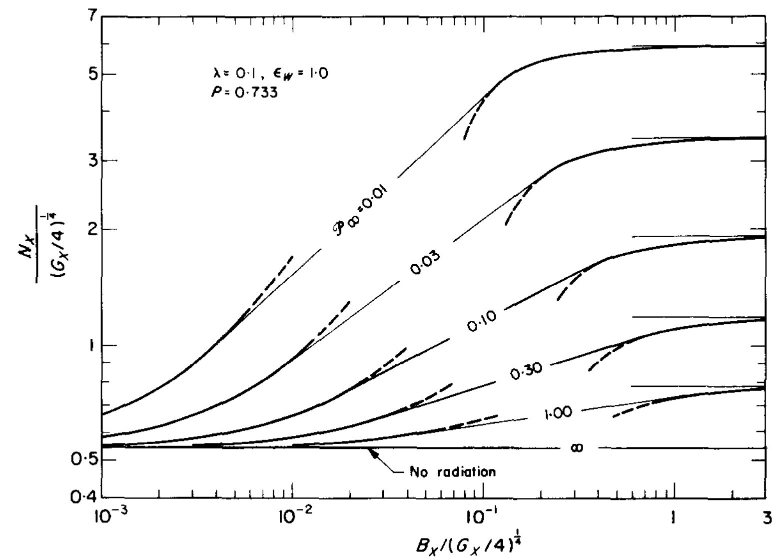

Fig. 1.

for thick gas with the Rosseland gas: rearrangement of equation (5) in terms of ξ' and $\xi = \eta/B$ yields

$$q^R = -4\sigma\{\int_0^\xi T^3(\partial T/\partial\xi')\exp[-3B(\xi-\xi')/2]\,d\xi' + \int_\xi^1 T^3(\partial T/\partial\xi')\exp[-3B(\xi'-\xi)/2]\,d\xi' - \rho_w\exp(-3B\xi/2)\int_0^1 T^3(\partial T/\partial\xi') \times \exp(-3B\xi'/2)\,d\xi'\}. \quad (53)$$

Expressing the assumed temperature profile, equation (11), in terms of ξ, ξ' and inserting the result into equation (53) results in

$$q^R = (16\sigma\lambda T_\infty^4/3B)\{2(1-\xi)[1+\lambda(1-\xi)^2]^3 - (1+\rho_w)(1+\lambda)^3\exp(-3B\xi/2) + O(1/B)\}. \quad (54)$$

Evaluation of the heat flux in terms of the Rosseland gas, $q_{Rs}^R = -(16\sigma/3\alpha)\,T^3(\partial T/\partial y)$, and the same temperature profile gives

$$q_{Rs}^R = (16\sigma\lambda T_\infty^4/3B)\,2(1-\xi)[1+\lambda(1-\xi)^2]^3. \quad (55)$$

After neglecting the terms of order $(1/B)^2$ and higher in equation (54), the ratio of equation (54) to equation (55) may be written as

$$q^R/q_{Rs}^R = 1 - (1-\epsilon_w/2)(1+\lambda)^3 \times \exp(-3B\xi/2)/(1-\xi)[1+\lambda(1-\xi)^2]^3. \quad (56)$$

Noting that near boundaries $\xi \ll 1$, and that $B\xi = \alpha y$, equation (56) may be approximated to yield

$$q^R = -\frac{16\sigma}{3\alpha}\left[1-\left(1-\frac{\epsilon_w}{2}\right) \times \exp(-\tfrac{3}{2}\alpha y)\right] T^3\frac{\partial T}{\partial y}. \quad (57)$$

When α becomes large while y remaining small but not zero, equation (57) approaches the Rosseland gas; when $y = 0$ and α is large but finite, equation (57) reduces to

$$q_w^R = -\epsilon_w(8\sigma/3\alpha)\,T_w^3(\partial T/\partial y)_w,$$

the result previously obtained by Deissler [9]. Since for the latter case $\exp(-3\alpha y/2) = 1$ rather than being zero, the omission of wall effect in the Rosseland gas leads to an erronious result near boundaries. It is apparent now that the expansion employed for the Rosseland approximation is not uniformly valid in the domain $0 \leqslant y < \infty$, and it can only be an outer expansion for the thick gas model. An inner expansion valid within the radiant attenuation depth where $y = O(1/\alpha)$ should be considered for the behavior of thick gas near boundaries. This problem is presently under investigation. For the time being however, equation (57) may be utilized in the solution of other boundary-layer problems because of its apparent independence from the present solution.

In Fig. 2 the effect of λ on the local Nusselt number is shown by holding $\mathscr{P}_\infty$ constant at 0·1; note that higher values of λ cannot be allowed because of the constant-property basis of the present formulation. The thin gas and thick gas approximations plotted in Figs. 1 and 2 are joined, arbitrarily to some extent, by their common tangent for a continuous representation.

Since the thin gas approaches the Squire problem as $B_x/(G_x/4)^{\frac{1}{4}} \to 0$, the effect of wall emissivity on Fig. 1 may be best understood by considering the limit of thick gas for $B_x/(G_x/4)^{\frac{1}{4}} \to \infty$ given by equation (52). Figure 3 shows the local Nusselt number of this equation versus ϵ_w for $0 < \mathscr{P}_\infty \leqslant \infty$, keeping λ and P constant.

The sketch of Fig. 4 summarizes the physics of the problem, by showing the effects of the temperature and absorptivity of gas, the emissivity of plate walls, and the ratio of the temperature difference between plate walls and ambient gas to the ambient gas temperature on the local Nusselt number.

A direct comparison of the results obtained from the present thin gas solution with those of reference [4] is not possible because of the different Prandtl numbers considered. An order of magnitude comparison, however, shows that

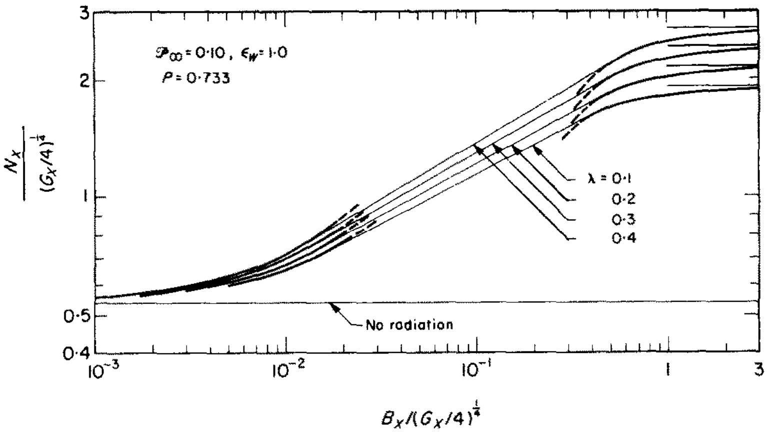

Fig. 2.

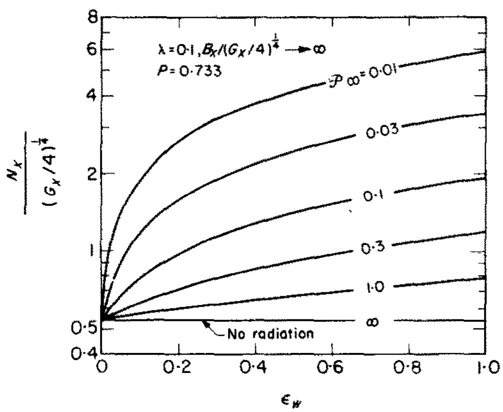

Fig. 3.

the two solutions do not deviate more than 5 per cent.

CONCLUSIONS

An attempt has been made to understand qualitatively the heat transfer from a heated vertical plate to a stagnant radiating gas. The study rests on the use of exponential approximation for the attenuation factor E_3. In accordance with this approximation, the integral formulation of the problem is considered, and first order profiles are employed.

It becomes apparent, after some attempts, that solutions of boundary-layer problems with

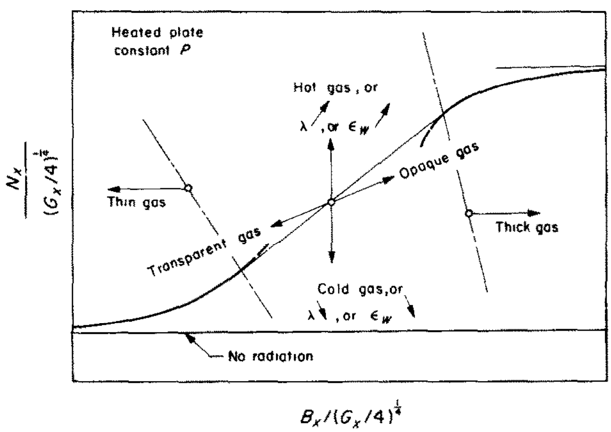

Fig. 4.

no restriction on the absorption of gas are rather involved. It may then be necessary, unless a numerical procedure is considered, to include the radiation effects into these problems on the basis of thin gas and thick gas approximations, and except that the results will be informative for the entire range of the absorption of gas. The Rosseland gas falls short in complementing the thin gas, and implies its oversimplified nature for boundary-layer problems. This difficulty is eliminated in the present work by including the wall effect in thick gas, and also that the absorption of thin gas in the boundary layer.

The most critical approximation in the present work is that of the gray gas. Since the dependence of absorption on the frequency is rather strong in actual cases, the problem should be reconsidered by including this dependence before any serious attempt to check the analytical results against experiment. Meanwhile the gray gas analyses should continue, at least in the near future, to provide further insight into heat-transfer problems in radiating gas.

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Résumé—Un essai théorique est effectué pour comprendre l'intéraction en non-équilibre entre le rayonnement thermique et la convection naturelle laminaire dans le cas d'une plaque verticale chauffée dans un gaz rayonnant au repos. L'effet du rayonnement est pris en compte dans la formulation intégrale du problème comme un flux de chaleur unidimensionnel, évalué en comprenant l'absorption dans l'approximation du gaz mince et l'effet de paroi dans l'approximation du gaz épais. Les nombres de Nusselt locaux obtenus ainsi aident à interpréter les domaines du gaz entre les cas transparent et opaque et entre le froid et le chaud.

Le modèle actuel du gaz épais donne le flux rayonnant d'une façon approchée sous la forme:

$$q^R = \frac{16\sigma}{3\alpha}\left[1 - \left(1 - \frac{\varepsilon_w}{2}\right)\exp\left(-\tfrac{3}{2}\alpha y\right)\right] T^3 \frac{\partial T}{\partial y}$$

dont la limite pour de grands α et des valeurs de y faibles mais non nulles est le gaz de Rosseland, $q^R_{Rs} = -(16\sigma/3\alpha)\, T^3(\partial T/\partial y)$ et celle pour $y = 0$ et pour des valeurs de α grandes mais finies est

$$q^R_w = -\varepsilon_w(8\sigma/3\alpha)\, T^3_w(\partial T/\partial y)_w.$$

Zusammenfassung—Es wird der analytische Versuch gemacht, die Nicht-Gleichgewichtswechselwirkung zwischen Wärmestrahlung und laminarer freier Konvektion in der gewohnten Art für eine beheizte senkrechte Platte in einem ruhenden, strahlenden Gas zu bestimmen. Der Strahlungseinfluss wird als eindimensionaler Wärmestrom in der Integralformulierung des Problems angesetzt und gelöst unter Berücksichtigung der Absorption in der Näherung für optisch dünne Gase und des Wandeffekts in der Näherung für optisch dicke Gase.

Mit Hilfe der so erhaltenen Nusselt-Zahlen, lässt sich der Einfluss von Durchlässigkeit und Temperatur des Gases ersehen. Das gegenwärtige Modell für ein optisch dickes Gas liefert für den Strahlungsstrom näherungsweise die Gleichung

$$q^R = -\frac{16\sigma}{3\alpha}\left[1-\left(1-\frac{\varepsilon_w}{2}\right)\exp\left(-\tfrac{3}{2}\alpha y\right)\right]T^3\frac{\partial T}{\partial y}$$

deren Grenzwert für grosses α und kleines aber endliches y das Rosseland Gas ist,

$$q^R_{Rs} = -(16\sigma/3\alpha)\,T^3(\partial T/\partial y)$$

und für $y = 0$ und grosses aber endliches α

$$q^R_w = -\varepsilon_w(8\sigma/3\alpha)T^3_w(\partial T/\partial y)_w \text{ ergibt.}$$

Аннотация—Сделана попытка аналитически описать неравновесное взаимодействие между тепловым излучением и свободной ламинарной конвекцией на примере нагретой вертикальной пластины в неподвижном излучающем газе. Влияние излучения учитывается при интегральной формулоровке задачи через одномерный тепловой поток, рассчитываемый с учетом абсорбции при тонкой аппроксимации газа и влиянии стенки при грубой аппроксимации газа. Полученные таким образом значения локальных чисел Нуссельта помогают описать состояния газа от прозрачного до непрозрачного и от холодного до горячего. Данная грубая модель газа дает следующую приближенную формулу для лучистого потока

$$q^R = \frac{16\sigma}{3\alpha}\left[1-\left(1-\frac{\epsilon_w}{2}\right)\exp\left(-\frac{3}{2}\alpha y\right)\right]T^3\frac{\partial T}{\partial y}$$

которая в пределе для больших α и малых, но равных нулю y переходит в формулу, справедливую для газа Росселанда $q^R_{Rs} = -(16\sigma/3\alpha)T^3(\partial T/\partial y)$ и для $y = 0$ и больших, но конечных значений α имеет вид

$$q^R_w = -\epsilon_w(8\sigma/3\alpha)\,T^3_w(\partial T/\partial y)_w.$$