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A Necessary and Sufficient Condition that an Operator be Normal

R. G. DOUGLAS* AND PETER ROSENTHAL

University of Michigan, Ann Arbor, Michigan 48104
and
University of Toronto, Toronto, Ontario, Canada

Submitted by Gian Carlo Rota

If A is a normal operator on a Hilbert space and if P is a polynomial in A and A^*, then P is normal. Thus, if r denotes spectral radius, $\| P \| = r(P)$ for every operator P that is a polynomial in A and A^*.

Our interest in the converse problem was stimulated by a conversation with C. R. MacCluer.

THEOREM. *If A is a bounded operator on a Hilbert space, and if $\| P \| = r(P)$ for every operator P that is a polynomial in A and A^*, then A is normal.*

PROOF. Translating A by an appropriate scalar we can assume that $\operatorname{Re} A = (A + A^*/2)$ and $\operatorname{Im} A = (A - A^*/2i)$ are strictly positive operators. It then follows by a well-known result (see [1]) that the spectrum of the product operator $B = (\operatorname{Re} A)(\operatorname{Im} A)$ is contained in the positive real axis. To show that A is normal it suffices to show that $\operatorname{Re} A$ and $\operatorname{Im} A$ commute, and this will follow immediately if we show that B is Hermitian.

If p is any polynomial in one variable then $\| p(B) \| = r(p(B))$ by hypothesis. Thus by the spectral mapping theorem

$$\sup_{z \in \sigma(B)} | p(z) | = \| p(B) \| .$$

It follows, since $\sigma(B)$ is real, that

$$\sup_{z \in \sigma(B)} | u(z) | = \| u(B) \|$$

for all rational functions u having no poles in $\sigma(B)$. Therefore $\sigma(B)$ is a spectral set of B, and hence so is the real axis. This implies that B is Hermitian (see [2], Section 155), and thus that A is normal.

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References

1. JAMES P. WILLIAMS. Spectra of products and numerical ranges. *J. Math. Anal. Appl.* **17** (1967), 214-220.
2. F. RIESZ AND B. SZ-NAGY. "Functional Analysis." Ungar, New York, 1955.