A Necessary and Sufficient Condition that an Operator be Normal

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If $A$ is a normal operator on a Hilbert space and if $P$ is a polynomial in $A$ and $A^*$, then $P$ is normal. Thus, if $r$ denotes spectral radius, $\| P \| = r(P)$ for every operator $P$ that is a polynomial in $A$ and $A^*$.

Our interest in the converse problem was stimulated by a conversation with C. R. MacCluer.

**THEOREM.** If $A$ is a bounded operator on a Hilbert space, and if $\| P \| = r(P)$ for every operator $P$ that is a polynomial in $A$ and $A^*$, then $A$ is normal.

**PROOF.** Translating $A$ by an appropriate scalar we can assume that $\text{Re } A = (A + A^*)/2$ and $\text{Im } A = (A - A^*)/2i$ are strictly positive operators. It then follows by a well-known result (see [1]) that the spectrum of the product operator $B = (\text{Re } A)(\text{Im } A)$ is contained in the positive real axis. To show that $A$ is normal it suffices to show that $\text{Re } A$ and $\text{Im } A$ commute, and this will follow immediately if we show that $B$ is Hermitian.

If $p$ is any polynomial in one variable then $\| p(B) \| = r(p(B))$ by hypothesis. Thus by the spectral mapping theorem

$$\sup_{\sigma(B)} |p(z)| = \| P(B) \| .$$

It follows, since $\sigma(B)$ is real, that

$$\sup_{\sigma(B)} |u(z)| = \| u(B) \|$$

for all rational functions $u$ having no poles in $\sigma(B)$. Therefore $\sigma(B)$ is a spectral set of $B$, and hence so is the real axis. This implies that $B$ is Hermitian (see [2], Section 155), and thus that $A$ is normal.

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