Characteristic Manifolds in Nonequilibrium Hydrodynamics*

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1. INTRODUCTION

We shall discuss two points: (1) a mathematical theory for determining the characteristic manifolds in nonequilibrium (relaxation) hydrodynamics, which is based on the Lichnerowicz concept of \((C^n, C^m)\) functions (cf. p. 5[1]), rather than the \textit{ad hoc} conditions A and B of our previous paper (cf. p. 113, [2]); (2) the correction of an erroneous statement in our previous work (cf. p. 117 [2]).

2. THE THEORY OF CHARACTERISTICS

We define the \((C^n, C^m)\) Lichnerowicz class of functions by

\textit{Definition 1a.} A function \(F(x_j, t), j = 1, 2, 3,\) will be called a \((C^n, C^m)\) function (with \(m > n\)) in the neighborhood of a three-dimensional differentiable manifold, \(S,\) of a Euclidean four-dimensional space, \(E,\) if:

\((1)\) the \(k\)th derivatives \(F^{(k)}, k = n + 1, \ldots, m\) are continuous at all points of two open sets \(E - S^{(n+1)}, E - S^{(n+1)}\), and are uniformly continuous in the corresponding closed sets with finite jumps along the common boundary \(S_n\) of these two sets (i.e. \(F^{(k)}, k = n + 1, \ldots, m\) are piecewise continuous or \(F\) is piecewise \(C^m\) in \(E\));

\((2)\) the \(k\)th derivatives \(F^{(k)}, k = 0, 1, 2, \ldots, n\) are continuous in \(E,\) or \(F\) is \(C^n\) in \(E\) (note, \(F^{(0)} = F\)).

\textit{Definition 1b.} When conditions (1) and (2) are satisfied, we write

\(F \in (C^n, C^m).\)

Now, we shall express the basic partial differential equations of relaxation

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hydrodynamics. Let \( \rho, q, s, v_j, T \) denote the density, relaxation variable, entropy, velocity vector, and temperature, respectively. Also, we introduce \( A, B, C, F, G \) which are known \( C^1 \) functions of \( \rho, q, s \) (cf. p. 271[3]). Note \( T \) is a known \( C^2 \) function of \( \rho, q, s \). Again, let \( x^k(f = 1, 2, 3) \) denote a system of Cartesian orthogonal coordinates in Euclidean three-space, \( E_3 \); let \( t = x^0 \) be the time variable, \( x^\alpha(\alpha = 0, 1, 2, 3) \) denote all these variables and

\[
\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} \quad \partial_j \equiv \frac{\partial}{\partial x^j}.
\]

We introduce the variables

**DEFINITION 2.**

\[
\begin{align*}
q^* & \equiv \partial q + v^j \partial_j q, \quad \rho^* \equiv \partial \rho + v^j \partial_j \rho, \\
\phi^* & \equiv \partial \phi + v^j \partial_j \phi, \\
v_j^* & \equiv \partial v_j + v^k \partial_k v_j,
\end{align*}
\]

(2.1)

\[
\begin{align*}
\bar{\phi} & \equiv \frac{q^*}{K}, \\
\bar{\phi} & \equiv \partial \phi^j,
\end{align*}
\]

(2.2)

(2.3)

where \( K \) is the relaxation scalar. These variables will belong to the following classes.

**DEFINITION 3.**

\[
\begin{align*}
v_j, \rho, q, s & \in (C^1, C^2), \\
v_j^*, \rho^*, q^*, s^*, \bar{\phi} & \in (C^0, C^2), q^* & \in (C^1, C^2).
\end{align*}
\]

(2.4)

(2.5)

**REMARK.** The class of \( q^*, s^*, \bar{\phi}, \rho^*, v_j^* \) is consistent with that of \( q, s, v_j, \rho \) (cf. Definition 2).

By differentiating the basic equations of continuity, motion, and energy of a nonheat conducting, perfect fluid of relaxation hydrodynamics (cf. p. 106[2]) and using Definition 2, we obtain the quasilinear second-order system. in \( \rho, q, s, v_j, \rho^*, q^*, s^*, \bar{\phi}, \bar{\phi} \)

\[
\begin{align*}
\partial_\alpha \rho^* + \partial_\alpha (\rho \bar{\phi}) & = 0 \\
\partial_\alpha (\rho v_j^*) + \partial_\alpha (A \partial_\alpha \rho + B \partial_\alpha s + C \partial_\alpha q) & = 0 \\
\partial_\alpha (T s^*) - q^* \partial_\alpha \bar{q} - \bar{q} \partial_\alpha q^* & = 0 \\
\partial_\alpha \left\{ (\bar{q})^* + \frac{C}{\rho^*} \phi^* + F s^* + G q^* \right\} & = 0.
\end{align*}
\]

(2.6)

(2.7)

(2.8)

(2.9)
In order to determine the characteristic three-dimensional manifolds $S_3$ of (2.6)-(2.9), we shall briefly study the Cauchy problem for this system along some given $S_3$. By such a study, we will determine the relation between the various unknown normal derivatives of 

$$\rho, q, s, v^*, \rho^*, q^*, s^*, \tilde{q}, \tilde{e}.$$

First, we introduce

**Definition 4.** If $S_3$ is given by the $C^2$ function of $x^\alpha$

$$\phi(x^\alpha) = c,$$

where $c$ is a constant then the space-time and space unitized normals, $n_\alpha$ and $n_j$ of $S_3$, respectively, are

$$n_\alpha = \frac{\phi_\alpha}{\Phi}, \quad n_j = \frac{\phi_j}{\Phi}.$$  \hspace{1cm} (2.10)

**Remark.** Note, by definition, $\phi_j = \partial \phi / \partial x^j$, $x^0$ is $t$ and, $\phi_\alpha$ is $\partial \phi / \partial x^\alpha$ (see the notation introduced before Definition 1). Further, if $g_{\alpha \beta}$ is the metric tensor of $E_3$, then

$$\Phi = (\phi_t^2 + g^{jk}\phi_j\phi_k)^{1/2}$$  \hspace{1cm} (2.11)

$$\tilde{\Phi} \equiv (g^{jk}\phi_j\phi_k)^{1/2}.$$  \hspace{1cm} (2.12)

Again, if $t_a(a = 1, 2, 3)$ denote any three mutually orthogonal unit vectors which span $S_3$ at any point and if $g_{\alpha \rho}$ is the metric tensor of $E_3$, then

$$g^{\alpha \beta}t_a n_\beta = t_0 n_0 + g^{\alpha \beta} t_a n_\beta = 0.$$  \hspace{1cm} (2.13)

We decompose the partial derivatives of any of the quantities of Definition 3 (represented by $u$) into their normal and tangential components by means of the following relations

**Definition 5.**

$$\left(U^a, U^a, U^a, U^a\right)$$

$$\partial_\beta U_a = n_\beta U_a + \sum_a t_\beta U_a$$  \hspace{1cm} (2.14)

$$\partial_\alpha \partial_\beta U_a - n_\alpha U_\beta + \sum_a t_\alpha U^a_\beta.$$  \hspace{1cm} (2.15)

**Remark.** $U^a$ and their tangential derivatives $t^a \partial_a^b t^a U^a_b$, are
known along $\tilde{S}_2$ in the Cauchy problem. Further, if $u \in (C^0, C^3)$ then $'U, 'U_\beta$ are unknown along $\tilde{S}_2$; if $u \in (C^1, C^0)$ or $u \in (C^1, C^2)$ then $'U$ (the normal derivative of $u$) is known but $'U_\beta$ (the normal component of the second derivatives of $u$) is unknown (cf. Section 3[4]).

Further, we introduce

**Definition 6.**

\[
(L, L, u^*), \quad (V_j, V_j)
\]

\[
L \equiv n_0 + \psi^i n_i, \quad L \equiv t_i v^i + t_0
\] (2.16)

\[
u^* \equiv \partial_i u + \psi^j \partial_j u, \quad \partial_j v_i \equiv n_\beta V_j + \sum_a n_a V_{ij}.
\] (2.17)

Now, we find the relations between the derivatives of $u$ and $u^*$.

**Theorem 1.** If $u \in (C^1, C^3)$ or $u \in (C^1, C^2)$, then

\[
n_{\alpha} \partial_\beta u^* = L' U_\beta n^\alpha + \sum_a L' U_{\alpha a} n^\alpha + V_j \partial_j u
\] (2.18)

\[
t_{\alpha} \partial_\beta u^* = L' U_\beta t^\alpha + \sum_a L' U_{\alpha a} t^\alpha + V_j \partial_j u.
\] (2.19)

**Proof.** The proof follows by direct expansion of $\partial_\beta(\partial_i + \psi^j \partial_j) u$, use of Definitions 5 and 6, Young’s theorem (cf. p. 145 [5]), and the determination of the scalar products of the resulting equation with $n^\alpha, t^\beta$.

We are now in a position to discuss the discontinuity theory for the derivatives of the variable $u$. First, we assume the variable $u$ is such that along some three-dimensional surface $S_3$, $u \in (C^1, C^3)$ or $u \in (C^1, C^2)$. We introduce (where brackets denote the jump)

**Definition 7.** $(U, U_\alpha, U_{\alpha a})$

\[
U \equiv [U], \quad U_{\alpha} \equiv [U_{\alpha}], \quad U_{\alpha a} \equiv [U_{\alpha a}]
\] (2.20)

In discontinuity theory the nonvanishing jumps are associated with the unknowns of the Cauchy problem. From our previous theory and Definition 3, we obtain

**Theorem 2.** If $u \in (C^1, C^3)$ or $u \in (C^1, C^2)$, then

\[
U = U_{\alpha} = U_{\alpha a} = 0
\] (2.22)

\[
[\partial_\alpha u^*] = LU_{\alpha}.
\] (2.23)

**Proof.** Relations (2.22) follow from the fact $u \in (C^1, C^3)$ or $u \in (C^1, C^2)$,
DEFINITION 7, and the fact that only \( \dot{U}_\beta \) is unknown for the corresponding Cauchy problem. Equation (2.23) follows from (2.22), after forming the jumps of (2.18), (2.19).

3. **The Jump Relations in Relaxation Hydrodynamic**

We shall now form the jumps of (2.6)-(2.9).

First, we note that \( \rho, q, s, v_j \) are \((C^1, C^3)\). We define (cf. (2.15), Definition 7, and (2.22) of Theorem 2)

**DEFINITION 8.**

\[
\begin{align*}
[\partial_\alpha \partial_\beta \mu] &\equiv n_\alpha P_\beta \quad (3.1) \\
[\partial_\alpha \partial_\beta \phi] &\equiv n_\alpha S_\beta \quad (3.2) \\
[\partial_\alpha \partial_\beta \rho] &\equiv n_\alpha Q_\beta \\
[\partial_\alpha \partial_\beta v_j] &\equiv n_\alpha V_\beta.
\end{align*}
\]

Hence, by use of (2.23), and Definition 3 we find that (2.6)-(2.9) become (as \([\partial_\alpha (\bar{q})^\alpha]\) is \(L[\partial_\alpha \bar{q}]\) and \(\bar{q} = q^* / K\))

\[
\begin{align*}
LP_\alpha + \rho V_j n_\alpha &= 0 \quad (3.5) \\
\rho LV_{\alpha j} + (AP_j + BS_j + CQ_j) n_\alpha &= 0 \quad (3.6) \\
L(TS_\alpha - 2qQ_\alpha) + \bar{q}^2[\partial_\alpha K] &= 0 \quad (3.7) \\
KLQ_\beta - q^*[\partial_\beta K] + \frac{K^2}{L} \left( \frac{CP_\beta}{\rho^2} + FS_\beta + GQ_\beta \right) &= 0. \quad (3.8)
\end{align*}
\]

We show that

**THEOREM 3.** *If the classes of \( \rho, s, q, v_j, K, \) etc. are determined by Definition 3, then scalars \( k, P, Q, S, V \) exist so that*

\[
\begin{align*}
[\partial_\alpha K] &= kn_\alpha \quad (3.9) \\
P_\alpha &= kn_\alpha, \quad Q_\alpha = Qn_\alpha, \quad S_\alpha = Sn_\alpha \quad (3.10) \\
V_{\alpha j} &= Vn_\alpha n_j. \quad (3.11)
\end{align*}
\]

**PROOF.** By use of Definition 3, we obtain (3.9). Use of Young's theorem leads to

\[
[\partial_\alpha \partial_\beta \mu] = [\partial_\beta \partial_\alpha \mu],
\]

where \( u \) is \( \rho, q, s, v_j \) of Definition 3. Hence, by (2.14), (2.22)

\[
n_\alpha U_\beta = n_\beta U_\alpha \quad (3.11a)
\]
Forming the scalar product of (3.11a) with \( n^* \), we obtain (3.10). Substituting (3.9), (3.10) into (3.6) leads to (3.11).

By use of Theorem 3, the system (3.5)–(3.8) reduces to

\[
LP + \rho V(n,n') = 0 \tag{3.12}
\]
\[
\rho LV + AP + BS + CQ = 0 \tag{3.13}
\]
\[
L(TS - 2\bar{q}Q) + \bar{q}^2k = 0 \tag{3.14}
\]
\[
LQ - \bar{q}k + \frac{K}{L} \left( \frac{CP}{\rho^2} + FS + GQ \right) = 0. \tag{3.15}
\]

The system (3.12)–(3.15) is a linear homogeneous set of four equations in the five unknowns \( P, Q, S, V, k \).

Now, we show

**Theorem 4.** If \( \bar{q}(or \ q^*/K) \) is of class \( C^1, C^2 \) then

\[
LQ = \bar{q}k \tag{3.16}
\]

**Proof.** The proof follows directly from

\[
[\partial_{\alpha}\bar{q}] = \frac{K[\partial_{\alpha}g^*] - g^*[\partial_{\alpha}K]}{K^2} \tag{3.17}
\]

by use of (3.9), and (2.23), when \( U_\alpha \) is replaced by (3.10).

**Remark.** The assumption of Theorem 4 is part of the Definition 3; the assumption merely further restricts the class of permissible \( q \).

Next, we define the coefficients \( \bar{a}, \bar{b} \) by

**Definition 8.**

\[
\bar{a} \equiv GT \tag{3.18}
\]
\[
\bar{b} \equiv \bar{q}F. \tag{3.19}
\]

Further, we define the speeds (cf. p. 115[2]) \( c_0^2, c_\infty^2 \) and \( c^2 \) by

**Definition 9.**

\[
c_\infty^2 \equiv A - \frac{BC}{\rho^2F}, \quad c_0^2 \equiv A - \frac{C^2}{\rho^2G}, \quad c^2 \equiv \frac{L^2}{n_jn^j}. \tag{3.20}
\]

**Remark.** By use of (2.11), (2.12), (2.16), we see that \( c^2 \) is

\[
(\psi + \tau^j\phi_j)^n/\Phi. \]
Finally, we prove

**THEOREM 5.** If the classes of \( \rho, q, s, v, s^*, q^*, K, \bar{q} \) are given by Definition 3, then the characteristic manifolds \( S_3 \) of relaxation hydrodynamics are determined by

\[
\bar{a}(c^2 - \bar{c}_a^2) + \bar{b}(c^2 - \bar{c}_b^2) = 0. \tag{3.21}
\]

**PROOF.** By substituting (3.16) into (3.14), (3.15), we obtain

\[
\frac{CP}{\rho^2} + FS + GQ = 0 \tag{3.22}
\]

\[
TS - \bar{q}Q = 0. \tag{3.23}
\]

Using this last relation to eliminate \( S \) in (3.22), we find

\[
\frac{CP}{\rho^2} + \left( \frac{F}{T} \bar{q} + G \right) Q = 0. \tag{3.24}
\]

Eliminating \( V \) from (3.12), (3.13), we obtain

\[
L^2 P - \langle n^2 n \rangle (AP + BS + CQ) = 0. \tag{3.25}
\]

Replacing \( S \) by \( \bar{q}Q/T \) and \( (-P) \) by \( \rho^2 Q(G\bar{q}F/T^{-1}) C^{-1} \) in (3.25), we find

\[
\left( \frac{L^2}{n^2} \right) - A \right) (TG + \bar{q}F) + \bar{q} BC \frac{BC}{\rho^2} + T \frac{C^2}{\rho^2} = 0. \tag{3.26}
\]

By factoring \( \bar{q} \) and \( T \) from the proper terms of (3.26) and using Definitions 8 and 9, we obtain (3.21).

4. A CORRECTION

In our previous paper [2], we stated (on p. 117 below (2.58)) "where \( 'c_x \) is obtained by replacing \( A \) by \(-A\) in \( c_x \) of (2.47)". This statement is incorrect; \( 'c_x \) is \( c_x \). The error is due to the fact that the first term of (2.59) should be \((L^2 - A\Phi) F\). It follows that the characteristics are the manifolds along which the Cauchy problem has no unique solution, as in the usual theory. This means that two of the statements on p. 103 must be revised (lines 1–3, 25–29).
REFERENCES