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INTERFERENCE BETWEEN RESONANCES IN A SLAB LATTICE*

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Abstract—In calculating resonance absorption for one resonance, the effect of neighbouring resonances is usually neglected. It is, therefore, customary to assume that the slowing-down flux takes its asymptotic value above the resonant energy and becomes independent of both lethargy and space. However, when the resonances are not widely separated, a resonance may lie in the region of energy oscillation due to other high-energy resonances. In this region the flux may depend on both lethargy and space. In the present paper a method has been developed for estimating this error. In this calculation the higher energy resonance has been replaced by a delta function sink, and the age–diffusion equation with modified source term has been solved to obtain the lethargy and space-dependent flux. The resonance integrals have then been calculated in the presence and in the absence of the higher energy resonance in an effort to estimate the importance of this effect. Various calculations of the effect of resonance interference have ignored spatial effects, but these may be important, particularly for tightly packed lattices.

1. INTRODUCTION

IN RESONANCE escape calculations, it is usually assumed that the flux feeding neutrons into the resonance is constant in space and lethargy. In the case of closely spaced resonances, the flux may not recover completely before reaching the next lower energy resonance and thus the flux may depend on both space and lethargy. When a resonance is in the transient region of the flux due to the presence of a higher energy resonance, the resonance integral of the lower energy resonance may differ from that calculated with the flat flux assumption.

A good deal of work has been carried out in the past for the case of overlapping resonances, both for the homogeneous and heterogeneous mixtures. For example, SCHERMER and CORNGOLD (1959) investigated the interference between resonances in an infinite homogeneous medium using a variational technique and concluded that the interference is negligibly small. The effect of overlapping resonances has been investigated by FOELL *et al.* (1963). Experimental work in this direction has been carried out by BROWN *et al.* (1962). An analytical expression has been derived by AMSTER (1965) to estimate the level separation corrections to resonance integrals. CODD and COLLINS (1963) calculated the influence of resonance overlapping on the Doppler effect in dilute fast reactors. The same effect has also been treated by HWANG (1963) and GREEBLER (1963). HWANG (1965) has considered the interference effect of overlapping resonances of a system having two or more resonance absorbers in evaluating the Doppler coefficient. STEVENS and SMITH (1965) have written a code—GAROL—for evaluating the resonance absorption taking into consideration the effect of resonance overlap. KIER (1966) calculated that resonance integrals of

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mixtures of ^{238}U and ^{232}Th for the energy range 1–1000 eV using the RIFF-RAFF code developed by KIER (1965). He found the resonance integrals of the mixtures were 2–4 per cent lower than the weighted resonances of individual absorbers. More recently, DE LOACH and SUICH (1966) and COHEN (1966) have studied the effect on resonance capture of non-uniform spatial distributions such as might be introduced by a nearby resonance at higher energy.

Our work is different from those referenced above as we neglect the overlapping effects. In our investigation we study the interference between non-overlapping close resonances in a heterogeneous lattice and try to present a unified analytical technique for studying all aspects of the problem. Our procedure has been first to find the neutron flux, as a function of energy and position, in the energy region near a given resonance due to the 'negative source' contribution of a higher energy resonance. This is done by utilizing the solutions of the age–diffusion equation.

Since we consider the interference only between two close resonances, it is assumed that the flux takes its asymptotic value before reaching the first (higher energy) resonance. Thus, the 'flat source' assumption is used in the calculation of the first resonance integral. The second resonance (lower energy) integral is evaluated in the absence and in the presence of the first resonance. The lethargy and spatially dependent flux due to the presence of the first resonance and the corresponding first-collision probabilities are used in the calculations of the second resonance integral.

The resonance integral, I, is defined as

$$I = \frac{1}{\Phi_{10}} \int \Phi_1(u) \sigma_a(u)\, du \tag{1}$$

where Φ_{10} is the flux in the fuel element which would exist at the resonance energy in the absence of the resonance. $\Phi_1(u)$ is the (spatially) averaged flux in the fuel element in the resonance region. The lethargy integral in equation (1) is over the resonance interval. In order to obtain an expression for $\Phi_1(u)$ in the resonance, we write the integral equations for $\Phi(u)$ in two media, following CHERNICK (1956), in the form given below (we assume a cell consisting of only two regions, fuel and moderator):

$$V_1 \sum_i \Sigma^i(u)\phi_1(u) = V_1 \sum_i P_1(u) \int_{u-\Delta_i}^{u} \frac{\Sigma_s^i \phi_1(u') \exp(u' - u)}{1 - \alpha^i}\, du' + V_2 \sum_j (1 - P_2(u)) \int_{u-\Delta_j}^{u} \frac{\Sigma_s^j \phi_2(u') \exp(u' - u)}{1 - \alpha^j}\, du' \tag{2a}$$

$$V_2 \sum_j \Sigma^j(u)\phi_2(u) = V_2 \sum_j P_2(u) \int_{u-\Delta j}^{u} \frac{\Sigma_s^j \phi_2(u') \exp(u' - u)}{1 - \alpha^j}\, du' + V_1 \sum_i (1 - P_1(u)) \int_{u-\Delta i}^{u} \frac{\Sigma_s^i \phi_1(u') \exp(u' - u)}{1 - \alpha^i}\, du'. \tag{2b}$$

The summations $\sum_i$ and $\sum_j$ are over the atoms in the fuel and moderator respectively. Σ^i and Σ_s^i are the total and scattering macroscopic cross sections respectively of the i-th atom, while $\phi_1(u)$ and $\phi_2(u)$ are the fluxes in the fuel and moderator respectively.

The maximum fractional energy loss of a neutron in an elastic collision with an atom of mass A_i is $(1 - \alpha^i)$ where

$$\alpha^i = \left(\frac{A_i - 1}{A_i + 1}\right)^2 \tag{3}$$

$$\Delta_i = \ln (1/\alpha^i). \tag{4}$$

The collision probabilities $P_1(u)$ and $P_2(u)$, which enable us to relate the collision densities $\Sigma_1(u)\phi_1(u)$ and $\Sigma_2(u)\phi_2(u)$ to the previous collisions at higher energy, are defined as the probability that neutrons of lethargy u originating in region $k(k = 1$ for fuel, $k = 2$ for moderator) will make their first collision in the same region k (probably after traversing other regions).

In order to obtain $\phi_1(u)$ from equations (2), so that the expression for I [equation (1)] may be evaluated, it is customary to use either the NR or IM approximations. In either case, the contribution to the integrals on the right side of equations (2) *from the resonance region itself* is ignored. Thus, the flux on the right side of these two equations is replaced with its value *above* the resonance, i.e. the unperturbed flux. If isolated resonances are considered, this unperturbed flux is a constant in both space and energy (the potential scattering cross section in both fuel and moderator is assumed constant). This leads to the 'normal' expressions for NR or IM resonance integrals as given, for example, by DRESNER (1960). In our case, the flux at the lower resonance is assumed not to have recovered from the perturbations introduced by the higher resonance. Thus, a spatial average must be used and, in addition, the spatially averaged flux is manifestly energy dependent. Expressions for this spatially averaged, energy-dependent flux are derived in Sections 2 and 5. The corresponding expressions for I, both in the NR and IM cases, are derived.

Our case differs from the 'normal' case in still another respect. Namely, the first-collision probabilities $P_K(u)$ which appear in equations (2) are, in the 'normal' case, those derived from a spatially flat source distribution and tabulated, for example, by CASE *et al.* (1953). In our case the collision densities are *not* flat, so that the $P_K(u)$ must be calculated for the actual source distributions. These quantities, using the fluxes calculated in Section 2, are presented in Section 3. In Section 4, the effective 'negative source' strength, i.e. the reduction in slowing down density due to absorption in the higher energy resonance, is obtained. In Section 6, the results of the collision probabilities, the source strength, and the resonance integral calculations are combined numerically to obtain interference corrections to the resonance integrals. We find, in certain cases, the interference effect can be quite large.

It must be stressed again that a major effect is the distortion in the spatial distribution of neutron flux feeding the resonance due to the high-energy resonance. Correct results cannot be obtained by treating the problem zero dimensionally.

Although our numerical results are restricted to one-dimensional lattices, similar results can be obtained for the two- and three-dimensional cases by utilizing the higher-dimensional solutions of the age equation given by MANNAN (1963) and ZWEIFEL and MANNAN (1967) plus the approximate expressions for the first-collision probabilities.

2. SOLUTION OF THE AGE-DIFFUSION EQUATION WITH DELTA FUNCTION SOURCE AND SINK OF NEUTRONS

Let us consider a slab cell of width $2b$ with a central fuel lump of width $2a$. A delta function negative source, S_0, at lethargy $u = u_0$ is added to the source term in the age–diffusion equation. We also assume that there is no absorption except the resonance. Thus, we write the age–diffusion equation as

$$-\frac{\partial}{\partial x}\left(D(x)\frac{\partial}{\partial x}\phi(x,u)\right) = -\xi(x)\Sigma_S(x)\frac{\partial\phi}{\partial u} + S\delta(u) - S_0\delta(u-u_0). \tag{5}$$

The one-dimensional case can be treated either by the method used in the two- and three-dimensional cases (MANNAN, 1963; and ZWEIFEL and MANNAN, 1967) i.e. by integrating over the singularity in $\partial D/\partial x$, or alternatively by measuring the distances in units of diffusion coefficients, $D(x)$. We adopt the latter approach. Let us introduce a new variable

$$y = \int_0^x \frac{\mathrm{d}x'}{D(x')}. \tag{6}$$

The boundaries $x = a$ and $x = b$ correspond to

$$y = \alpha = \frac{a}{D_1}$$

and

$$y = \beta = \frac{a}{D_1} + \frac{b-a}{D_2} \tag{7}$$

respectively. The flux is expanded in the following way:

$$\phi(y,u) = \frac{a_0(u)}{2} + \sum_{n=1}^{\infty} a_n(u)\cos\frac{n\pi y}{\beta}. \tag{8}$$

We put the above value of $\phi(y,u)$ in equation (5) and multiply by $\cos m\pi y/\beta$ and integrate over the cell. Taking the Laplace transform with respect to u (λ = transform variable), we obtain the matrix equation for the expansion coefficients as

$$\bar{a}_0(\lambda)\lambda\delta_{0m} + \sum_{n=1}^{\infty}\bar{a}_n(\lambda)[A_{mn} + \lambda\delta_{mn}] = S(m) - S_0(m)\exp(-\lambda u_0) \tag{9}$$

where

$$A_{mn} = \pi\left(\frac{n}{\beta}\right)^2\left[(\omega_1-\omega_2)\left\{\frac{\sin\{(m+n)\pi\alpha/\beta\}}{m+n} + \frac{\sin\{(m-n)\pi\alpha/\beta\}}{m-n}\right\} + \omega_2\pi\delta_{mn}\right] \tag{10}$$

$$S(m) = \frac{2}{\pi}\left(\frac{S_1}{\xi_1\Sigma_{S1}} - \frac{S_2}{\xi_2\Sigma_{S2}}\right)\frac{\sin(m\pi\alpha/\beta)}{m} + \frac{2S_2}{\xi_2\Sigma_{S2}}\delta_{0m}. \tag{11}$$

$m = 0, 1, 2, \ldots,$ etc.

$$\omega_i = \frac{3(1-\bar{u}_i)}{\xi_i}. \tag{12}$$

$S_0(m)$ has exactly the same form as $S(m)$ except that S_{01} is the sink strength in the fuel and $S_{02} = 0$, where the subscripts 1 and 2 stand for the fuel and the moderator regions respectively.

Solving equation (9) for $\bar{a}_n(\lambda)$ and taking the inverse Laplace transform, we obtain the expansion coefficients for the case in which there is no absorption during slowing down. Since at high energies resonance absorption is the most dominant absorption term during slowing down, the neglect of non-resonance absorption is a good approximation.

We note that the roots of the following determinant, equation (13), multiplied by u or $(u - u_0)$ will appear in the expansion coefficients $a_n(n)$ as negative exponents:

$$\begin{vmatrix} \lambda & A_{01} & A_{02} & \cdots & A_{0n} \\ 0 & A_{11}+\lambda & A_{12} & \cdots & A_{1n} \\ 0 & A_{21} & A_{22}+\lambda & \cdots & A_{2n} \\ & \cdots & \cdots & \cdots & \cdots \\ 0 & A_{m1} & A_{m2} & A_{mm}+\lambda & A_{mn} \\ & \cdots & \cdots & \cdots & \cdots \\ 0 & A_{n1} & A_{n2} & \cdots & A_{nn}+\lambda \end{vmatrix} = 0 \qquad (13)$$

Equation (13) is an infinity by infinity determinant. Taking different values of n, we have solved equation (13) for λ. Table 1 shows that the roots converge rapidly

TABLE 1.—ROOTS OF DETERMINANTS OF SEVERAL NATURAL URANIUM–GRAPHITE SLAB LATTICES*

Half-width of fuel = a cm

Half-width of moderator = $(b - a)$ cm

$\sigma_p^u = 10{\cdot}75$ barns $\quad \sigma_p^c = 4{\cdot}5$ barns

a	b	λ_1	λ_2	λ_3	λ_4
		1126·789			
0·5	1·0	1114·54	5905·7		
		1093·19	5905·1	10999·9	
		1092·75	5865·9	10954·9	21876·0
		281·7			
1·0	2	278·6	1476·4		
		273·3	1476·2	2749·9	
		273·2	1466·4	2738·7	5469·0
		125·2			
1·5	3	123·8	656·2		
		121·5	656·1	1222·2	
		121·4	651·7	1217·2	2430·7
		70·4			
2·0	4	69·6	369·1		
		68·3	369·0	687·5	
		68·29	366·6	684·7	1367·2
		33·1			
2·5	6	31·7	143·5		
		31·6	140·7	301·5	
		31·2	140·6	298·1	571·8

* The first row for each value of (a, b) corresponds to using a 2×2 approximation for equation (13). The second corresponds to a 3×3 approximation, and so on.

and the smallest root may be approximated by A_{11}, i.e. by the solution of a 2×2 determinant. The actual value of the smallest root, λ_1, is about three per cent less than A_{11}. Let us also note that the next higher root, λ_2, is about five times as large as λ_1. Since the λ_i appear in the expansion coefficients as negative exponents, we can neglect all the terms involving λ_i except for $i = 1$. In addition, we have found by considering the one-speed case (MANNAN, 1963) that the first two terms in the trigonometric series give an excellent representation of the flux. Thus, only $a_0(u)$ and $a_1(u)$ are needed:

$$\phi(y, u) \approx \frac{a_0(u)}{2} + a_1(u) \cos \frac{\pi y}{\beta} \tag{14}$$

where

$$y = \begin{cases} \dfrac{x}{D_1}, & \text{for } -a < x < a \\ \dfrac{a}{D_1} + \dfrac{x - a}{D_2}, & \text{for } \quad a < x < b. \end{cases} \tag{15}$$

The expansion coefficients are*

$$a_0(u) = S(0) - \frac{A_{01}S(1)}{A_{11}}(1 - \exp -(\lambda_1 u) - H(u - u_0)\left\{S_0(0) - \frac{A_{01}S_0(1)}{A_{11}}(1 - \exp\left[-\lambda_1(u - u_0)\right]\right\} \tag{16}$$

and

$$a_1(u) = S(1) \exp(-\lambda_1 u) - H(u - u_0)S_0(1) \exp\left[-\lambda_1(u - u_0)\right] \tag{17}$$

where

$$H(u - u_0) = \begin{cases} 0, & \text{for } u < u_0 \\ 1, & \text{for } u \geq u_0 \end{cases} \tag{18}$$

is a unit step function.

The terms containing $\exp(-\lambda_1 u)$ can be neglected. The term containing $\exp\left[-\lambda_1(u - u_0)\right]$ will be important only when $\lambda_1(u - u_0)$ is small. For widely separated resonances, $\lambda_1(u - u_0)$ at the second resonance would be large, and consequently $\exp\left[-\lambda_1(u - u_0)\right] \to 0$, i.e. $a_1 \to 0$ and as expected the flux would be constant in lethargy and in space. The flux above the second resonance becomes space and lethargy dependent due to the presence of a close first (higher energy) resonance only.

*As $u \to \infty$, $a_0(u)$ should approach twice the result of equation (35), as can easily be seen from neutron conservation agreement. The difference between that equation and our solution

$$S(0) - \frac{A_{01}S(1)}{A_{11}}$$

is due to that approximate (2×2 matrix) solution of equation (9).

3. CALCULATION OF THE FIRST-COLLISION PROBABILITIES

The first-collision probabilities are usually calculated assuming a flat source distribution. We derive expressions for the first-collision probabilities for source distributions represented by the spatially dependent flux, $\phi(y, u)$, shown in equation (14). The probability, $P_1(u)$, that a neutron of lethargy u originating in the lump $(-a < x' < a)$ of region 1 will make its first collision anywhere in the same region 1 of the slab lattice has been calculated by MANNAN (1963), who obtains (neglecting flux anisotropy)

$$P_1(u) = 1 - \frac{a_0(u)}{2d_1\Sigma_1\bar{\phi}_1}\left[0{\cdot}5 - \sum_{n=0}^{\infty}\{E_3(l_{1n}) - 2E_3(l_{2n}) + E_3(l_{3n})\}\right]$$

$$+ \sum_{n=0}^{\infty}\frac{a_1(u)}{d_1\bar{\phi}_1}[\Sigma_1 \cos(\alpha' a)\{\psi_1(n) - 2\psi_2(n) + \psi_3(n)\}$$

$$+ \alpha' \sin(\alpha' a)\{\psi_1'(n) - \psi_3'(n)\}] \quad (19)$$

where $E_n(x)$ is defined as (CASE *et al.*, 1953)

$$E_n(x) = \int_0^1 u^{n-2} \exp -(x/u)\, du \quad (20)$$

and

$$\left.\begin{aligned} d_1 &= 2a \\ d_2 &= 2(b-a) \end{aligned}\right\} \quad (21)$$

$$\psi_i(n) = \int_0^1 \frac{t \exp -(l_{in}/t)\, dt}{\Sigma_2^2 + \alpha'^2 t^2} \quad (22)$$

$$\psi_i'(n) = \int_0^1 \frac{t^2 \exp -(l_{in}/t)\, dt}{\Sigma_2^2 + \alpha'^2 t^2} \quad (23)$$

$$\alpha' = \frac{\pi D_2}{D_2 a + D_1(b-a)} \quad (24)$$

$$\bar{\phi}_1 = \frac{1}{2a}\int_{-a}^{a} \phi(x')\, dx' \quad (25)$$

$$\left.\begin{aligned} l_{1n} &= (n+1)d_2\Sigma_2 + nd_1\Sigma_1 \\ l_{2n} &= (n+1)(d_2\Sigma_2 + d_1\Sigma_1) \\ l_{3n} &= (nd_2\Sigma_2 + (n+1)d_1\Sigma_1 \end{aligned}\right\}. \quad (26)$$

Similarly, we find the probability, $P_2(u)$, that the neutrons of lethargy u originating in the moderator lump $(a < x' < 2b - a)$ of region 2 will make their first collision in the same region 2 anywhere in the lattice.

$$P_2(u) = 1 - \frac{a_0(u)}{2d_2\Sigma_2\bar{\phi}_2}\left[0{\cdot}5 - \sum_{n=0}^{\infty}\{E_3(l_{1n}') - 2E_3(l_{2n}') + E_3(l_{3n}')\}\right]$$

$$- \frac{a_1(u)\cos\theta}{2d_2\bar{\phi}_2}\left[\frac{\Sigma_2}{2\beta'^2}\ln\left(1 + \frac{\beta'^2}{\Sigma_2{}^2}\right)\{\cos\beta'(2b-a) + \cos\beta' a\}\right.$$

$$\left. + \frac{1}{\beta'}\left(1 - \frac{\Sigma_2}{\beta'}\tan^{-1}\frac{\beta'}{\Sigma_2}\right)\{\sin\beta'(2b-a) - \sin\beta' a\}\right]$$

$$+ \frac{a_1(u)\sin\theta}{2d_2\bar{\phi}_2}\left[\frac{\Sigma}{2\beta'^2}\ln\left(1 + \frac{\beta'^2}{\Sigma_2{}^2}\right)\{\sin\beta'(2b-a) + \sin\beta' a\}\right.$$

$$\left. + \frac{1}{\beta'}\left(1 - \frac{\Sigma_2}{\beta'}\tan^{-1}\frac{\beta'}{\Sigma_2}\right)\{\cos\beta'(2b-a) + \cos\beta' a\}\right]$$

$$+ \sum_{n=0}^{\infty}\frac{a_1(u)\cos\theta}{2d_2\bar{\phi}_2}[\Sigma_2\{\cos\beta'(2b-a) + \cos\beta' a\}\{\Pi_1(n) - 2\Pi_2(n)$$

$$+ \Pi_3(n)\} + \beta'\{\sin\beta'(2b-a) - \sin\beta' a\}\{\Pi_1'(n) - \Pi_3'(n)\}]$$

$$+ \sum_{n=0}^{\infty}\frac{a_1(u)\sin\theta}{2d_2\bar{\phi}_2}[\Sigma_2\{\sin\beta'(2b-a) + \sin\beta' a\}\{\Pi_1(n) - 2\Pi_2(n) + \Pi_3(n)\}$$

$$- \beta'\{\cos\beta'(2b-a) - \cos\beta' a\}\{\Pi_1'(n) - \Pi_3'(n)\}] \tag{27}$$

where

$$\theta = \frac{\pi a}{\beta}\left(\frac{1}{D_1} - \frac{1}{D_2}\right) \tag{28}$$

$$\beta' = \frac{\pi}{\beta D_2} \tag{29}$$

$$\bar{\phi}_2 = \frac{1}{2(b-a)}\int_a^{2b-a}\phi(x')\,dx', \tag{30}$$

$$\Pi_i(n) = \int_0^1 \frac{t\exp -(l_{in}'/t)\,dt}{\Sigma_2{}^2 + \beta'^2t^2}, \tag{31}$$

$$\Pi_i'(n) = \int_0^1 \frac{t^2\exp -(l_{in}'/t)\,dt}{\Sigma_2{}^2 + \beta'^2t^2}, \tag{32}$$

and

$$\begin{aligned} l_{1n}' &= (n+1)d_1\Sigma_1 + nd_2\Sigma_2 \\ l_{2n}' &= (n+1)(d_1\Sigma_1 + d_2\Sigma_2) \\ l_{3n}' &= nd_1\Sigma_1 + (n+1)d_2\Sigma_2 \end{aligned} \tag{33}$$

It may be noted that the first two terms of $P_1(u)$ and $P_2(u)$ are exactly the same as derived by ROTHENSTEIN (1959, 1960) with a flat source assumption. The remaining terms are due to the presence of the cosine term in the flux.

4. CALCULATION OF THE STRENGTH OF THE SINK

The strength of the sink, S_0, is given by the rate of adsorption of neutrons per unit volume in the fuel element due to the first (higher energy) resonance.

$$S_0 = NI\phi_{\text{asy}} \tag{34}$$

where N is the number of absorber nuclei per unit volume of the fuel; I, the resonance integral defined in equation (1); and ϕ_{asy} is the asymptotic flux.

$$\phi_{\text{asy}} = \frac{q(V_1 + V_2)}{V_1\xi_1\Sigma_{P1} + V_2\xi_2\Sigma_{P2}} \tag{35}$$

where q is the slowing down density, V_1, Σ_{P1} and V_2, Σ_{P2} are the volumes and macroscopic potential scattering cross sections of the fuel and the moderator respectively.

The resonance integral I, for the two extreme cases, can be found easily. When the maximum energy loss in a collision with an absorber atom is greater than the practical width, the interval over which the resonance cross section is larger than the potential cross section of the resonance, the narrow resonance approximation is used. In this case the energy-dependent flux inside the integral is taken outside the integral sign and the asymptotic value is used. Let us assume that the fuel elements may contain U, Th, and O atoms only. The resonance integral for the i-th absorber, $I_{\text{NR}}{}^i$, is found as (MANNAN, 1963)

$$I_{\text{NR}}{}^i = (\Sigma_S{}^{\text{U}} + \Sigma_S{}^{\text{Th}} + \Sigma_S{}^0)\int_{\text{RES}} \frac{\sigma_a{}^i(u)P_i(u)\,du}{(\Sigma^{\text{U}} + \Sigma^{\text{Th}} + \Sigma^0)} - \int_{\text{RES}} \sigma_a{}^i(u)P_1{}^0(u)\,du + \int_{\text{RES}} \sigma_a{}^i(u)\,du. \tag{36}$$

When the practical width is larger than the maximum energy in a collision with the absorber atom, the infinite mass (IM) approximation is used. But the NR approximation is used for oxygen and the moderator atoms. For this particular case, it is assumed α^{U}, $\alpha^{\text{Th}} \to 1$ and we find

$$I_{\text{IM}}{}^i = \int \frac{(\Sigma^{\text{U}} + \Sigma^{\text{Th}} + \Sigma^0)\sigma_a{}^i(u)\,du}{(\Sigma^{\text{U}} + \Sigma^{\text{Th}} + \Sigma^0) - P_1{}^0(u)(\Sigma_S{}^{\text{Th}} + \Sigma_S{}^{\text{U}})} - \int \frac{(\Sigma^{\text{U}} + \Sigma^{\text{Th}})\sigma_a{}^i(u)P_1{}^0(u)\,du}{(\Sigma^{\text{U}} + \Sigma^{\text{Th}} + \Sigma^0) - P_1{}^0(u)(\Sigma_S{}^{\text{Th}} + \Sigma_S{}^{\text{U}})}. \tag{37}$$

In equations (36) and (37) the reciprocity relation (CASE *et al.*, 1953)

$$V_1\Sigma_1(1 - P_1{}^0) = V_2\Sigma_2(1 - P_2{}^0) \tag{38}$$

has been used to eliminate $P_2{}^0(u)$. Here $P_1{}^0$ and $P_2{}^0$ are the collision probabilities with a flat source assumption. Let us note that when $i = \text{Th}$, $\Sigma^{\text{U}} = \Sigma_P{}^{\text{U}}$ and when $i = \text{U}$, $\Sigma^{\text{TH}} = \Sigma_P{}^{\text{TH}}$ unless there is an overlapping of resonances.

5. CALCULATION OF THE RESONANCE INTEGRAL OF THE SECOND (LOWER ENERGY) RESONANCE

In order to calculate the second (lower energy) resonance integral in the presence of the first resonance, we consider the first resonance as a delta function sink. The solution of the age–diffusion equation with this approximation has been obtained

in Section 2. The average fluxes $\bar{\phi}_1(u)$ and $\bar{\phi}_2(u)$ in the fuel, moderator, respectively, are used in the calculation. The average first-collision probabilities $P_1(u)$ and $P_2(u)$, derived in Section 3, due to the spatially dependent source distribution are used in this case. In the absence of the first resonance, the uninterfered second resonance integral will have the same form as shown in Section 4.

If the first resonance is at lethargy u_0, the average fluxes $\bar{\phi}_1(u)$ and $\bar{\phi}_2(u)$ above the second resonance are

$$\begin{aligned} \bar{\phi}_1(u) &= \bar{\phi}_0 + \phi_1' \exp[-A_{11}(u-u_0)], && \text{for } u \geq u_0 \\ &= \phi_{\text{asy}}, && \text{for } u < u_0 \end{aligned} \tag{39}$$

$$\begin{aligned} \bar{\phi}_2(u) &= \phi_0 + \phi_2' \exp[-A_{11}(u-u_0)], && \text{for } u \geq u_0 \\ &= \phi_{\text{asy}}, && \text{for } u < u_0 \end{aligned} \tag{40}$$

where

$$\phi_0 = \frac{1}{2}\left[S(0) - \frac{A_{02}S(1)}{A_{11}} - H(u-u_0)\left\{S_0(0) - \frac{A_{01}S_0(1)}{A_{11}}\right\}\right] \tag{41}$$

$$\phi_1' = -H(u-u_0)\left[\frac{A_{01}S_0(1)}{2A_{11}} + \frac{S_0(1)}{\alpha'\alpha}\sin(\alpha'\alpha)\right] \tag{42}$$

$$\phi_2' = -H(u-u_0)\left[\frac{A_{01}S_0(1)}{2A_{11}} - \frac{S_0(1)\beta}{\pi(\beta-\alpha)}\sin\left(\frac{\pi\alpha}{\beta}\right)\right]. \tag{43}$$

[The average fluxes ϕ_1' and ϕ_2' have been obtained by averaging the expression for the flux given in equation (14).]

Let us assume that the second resonance is at lethargy u_r. In the NR approximation we find the interfered resonance integral for the i-th absorber as

$$\begin{aligned} (I_{\text{NR}}^i)_{\text{int}} = {} & \frac{1}{\bar{\phi}_1(u_r)}[\Sigma_S{}^{\text{U}}R_{\text{U}}(u_r) + \Sigma_S{}^{\text{Th}}R_{\text{Th}}(u_r) + \Sigma_S{}^{0}R_0(u_r)] \\ & \times \int_{\text{RES}} \frac{P_1(u)\sigma_a^i(u)\,du}{(\Sigma^{\text{U}} + \Sigma^{\text{Th}} + \Sigma^0)} + \frac{V_2}{V_1\bar{\phi}_1}\sum_i [\Sigma_S{}^i R_i(u_r)] \int_{\text{RES}} \frac{(1-P_2(u))\sigma_a^i(u)\,du}{(\Sigma^{\text{U}} + \Sigma^{\text{Th}} + \Sigma^0)}. \end{aligned} \tag{44}$$

where

$$R_i(u) = \phi_0 + \phi_0'\left\{\left(\frac{1}{\alpha^i}\right)^{A_{11}-1} - 1\right\}\frac{\exp[-A_{11}(u-u_0)]}{(1-\alpha^i)(A_{11}-1)} \tag{45}$$

when $\ln(1/\alpha^i) > u_r - u_0$.

$$\begin{aligned} R_i(u) = {} & \phi_{\text{asy}}\{\exp[-(u-u_0)] - \alpha^i\} + \frac{\phi_0}{1-\alpha^i}\{1 - \exp[-(u-u_0)]\} \\ & + \phi_1'\frac{\exp[-(u-u_0)] - \exp[-A_{11}(u-u_0)]}{(1-\alpha^i)(A_{11}-1)} \end{aligned} \tag{46}$$

when $\ln(1/\alpha^i) \leq u_r - u_0$.

$R_j(u)$ has the same form as $R_i(u)$ except that ϕ_2' should be written in place of ϕ_1'.

In the IM approximation, we obtain the interfered resonance integral for the i-th absorber as

$$(I_{\mathrm{IM}}{}^{i})_{\mathrm{int}} = \frac{\Sigma_S{}^{0}R_0(u_r)}{\bar{\phi}_1}\int_{\mathrm{RES}} \frac{P_1(u)\sigma_a{}^{i}(u)\,\mathrm{d}u}{(\Sigma^{\mathrm{U}}+\Sigma^{\mathrm{Th}}+\Sigma^{0}) - P_1(u)(\Sigma_S{}^{\mathrm{Th}}+\Sigma_S{}^{\mathrm{U}})} + \frac{V_2}{V_1\bar{\phi}_1}\sum_j[\Sigma_S{}^{j}R_j(u_r)]\int \frac{(1-P_2(u))\sigma_a{}^{i}(u)\,\mathrm{d}u}{(\Sigma^{\mathrm{U}}+\Sigma^{\mathrm{Th}}+\Sigma^{0}) - P_1(u)(\Sigma_S{}^{\mathrm{Th}}+\Sigma_S{}^{\mathrm{U}})}. \quad (47)$$

6. CALCULATION AND DISCUSSIONS

The cross sections at the resonances have been calculated using the Breit–Wigner one-level formula. The integration over the resonances have been performed numerically using Simpson's rule. The range of integration has been taken six times in practical width of the resonance. No correction has been made for the adsorption in the wings of the resonance. It has been shown (NORDHEIM, 1961) that even when the range of integration is five times the practical width the wing correction is less than a few per cent. The potential scattering cross sections and resonance parameters used in the calculations have been taken from Nordheim. All the numerical results were obtained on The University of Michigan IBM 7090 computer.

In Tables 2–6 we have shown the resonance integrals of several pairs of resonances for different lattices. The second (lower energy) resonances have two values for the resonance integral, one assuming no absorption of neutrons during slowing down before reaching the resonance and the other assuming a delta function sink above it. Therefore, the absorption rate by the second resonance in the absence of the first resonance is

$$\rho_1{}' = \phi_{10}N_i(I)_{\mathrm{unint}} \quad (48)$$

where ϕ_{10} is the asymptotic flux above the second resonance, which has been reduced

TABLE 2.—RESONANCE INTERFERENCE BETWEEN THE 116·5 eV AND 102·5 eV RESONANCES FOR ^{238}U–GRAPHITE CELLS

Thickness of the absorber lump = $2a$. Thickness of the moderator lump = $2(b-a)$

$2a$ (cm)	$2(b-a)$ (cm)	E_0 (eV)	$(I)_{\mathrm{unint}}$ (barns)	$(I)_{\mathrm{int}}$ (barns)	$\frac{(I)_{\mathrm{int}}-(I)_{\mathrm{unint}}}{(I)_{\mathrm{int}}}\times 100$	ρ
		116·5	0·136624			
1·0	9·0				2·900018	−0·032023
		102·5	0·175556	0·180799		
		116·5	0·117069			
2·0	18·0				6·197581	−0·062636
		102·5	0·134082	0·142941		
		116·5	0·117076			
2·0	28·0				6·592241	−0·103103
		102·5	0·134096	0·143560		
		116·5	0·106046			
4·0	26·0				0·220431	−0·098395
		102·5	0·126165	0·138980		
		116·5	0·106046			
4·0	36·0				11·637581	−0·096810
		102·5	0·126164	0·142780		
		116·5	0·103780			
5·0	45·0				18·543231	−0·036179
		102·5	0·150339	0·184563		

TABLE 3.—RESONANCE INTERFERENCE BETWEEN SEVERAL PAIRS OF RESONANCES IN A ^{238}U–GRAPHITE CELL

Thickness of absorber = 2 cm. Thickness of moderator = 18 cm

E_0 (eV)	$(I)_{\text{unint}}$ (barns)	$(I)_{\text{int}}$ (barns)	$\frac{(I)_{\text{int}} - (I)_{\text{unint}}}{(I)_{\text{int}}} \times 100$	ρ
90·0	0·015188			
			0·493352	−0·013194
81·3	0·088780	0·089220		
208·5	0·068054			
			4·862141	−0·037883
189·6	0·059805	0·062861		
274·0	0·038607			
			1·393043	−0·056080
264·5	$4{\cdot}757314 \times 10^{-3}$	$4{\cdot}835669 \times 10^{-3}$		
411·0	0·01833			
			0·801507	−0·027134
398·5	0·015957	0·016086		

TABLE 4.—RESONANCE INTERFERENCE BETWEEN SEVERAL PAIRS OF RESONANCES IN A UO_2–HEAVY WATER CELL

Thickness of absorber = 1·0 cm

Thickness of moderator = 3·0 cm

E_0 (eV)	$(I)_{\text{unint}}$ (barns)	$(I)_{\text{int}}$ (barns)	$\frac{(I)_{\text{int}} - (I)_{\text{unint}}}{(I)_{\text{int}}} \times 100$	ρ
90·0	0·022161			
			−3·796641	−0·036736
81·3	0·138441	0·133377		
208·5	0·102702			
			−4·759987	−0·046174
189·6	0·112919	0·107788		
291·6	0·048700			
			−2·251708	−0·022454
274·0	0·058837	0·057541		
411·0	0·028091			
			−1·085112	−0·011338
398·5	0·024579	0·024315		

by an amount equal to absorption by the first resonance, and N_i is the density of the absorber atoms in the fuel. The absorption by the second resonance in the presence of the first resonance is

$$\rho_2' = \bar{\phi}_2 N_i (I)_{\text{int}} \tag{49}$$

where $\bar{\phi}_2$ is the average flux which would be present at the second resonance if the resonance were not there. $(I)_{\text{int}}$ is the interfered resonance integral given by equations (44) or equation (47). Let us define

$$\rho = \frac{\rho_2' - \rho_1'}{\rho_1'} \tag{50}$$

as the ratio of excess capture due to the interference to non-interference capture.

TABLE 5.—RESONANCE INTERFERENCE BETWEEN SEVERAL PAIRS OF RESONANCES IN A ^{232}Th–GRAPHITE CELL

Thickness of absorber = 2 cm
Thickness of moderator = 18·0 cm

E_0 (eV)	$(I)_{unint}$ (barns)	$(I)_{int}$ (barns)	$\frac{(I)_{int} - (I)_{unint}}{(I)_{int}} \times 100$	ρ
23·48	0·650028			
			43·312819	−0·081550
21·84	0·568972	1·003704		
121·0	0·189870			
			4·763815	−0·065327
113·15	0·174218	0·182933		
129·4	0·079632			
			7·213855	−0·007347
128·5	0·012501	0·013473		
203·0	0·087732			
			3·601960	−0·045953
199·8	0·067155	0·069664		
413·4	0·033996			
			1·23086	−0·015349
402·8	0·023351	0·023642		
465·0	0·033140			
			1·277404	−0·016149
456·4	0·029775	0·030160		

TABLE 6.—INTERFERENCE BETWEEN RESONANCES IN A ^{232}Th–HEAVY WATER CELL

Thickness of absorber = 2·0 cm
Thickness of moderator = 6·0 cm

E_0 (eV)	$(I)_{unint}$ (barns)	$(I)_{int}$ (barns)	$\frac{(I)_{int} - (I)_{unint}}{(I)_{int}} \times 100$	ρ
23·48	0·630786			
			−2·837988	−0·03323
21·84	0·551959	0·536727		
121·0	0·187481			
			−1·723837	−0·018950
113·15	0·171990	0·169076		
129·4	0·078597			
			0·483584	−0·003332
128·5	0·012371	0·012432		
203·0	0·086629			
			−0·410150	−0·008242
199·8	0·066294	0·066023		
371·0	0·038097			
			−0·270201	−0·004902
366·0	0·043781	0·043663		
413·4	0·033575			
			−0·728074	−0·008029
402·8	0·023053	0·022887		
465·0	0·032739			
			−0·522162	−0·006460
456·4	0·029407	0·029254		

From the tables we may observe that the interference is larger in thicker lumps. It is also found that the interference depends upon the resonance integral of the higher resonance, which is proportional to the strength of the sink, and also upon the lethargy separation between the first and second resonance. It may be observed that the interfered resonance integral is larger than the resonance integral with a flat flux approximation in the graphite moderated cells, i.e. defining

$$\Delta I = (I)_{\text{int}} - (I)_{\text{unint}}.$$

ΔI is positive for the graphite-moderated lattices, while for the heavy-water lattices ΔI is generally a negative quantity. But this should not give one a wrong impression. The presence of a higher energy resonance decreases as expected the absorption by the second resonance in both the graphite and heavy-water-moderated lattices as the negative value of ρ shows clearly.

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