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Characteristic Manifolds in Nonequilibrium Hydrodynamics*

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1. INTRODUCTION

We shall discuss two points: (1) a mathematical theory for determining the characteristic manifolds in nonequilibrium (relaxation) hydrodynamics, which is based on the Lichnerowicz concept of (C^n, C^m) functions (cf. p. 5[1]), rather than the *ad hoc* conditions A and B of our previous paper (cf. p. 113, [2]); (2) the correction of an erroneous statement in our previous work (cf. p. 117 [2]).

2. THE THEORY OF CHARACTERISTICS

We define the (C^n, C^m) Lichnerowicz class of functions by

DEFINITION 1a. *A function $F(x^j, t), j = 1, 2, 3$, will be called a (C^n, C^m) function (with $m > n$) in the neighborhood of a three-dimensional differentiable manifold, S_3, of a Euclidean four-dimensional space, E, if:*

(1) *the kth derivatives $F^{(k)}$, $k = n + 1, \ldots, m$ are continuous at all points of two open sets $E - S_3^{(+)}$, $E - S_3^{(-)}$, and are uniformly continuous in the corresponding closed sets with finite jumps along the common boundary S_3 of these two sets (i.e. $F^{(k)}$, $k = n + 1, \ldots, m$ are piecewise continuous or F is piecewise C^m in E);* (2) *the kth derivatives $F^{(k)}$, $k = 0, 1, 2, \ldots, n$ are continuous in E, or F is C^n in E (note, $F^{(0)} \equiv F$).*

DEFINITION 1b. *When conditions* (1) *and* (2) *are satisfied, we write*

$$F \in (C^n, C^m).$$

Now, we shall express the basic partial differential equations of relaxation

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hydrodynamics. Let ρ, q, s, v_j, T denote the density, relaxation variable, entropy, velocity vector, and temperature, respectively. Also, we introduce A, B, C, F, G which are known C^1 functions of ρ, q, s (cf. p. 271[3]). Note T is a known C^2 function of ρ, q, s. Again, let $x^j (j = 1, 2, 3)$ denote a system of Cartesian orthogonal coordinates in Euclidean three-space, E_3 ; let $t \equiv x^\circ$ be the time variable, $x^\alpha (\alpha = 0, 1, 2, 3)$ denote all these variables and

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} \qquad \partial_j \equiv \frac{\partial}{\partial x^j} .$$

We introduce the variables

DEFINITION 2.

$$q^* \equiv \partial_t q + v^j \, \partial_j q, \qquad \rho^* \equiv \partial_t \rho + v^j \, \partial_j \rho, \tag{2.1}$$

$$s^* \equiv \partial_t s + v^j \, \partial_j s, \qquad v_j^* \equiv \partial_t v_j + v^k \, \partial_k v_j , \tag{2.2}$$

$$\tilde{q} \equiv \frac{q^*}{K}, \qquad \tilde{v} \equiv \partial_j v^j, \tag{2.3}$$

where K is the relaxation scalar. These variables will belong to the following classes.

DEFINITION 3.

$$v_j , \rho, q, s \in (C^1, C^3), \tag{2.4}$$

$$v_j^*, \rho^*, q^*, s^*, \tilde{v}, K \in (C^0, C^2), \tilde{q} \in (C^1, C^2). \tag{2.5}$$

REMARK. The class of q^*, s^*, $\tilde{v}$, ρ^*, v_j^* is consistent with that of q, s, v_j, ρ (cf. Definition 2).

By differentiating the basic equations of continuity, motion, and energy of a nonheat conducting, perfect fluid of relaxation hydrodynamics (cf. p. 106[2]) and using Definition 2, we obtain the quasilinear second-order system. in ρ, q, s, v_j, ρ^*, q^*, s^*, $\tilde{q}$, $\tilde{v}$

$$\partial_\alpha \rho^* + \partial_\alpha(\rho \tilde{v}) = 0 \tag{2.6}$$

$$\partial_\alpha(\rho v_j^*) + \partial_\alpha(A \, \partial_j \rho + B \, \partial_j s + C \, \partial_j q) = 0 \tag{2.7}$$

$$\partial_\alpha(T s^*) - q^* \, \partial_\alpha \tilde{q} - \tilde{q} \, \partial_\alpha q^* = 0 \tag{2.8}$$

$$\partial_\beta \left\{ (\tilde{q})^* + \frac{C}{\rho^2} \rho^* + F s^* + G q^* \right\} = 0. \tag{2.9}$$

In order to determine the characteristic three-dimensional manifolds S_3 of (2.6)–(2.9), we shall briefly study the Cauchy problem for this system along some given $\tilde{S}_3$. By such a study, we will determine the relation between the various *unknown normal derivatives* of

$$\rho, q, s, v_j^*, \rho^*, q^*, s^*, \tilde{q}, \tilde{v}.$$

First, we introduce

DEFINITION 4. *If $\tilde{S}_3$ is given by the C^2 function of x^α*

$$\phi(x^\alpha) = c,$$

where c is a constant then the space-time and space unitized normals, n_α and $\bar{n}_j$ of $\tilde{S}_3$, *respectively, are*

$$n_\alpha \equiv \frac{\phi_\alpha}{\Phi}, \qquad \bar{n}_j \equiv \frac{\phi_j}{\bar{\Phi}}. \tag{2.10}$$

REMARK. Note, by definition, ϕ_j is $\partial\phi/\partial x^j$, x^0 is t and, ϕ_α is $\partial\phi/\partial x^\alpha$ (see the notation introduced before Definition 1). Further, if g_{jk} is the metric tensor of E_3, then

$$\Phi \equiv (\phi_t^2 + g^{jk}\phi_j\phi_k)^{1/2} \tag{2.11}$$

$$\bar{\Phi} \equiv (g^{jk}\phi_j\phi_k)^{1/2}. \tag{2.12}$$

Again, if $\underset{a}{t_\alpha}(a = 1, 2, 3)$ denote any three mutually orthogonal unit vectors which span $\tilde{S}_3$ at any point and if $g_{\alpha\rho}$ is the metric tensor of E, then

$$g^{\alpha\beta}\underset{a}{t_\alpha} n_\beta \equiv \underset{a}{t_0} n_0 + g^{jk}\underset{a}{t_j} n_k = 0. \tag{2.13}$$

We decompose the partial derivatives of any of the quantities of Definition 3 (*represented by u*) into their normal and tangential components by means of the following relations

DEFINITION 5.

$$('U, '\underset{a}{U}, '\underset{a}{U_\alpha}, '\underset{a}{U_\alpha})$$

$$\partial_\beta u = n_\beta\, 'U + \sum_a \underset{a}{t_\beta}\, '\underset{a}{U} \tag{2.14}$$

$$\partial_\alpha\, \partial_\beta u = n_\alpha\, 'U_\beta + \sum_a \underset{a}{t_\alpha}\, '\underset{a}{U_\beta}. \tag{2.15}$$

REMARK. $'\underset{a}{U}, '\underset{a}{U_\alpha}$ and their *tangential* derivatives $t^\alpha\partial_\alpha' \underset{b}{U}$, $\underset{a}{t^\alpha}\partial' \underset{\alpha\, a}{U_\beta}$, are *known* along $\tilde{S}_3$ in the Cauchy problem. Further, if $u \in (C^0, C^2)$ then $'U$, $'U_\beta$ are *unknown* along $\tilde{S}_3$; if $u \in (C^1, C^3)$ or $u \in (C^1, C^2)$ then $'U$ (the normal derivative of u) is *known* but $'U_\beta$ (the normal component of the second derivatives of u) is *unknown* (cf. Section 3[4]).

Further, we introduce

DEFINITION 6.

$$(L, \underset{a}{L}, u^*), \quad ('V_j, \underset{a}{V_j})$$

$$L \equiv n_0 + v^j n_j, \qquad \underset{a}{L} \equiv \underset{a}{t_j}\, v^j + \underset{a}{t_0} \tag{2.16}$$

$$u^* \equiv \partial_t u + v^j \partial_j u, \qquad \partial_\beta v_j \equiv n_\beta\, 'V_j + \sum_a \underset{a}{t_\beta}\, '\underset{a}{V_j}. \tag{2.17}$$

Now, we find the relations between the derivatives of u and u^*.

THEOREM 1. *If* $u \in (C^1, C^3)$ *or* $u \in (C^1, C^2)$, *then*

$$n^\beta \partial_\beta u^* = L\, 'U_\beta\, n^\beta + \sum_a \underset{a}{L}\, '\underset{a}{U_\beta}\, n^\beta + 'V^j \partial_j u \tag{2.18}$$

$$\underset{c}{t_\beta}\, \partial_\beta u^* = L\, 'U_\beta\, \underset{c}{t^\beta} + \sum_a \underset{a}{L}\, '\underset{a}{U_\beta}\, \underset{c}{t^\beta} + '\underset{c}{V^j}\, \partial_j u. \tag{2.19}$$

PROOF. The proof follows by direct expansion of $\partial_\beta(\partial_t + v^j\partial_j)\, u$, use of Definitions 5 and 6, Young's theorem (cf. p. 145 [5]), and the determination of the scalar products of the resulting equation with n^β, $\underset{c}{t^\beta}$.

We are now in a position to discuss the discontinuity theory for the derivatives of the variable u. First, we *assume* the variable u is such that along some three-dimensional surface S_3, $u \in (C^1, C^2)$ or $u \in (C^1, C^3)$. We introduce (where brackets denote the jump)

DEFINITION 7. $(U, U_\beta, \underset{a}{U}, \underset{a}{U_\beta})$

$$U \equiv [U], \quad U_\beta \equiv ['U_\beta], \tag{2.20}$$

$$\underset{a}{U} \equiv ['\underset{a}{U}], \quad \underset{a}{U_\beta} \equiv ['\underset{a}{U_\beta}]. \tag{2.21}$$

In discontinuity theory the nonvanishing jumps are associated with the unknowns of the Cauchy problem. From our previous theory and Definition 3, we obtain

THEOREM 2. *If* $u \in (C^1, C^3)$ *or* $u \in (C^1, C^2)$, *then*

$$U = \underset{a}{U} = \underset{a}{U_\alpha} = 0 \tag{2.22}$$

$$[\partial_\alpha u^*] = L U_\alpha. \tag{2.23}$$

PROOF. Relations (2.22) follow from the fact $u \in (C^1, C^3)$ or $u \in (C^1, C^2)$, DEFINITION 7, and the fact that only $'U_\beta$ is unknown for the corresponding Cauchy problem. Equation (2.23) follows from (2.22), after forming the jumps of (2.18), (2.19).

3. THE JUMP RELATIONS IN RELAXATION HYDRODYNAMIC

We shall now form the jumps of (2.6)–(2.9).

First, we note that ρ, q, s, v_j are (C^1, C^3). We define (cf. (2.15), Definition 7, and (2.22) of Theorem 2)

DEFINITION 8.

$$[\partial_\alpha \partial_\beta \rho] \equiv n_\alpha P_\beta \tag{3.1}$$

$$[\partial_\alpha \partial_\beta s] \equiv n_\alpha S_\beta \tag{3.2}$$

$$[\partial_\alpha \partial_\beta q] \equiv n_\alpha Q_\beta \tag{3.3}$$

$$[\partial_\alpha \partial_\beta v_j] \equiv n_\alpha V_{\beta j}. \tag{3.4}$$

Hence, by use of (2.23), and Definition 3 we find that (2.6)–(2.9) become (as $[\partial_\alpha(\tilde{q}/)^*]$ is $L[\partial_\alpha \tilde{q}]$ and $\tilde{q}$ is q^*/K)

$$LP_\alpha + \rho V_j{}^j n_\alpha = 0 \tag{3.5}$$

$$\rho L V_{\alpha j} + (AP_j + BS_j + CQ_j)\, n_\alpha = 0 \tag{3.6}$$

$$L(TS_\alpha - 2\tilde{q}Q_\alpha) + \tilde{q}^2[\partial_\alpha K] = 0 \tag{3.7}$$

$$KLQ_\beta - q^*[\partial_\beta K] + \frac{K^2}{L}\left(\frac{CP_\beta}{\rho^2} + FS_\beta + GQ_\beta\right) = 0. \tag{3.8}$$

We show that

THEOREM 3. *If the classes of ρ, s, q, v_j, K, etc. are determined by Definition 3, then scalars k, P, Q, S, V exist so that*

$$[\partial_\alpha K] = kn_\alpha \tag{3.9}$$

$$P_\alpha = Pn_\alpha, \qquad Q_\alpha = Qn_\alpha, \qquad S_\alpha = Sn_\alpha \tag{3.10}$$

$$V_{\alpha j} = Vn_\alpha n_j. \tag{3.11}$$

PROOF. By use of Definition 3, we obtain (3.9). Use of Young's theorem leads to

$$[\partial_\alpha \partial_\beta u] = [\partial_\beta \partial_\alpha u],$$

where u is ρ, q, s, v_j of Definition 3. Hence, by (2.14), (2.22)

$$n_\alpha U_\beta = n_\beta U_\alpha \tag{3.11a}$$

Forming the scalar product of (3.11a) with n^α, we obtain (3.10). Substituting (3.9), (3.10) into (3.6) leads to (3.11).

By use of Theorem 3, the system (3.5)–(3.8) reduces to

$$LP + \rho V(n_j n^j) = 0 \tag{3.12}$$

$$\rho LV + AP + BS + CQ = 0 \tag{3.13}$$

$$L(TS - 2\tilde{q}Q) + \tilde{q}^2 k = 0 \tag{3.14}$$

$$LQ - \tilde{q}k + \frac{K}{L}\left(\frac{CP}{\rho^2} + FS + GQ\right) = 0. \tag{3.15}$$

The system (3.12)–(3.15) is a linear homogeneous set of *four equations* in the *five unknowns* P, Q, S, V, k.

Now, we show

THEOREM 4. *If* $\tilde{q}$(or q^*/K) *is of class* (C^1, C^2) *then*

$$LQ = \tilde{q}k \tag{3.16}$$

PROOF. The proof follows directly from

$$[\partial_\alpha \tilde{q}] = \frac{K[\partial_\alpha q^*] - q^*[\partial_\alpha K]}{K^2} \tag{3.17}$$

by use of (3.9), and (2.23), when U_α is replaced by (3.10).

REMARK. The assumption of Theorem 4 is part of the Definition 3; the assumption merely further restricts the class of permissable q.

Next, we define the coefficients $\bar{a}$, $\bar{b}$ by

DEFINITION 8.

$$\bar{a} \equiv GT \tag{3.18}$$

$$\bar{b} \equiv \tilde{q}F. \tag{3.19}$$

Further, we define the speeds (cf. p. 115[2]) $c_0{}^2$, $c_\infty{}^2$ and c^2 by

DEFINITION 9.

$$c_\infty^2 \equiv A - \frac{BC}{\rho^2 F}, \quad c_0{}^2 \equiv A - \frac{C^2}{\rho^2 G}, \quad c^2 \equiv \frac{L^2}{n_j n^j}. \tag{3.20}$$

REMARK. By use of (2.11), (2.12), (2.16), we see that c^2 is

$$(\phi_t + v^j \phi_j)^2/\bar{\Phi}.$$

Finally, we prove

THEOREM 5. *If the classes of $\rho, q, s, v_j, s^*, q^*, K, \tilde{q}$ are given by Definition* 3, *then the characteristic manifolds S_3 of relaxation hydrodynamics are determined by*

$$\bar{a}(c^2 - c_0{}^2) + \bar{b}(c^2 - c_\infty{}^2) = 0. \tag{3.21}$$

PROOF. By substituting (3.16) into (3.14), (3.15), we obtain

$$\frac{CP}{\rho^2} + FS + GQ = 0 \tag{3.22}$$

$$TS - \tilde{q}Q = 0. \tag{3.23}$$

Using this last relation to eliminate S in (3.22), we find

$$\frac{CP}{\rho^2} + \left(\frac{F}{T}\tilde{q} + G\right)Q = 0. \tag{3.24}$$

Eliminating V from (3.12), (3.13), we obtain

$$L^2P - (n^j n_j)(AP + BS + CQ) = 0. \tag{3.25}$$

Replacing S by $\tilde{q}Q/T$ and $(-P)$ by $\rho^2 Q(G\tilde{q}F/T^{-1})\, C^{-1}$ in (3.25), we find

$$\left(\frac{L^2}{n_j n^j} - A\right)(TG + \tilde{q}F) + \tilde{q}\frac{BC}{\rho^2} + \frac{TC^2}{\rho^2} = 0. \tag{3.26}$$

By factoring $\tilde{q}$ and T from the proper terms of (3.26) and using Definitions 8 and 9, we obtain (3.21).

4. A CORRECTION

In our previous paper [2], we stated (on p. 117 below (2.58)) "where $'c_\infty$ is obtained by replacing A by $-A$ in c_∞ of (2.47". This statement is incorrect; $'c_\infty$ is c_∞. The error is due to the fact that the first term of (2.59) should be $(L^2 - A\Phi)\,\bar{P}$. It follows that the characteristics are the manifolds along which the Cauchy problem has no unique solution, as in the usual theory. This means that two of the statements on p. 103 must be revised (lines 1–3, 25–29).

References

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