Some Remarks on Critical Point Theory in Hilbert Space  
(Continuation)

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This paper is a continuation of the paper of the same title in the Proceedings of the Symposium on Nonlinear Problems, held at the Mathematical Research Center, U.S. Army, at the University of Wisconsin April 30 to May 2, 1962 (University of Wisconsin Press, 1963, pp. 233-256). References to definitions, lemmas etc. always refer to that paper as do bibliographical references (numbers in brackets). The particular subject matter of the present paper is indicated in the title of the following Section 7.

7. Localization and addition theorems.

**Theorem 7.1.** Let the scalar \( i = i(x) \) satisfy the assumptions (A)-(E) of Section 4. Let \( c \) be a critical level of \( i \), and let \( a, b \) be two numbers such that \( a < c < b \) and such that \( c \) is the only critical level in the closed interval \([a, b]\) while \( i_b \) is not empty. Let \( a \) be the critical set at level \( c \). Then every open neighborhood \( U \) of \( D \) contains a neighborhood \( W \) of \( T \) such that the critical group of dimension \( q \) at level \( c \) (see Definition 4.4) is isomorphic to the homology group of the same dimension of the couple \((W, W \cap i_a)\).

(See [6, Theorem 8.2] for the finite dimensional case).

For the proof we need some lemmas.

**Lemma 7.1.** For any \( \varepsilon \) with \( 0 < \varepsilon < b - c \) the critical group of dimension \( q \) at the level \( c \) is isomorphic to

\[
H_q(\tilde{i}_{c+\varepsilon}, \tilde{i}_c).
\]  

**Proof.** By Definition 4.4 and Theorem 4.2 the critical group in question is isomorphic to

\[
H_q(\tilde{i}_{c+\varepsilon}, \tilde{i}_a).
\]  

Now by Lemma 5.6 the pair \((\tilde{i}_c, \tilde{i}_a)\) is homotopically trivial, and since \( \tilde{i}_{c+\varepsilon} \supseteq i_c \supseteq \tilde{i}_a \) the isomorphism of the groups (7.1), (7.2) follows from Lemma 2.4.
Lemma 7.2.  (α). There exists an open neighborhood $U_2 \subset S_{a,b}$ of $\sigma$ where
$S_{a,b}$ is defined as in Lemma 4.8.

(β). For $0 \leq t \leq T$ let $\Delta(x_0, t)$ be the deformation defined in Definition 4.3, and for any set $S \subset V$ let $\Delta_s S$ denote the image of $S$ under the map $\Delta(\cdot, t)$. If then $U$ is an open neighborhood of $\sigma$ there exists an open neighborhood $U_1 \subset U$ of $\sigma$ such that

$$\Delta_s U_1 \subset U, \quad 0 \leq t \leq T.$$  \hfill (7.3)

Proof of (α). The set $\Gamma$ of all critical points of $i$ in $V$ is compact (Lemma 4.3) and has no points in common with the boundary $\hat{V}$ of $V$ (by assumption D). As a closed subset of $\Gamma$, the set $\sigma$ has also these properties; moreover $\sigma$ has no points in common with the closure $\hat{S}_{a,b}$ of the complement $S_{a,b}'$ of $S_{a,b}$. Therefore the compact set $\sigma$ has a positive distance from $\hat{V} \cup \hat{S}_{a,b}'$, which proves the assertion (α).

Proof of (β). By Definition 4.3

$$\Delta(\xi, t) = \xi \quad \text{for} \quad \xi \in \sigma.$$  \hfill (7.4)

Now $\Delta(x_0, t)$ depends continuously on $x_0$ uniformly in $t$ (see e.g. [9, Lemma 4.4]). Therefore (7.4) implies the existence of a positive $\rho = \rho(\xi)$ such that $\Delta(x, t) \in U$ for $0 \leq t \leq T$ and for $x_0$ in the ball $V_{\rho}(\xi)$ of center $\xi$ and radius $\rho$. Since $\sigma$ is contained in the open set $U$ we may, in addition, require $\rho$ to be such that $V_{\rho}(\xi) \subset U$. But the compactness of $\sigma$ assures the existence of a finite number of points $\xi_1, \ldots, \xi_r$ in $\sigma$ such that $\sigma \subset \cup V_{\rho_i}(\xi_i) \subset U$ where $\rho_i = \rho(\xi_i)$. Then $U_1 = \cup V_{\rho_i}(\xi_i)$ satisfies the requirements of our assertion (β).

Lemma 7.3. Let $U$ be an open neighborhood of $\sigma$. Then there exist positive numbers $e$ and $\epsilon$ such that

$$\Delta(x_0, T) \in U \cup \tilde{S}_{a,e} \quad \text{for} \quad x_0 \in \tilde{i}_{e+\epsilon}.$$  \hfill (7.5)

Proof. Let first $e$ be any number satisfying

$$0 < \epsilon < b - c.$$  \hfill (7.6)

Now $\tilde{i}_{e+\epsilon} = S_{a,c+\epsilon} \cup i_a$. If $x_0 \in i_a$ then (7.5) is true for any $e$ satisfying

$$0 < \epsilon < c - a,$$  \hfill (7.7)

since $i$ is not increasing under $\Delta$ (Lemma 4.11). Let now $x_0 \in S_{a,c+\epsilon}$, and let $U_1$ be as in the (β)-part of Lemma 7.2. Then (7.5) is true for $x_0 \in S_{a,c+\epsilon} \cap U_1$. It remains to prove (7.5) for $x_0 \in S_{a,c+\epsilon} \setminus U_1$ where $U_1$ is the complement of $U_1$. We note that by (7.6)

$$S_{a,c+\epsilon} \cap U_1' \subset \tilde{S}, \quad \text{where} \quad \tilde{S} = S_{a,b} \cap U_1'.$$  \hfill (7.8)
Now
\[ \tilde{S} \cap \Gamma = \emptyset \quad (\emptyset \text{ the empty set}). \quad (7.9) \]
Indeed: \( \sigma \cap U_1 = \emptyset \) since \( \sigma \in U_1 \). On the other hand \( (\Gamma - \sigma) \cap S_{a,b} = \emptyset \) since \( c \) is the only critical level in \([a, b]\). Equation (7.9), together with the compactness of \( \Gamma \) and the closedness of \( \tilde{S} \), implies that these two sets have a positive distance. If we denote this distance by \( 3\rho \) then the \( \rho \)-neighborhood \( \tilde{S}_\rho \) of \( \tilde{S} \) has a distance greater than \( \rho \) from \( \Gamma \). Consequently if \( M \) is a constant satisfying
\[ M > \left\{ \left\| g(x) \right\| \right\} \rho/T, \quad \text{for} \quad x \in V, \quad (7.10) \]
if \( \mu = \rho M \), and if \( \Gamma_\rho \) is as defined in Lemma 4.5, then, noting that \( \tilde{S} \subset \tilde{S}_\rho \subset \Gamma_\rho \), we see from Lemma 4.6 that there exists a constant \( m \) such that
\[ \left\| g(x) \right\| > m > 0 \quad \text{for} \quad x \in \tilde{S}_\rho. \quad (7.11) \]
Let now \( x_0 \in \tilde{S} \). Then, because of (7.9), \( \Delta(x_0, t) \) is by its Definition 4.3 the solution \( x(t) \) of the problem \( dx/dt = -g(x), \ x(0) = x_0 \). From this and from (7.10) we see that
\[ \left\| \Delta(x_0, t) - x_0 \right\| = \left\| \int_0^t g(x(\tau)) \, d\tau \right\| \leq Mt \leq \rho \quad \text{for} \quad 0 \leq t \leq \rho/M. \]
This shows that \( \Delta(x_0, t) \in \tilde{S}_\rho \) for the \( t \)-values indicated such that for these \( t \), (7.11) holds with \( x = x(t) = \Delta(x_0, t) \). We see therefore from (4.15) that
\[ i(\Delta x_0, \rho M^{-1}) = i(x_0) - \int_0^{\rho M^{-1}} \left\| g(x(\tau)) \right\|^2 \, d\tau < i(x_0) - \frac{m^2 \rho}{M}. \]
But, by (7.10), \( T > \rho/M \), and since \( i(\Delta x_0, t) \) is not increasing in \( t \) we see that
\[ i(\Delta x_0, T) < i(x_0) - \frac{m^2 \rho}{M} \quad \text{for} \quad x_0 \in \tilde{S}. \quad (7.12) \]
Because of (7.8) this inequality holds for \( x_0 \in S_{a, c+\epsilon} \cap U_1 \). For such \( x_0 \), \( i(x_0) \leq c + \epsilon \). Therefore by (7.12)
\[ i(\Delta(x_0, T)) < c - \left( \frac{m^2 \rho}{M} - \epsilon \right). \]
But this implies \( \Delta(x_0, t) \in i_{c+\epsilon} \) if we choose
\[ \epsilon = \epsilon = \min \left( b - c, c - a, \frac{m^2 \rho}{2M} \right). \]
This finishes the proof of (7.5).
Lemma 7.4. In the notation of Lemma 7.3 the couples \((i_{c+\epsilon}, i_c)\) and \((\Delta i_{c+\epsilon}, i_c \cap \Delta i_{c+\epsilon})\) are homotopically equivalent, and therefore (Lemma 2.2)

\[ H_q(i_{c+\epsilon}, i_c) \cong H_q(\Delta i_{c+\epsilon}, i_c \cap \Delta i_{c+\epsilon}). \]  

(7.13)

Proof. It is easily verified from the properties of the deformation \(\Delta t(0 \leq t \leq T)\) that the assumptions of Lemma 2.1 are satisfied with

\[ B = i_c, \quad A = \Delta i_{c+\epsilon}, \quad D = i_c, \quad C = i_c \cap D. \]  

(7.14)

Proof of Theorem 7.1. Let \(\epsilon\) and \(\epsilon\) be as in Lemma 7.3. Using the notation of (7.14) we set

\[ W = U \cap D, \quad W_1 = U' \cap D, \]  

(7.15)

where \(U'\) is the complement of \(U\). The proof consists in excising \(W_1\) from the couple \((D, C)\). Obviously

\[ D - W_1 = W. \]  

(7.16)

We assert moreover that

\[ C - W_1 = i_c \cap W. \]  

(7.17)

Indeed we see from (7.5) and (7.14) that \(D \subset U \cup i_{c-\epsilon}\), and therefore from (7.15)

\[ W_1 \subset U' \cap (U \cup i_{c-\epsilon}) = U' \cap i_{c-\epsilon} \subset i_{c-\epsilon} \subset i_c. \]  

(7.18)

Thus \(W_1 = i_c \cap W_1\), and the left member of (7.17) may be written as \(i_c \cap (D - W_1)\), and (7.17) now follows from (7.16).

In order to apply the excision theorem for singular homology theory ([3, VII, Theorem 9.1]) we have to verify that in the relative topology of \(D\) the interior of \(C\) contains the closure of \(W_1\). Now in this topology \(C = i_c \cap D\) is open since \(i_c\) is (absolutely) open. Thus we have to prove \(W_1 \subset C\). But we see from (7.18) that \(W_1 \subset i_{c-\epsilon} \subset i_c\) and therefore from (7.14), (7.15)

\[ W_1 = i_c \cap W_1 = i_c \cap (U' \cap D) = i_c \cap U' \cap D \subset i_c \cap D = C. \]  

(7.19)

We thus may apply the excision theorem which by (7.16), (7.17) yields the isomorphism \(H_q(D, C) \cong H_q(W, i_c \cap W)\). Combining this with (7.13) we see (using (7.14)) that \(H_q(i_{c+\epsilon}, i_c) \cong H_q(W, i_c \cap W)\). By Lemma 7.1 this isomorphism proves Theorem 7.1.

Theorem 7.2. Let \(\sigma\) be as in Theorem 7.1. Assume \(\sigma = \sigma_1 \cup \sigma_2\), where \(\sigma_1\) and \(\sigma_2\) have a positive distance from each other, and let \(U\) be an open neighborhood of \(\sigma\). Then there exist disjoint neighborhoods \(W_j\) of \(\sigma_j\) \((j = 1, 2)\) such that \(W = W_1 \cup W_2 \subset U\) and such that the \(q\)th critical group of \(i\) at level \(c\) is isomorphic to the direct sum

\[ H_q(W_1, i_c \cap W_1) \oplus H_q(W_2, i_c \cap W_2). \]  

(7.19)
PROOF. Since $\sigma$ is compact and $\sigma_1$ and $\sigma_2$ have a positive distance it follows that $\sigma_1$ and $\sigma_2$ are compact. Therefore there exist open neighborhoods $U_1$ and $U_2$ of $\sigma_1$ and $\sigma_2$ respectively with a positive distance from each other. Obviously we may also require that these neighborhoods are subsets of $U$. Then their union is contained in $U$, and for the purpose of our proof it is no loss of generality to assume that $U = U_1 \cup U_2$. Let now $W \subset U$ be a neighborhood of $\sigma$ which satisfies the assertions of Theorem 7.1, and let $W_j = W \cap U_j$. Then $W = W_1 \cup W_2$, and by the direct sum theorem ([3; p. 33]) the direct sum (7.19) is isomorphic to $H_\varphi(W, i_c \cap W)$. Theorem 7.2 now follows from Theorem 7.1.

THEOREM 7.3. In addition to our previous assumptions let $\sigma$ consist of a finite number of distinct critical points $\sigma_1, \ldots, \sigma_p$. Then there exist (arbitrarily small) disjoint neighborhoods $Z_j$ of $\sigma_j$ such that the $q$th critical group of $i$ at level $c$ is isomorphic to the direct sum of the groups

$$H_\varphi(\sigma_j \cup Z_j, Z_j), \quad j = 1, \ldots, p.$$ (7.20)

PROOF. By Theorem 5.2 our critical group is isomorphic to $H_\varphi(\sigma \cup i_c, i_c)$. We will prove that every open neighborhood $U$ of $\sigma$ contains a neighborhood $Z$ of $\sigma$ such that

$$H_\varphi(\sigma \cup i_c, i_c) \approx H_\varphi(\sigma \cup Z, Z).$$ (7.21)

Let us show first that this assertion implies our theorem. For $j = 1, \ldots, p$, let $U_j \subset U$ be disjoint open neighborhoods of $\sigma_j$. It is then no loss of generality to assume that $U = \cup_1^p U_j$. If now $Z \subset U$ satisfies (7.21) and if $Z_j = Z \cap U_j$ such that $Z = \cup_1^p Z_j$, then by the direct sum theorem the right member of (7.21) is isomorphic to the direct sum of the groups (7.20).

Now the proof of (7.21) follows closely that of Theorem 7.1 if instead of (7.14) we introduce the notation

$$B = \sigma \cup i_c, \quad A = i_c, \quad D = \Delta TB, \quad C = i_c \cap D.$$ (7.22)

Noting that

$$\Delta_T \sigma = \sigma, \quad i_c \cap \sigma = \emptyset, \quad \Delta_T i_c \subset i_c,$$ (7.23)

we see that

$$C = \Delta_T A.$$ (7.24)

This together with (7.22) shows that the couple $(D, C)$ is obtained from the couple $(B, A)$ by the deformation $\Delta_T$. Therefore

$$H_\varphi(\sigma \cup i_c, i_c) = H_\varphi(B, A) \approx H_\varphi(D, C).$$ (7.25)
If we define again $W$ and $W_1$ by (7.15) we see that (7.16) and (7.17) hold. Consequently by excising $W_1$ we obtain from (2.25)

$$H_g(\sigma \cup i_e, i_e) \approx H_g(W, i_e \cap W). \quad (7.26)$$

We now set $Z = U \cap \Delta i_e$. We then see from (7.15), (7.22) and (7.23) that $W = \sigma \cup Z$ and $i_e \cap W = Z$. Therefore (7.21) follows from (7.26).