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FIVE-DIMENSIONAL QUASI-SPIN THE n, T DEPENDENCE OF SHELL-MODEL MATRIX ELEMENTS IN THE SENIORITY SCHEME

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Abstract: The five-dimensional quasi-spin formalism is used to factor out the n, T dependent parts of shell-model matrix elements in the seniority scheme and derive reduction formulae which make it possible to express matrix elements for states of definite isospin T in the configuration j^n in terms of the corresponding matrix elements for the configuration j^v. The n, T dependent factors for one- and two-nucleon c.f.p. and for the matrix elements of one-body operators and the two-body interaction are expressed in terms of generalized R(5) Wigner coefficients. The needed R(5) Wigner coefficients are calculated in the form of general algebraic expressions for the seniorities v and reduced isospins t corresponding to the simpler R(5) irreducible representations. In this first contribution, the R(5) representations $(\omega_1 t) = (j+\frac{1}{2}-\frac{1}{2}v, t)$ are restricted to $(\omega_1 0)$, $(\omega_1 \frac{1}{2})$, (tt), and the states of $(\omega_1 1)$ with $n-v = 4k-2T$, (k is an integer). Explicit expressions are given for the diagonal matrix elements of the general, charge-independent, two-body interaction and the isovector and isotensor parts of the Coulomb interaction for seniorities $v = 0$ and 1, and the $v = 2$ states with $n = 4k+2-2T$.

1. Introduction

For configurations of identical nucleons, the three-dimensional, quasi-spin formalism [1]) has been used by Lawson and Macfarlane[2]) [††] to factor out the n-dependent parts of nuclear matrix elements. The resulting reduction formulae make it possible to express matrix elements involving states for the configuration j^n with seniority v in terms of the corresponding matrix elements for the configuration j^v. For the most part, however, the three-dimensional, quasi-spin formalism merely furnishes a simple and elegant way to understand well-known results [5]). Expressions for the n-dependent factors of nuclear matrix elements for configurations j^n of identical nucleons have been derived without the use of the quasi-spin formalism. For configurations of both protons and neutrons on the other hand, analogous reduction formulae which give the dependence of nuclear matrix elements on nucleon number n and isospin T are much more complicated, and the generalization of the quasi-spin formalism to five dimensions [6-12]) now constitutes a valuable tool in deriving such formulae. The five-dimensional, quasi-spin formalism makes it possible to give the explicit n, T dependence of all nuclear matrix elements in the seniority scheme in terms of generalized Wigner coefficients for a five-dimensional rotation group. With the calculation of these R(5)

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†† See also ref. [3]) and for extensions to mixed configurations, see refs. [2-4]).

Wigner coefficients, the basic purpose of the seniority classification can be achieved; i.e. the influence on all nuclear properties of nucleon pairs coupled to $J = 0$ can be expressed in terms of explicit n, T dependent factors. Some progress has been made toward the achievement of this goal. The R(5) Wigner coefficients needed for the n, T dependent factors of one-nucleon fractional parentage coefficients for $j \leqq \frac{7}{2}$ have been tabulated, partly numerically, by Ginocchio [12]), while the R(5) coefficients needed for the $J = 0$ coupled two-nucleon c.f.p. have been tabulated numerically by Ichimura [13]). It is the purpose of this and a succeeding investigation [27]) to extend this work to give general algebraic formulae for the n, T dependent factors for all one- and two-nucleon c.f.p., all matrix elements of one-body operators and the diagonal matrix elements of the general two-body interaction including the isovector and isotensor parts of the Coulomb interaction [14]), for all seniorities and reduced isospins of possible interest in shell-model calculations.

In order to review the power of the method it may be useful to compare the "quasi-spin spectroscopy" of Helmers and others [6-12]) with the conventional seniority spectroscopy of Racah and Flowers [5]). In the conventional spectroscopy, states for a simple configuration j^n of both neutrons and protons are classified according to the group chain $U(2j+1) \supset Sp(2j+1) \supset R(3)$. Some disadvantages are inherent in this chain. (i) The highest symmetry which preferably should be associated with the most powerful quantum numbers actually has irreducible representations characterized solely by the trivial quantum numbers n and T. (ii) The highest group in the chain is unnecessarily complicated. For large j the rank of the highest group is unnecessarily large since the irreducible representations of actual interest are labelled by merely two quantum numbers, n and T. (iii) A different symmetry group is necessary as starting point of the group chain for each j, and in particular for mixed configurations. In quasi-spin spectroscopy on the other hand, the classification scheme is based on two parallel group chains starting with the direct product of the quasi-spin group and the symplectic group in $(2j+1)$ dimensions, i.e. $R(5) \times Sp(2j+1)$. The set of ten infinitesimal operators which generate R(5) commute with the infinitesimal operators which generate $Sp(2j+1)$. The irreducible representations of both R(5) and $Sp(2j+1)$ are labelled by seniority v and reduced isospin t [ref. [6])]. The group chain based on Sp $(2j+1)$ is that associated with the conventional spectroscopy, $Sp(2j+1) \supset R(3)$, but is now completely specified by the simple configuration j^v. The group chain based on R(5) can be chosen to include the subgroup SU(2) generated by the three components of isospin $\boldsymbol{T}$ which commute with the number operator [generator of U(1)] : $R(5) \supset SU(2) \times U(1)$. Quasi-spin spectroscopy thus achieves the following aims. (i) Nucleon number n and isospin T are now associated with the lowest subgroups in one of the chains. They play the same role as the magnetic quantum number M_J of ordinary angular momentum theory. Dependence on n and T can thus be factored out of any matrix element by application of a generalized Wigner-Eckart theorem and knowledge of the associated generalized R(5) Wigner coefficients. (ii) The highest symmetry in the classification scheme is now as simple as possible. The group R(5) which is the starting point of one of the two parallel group chains is a simple group of rank 2 for which explicit properties including the needed Wigner coefficients can be worked out in detail. (iii) The starting point of that branch of the group chain containing the isospin and number operators is always R(5) so that the same symmetry group serves for all j as well as for mixed configurations.

The quasi-spin technique is thus a natural tool for deriving reduction formulae through which matrix elements for the configuration j^n can be expressed in terms of corresponding matrix elements for the simpler configuration j^v. The formulae are easily generated to mixed configurations. The quasi-spin technique is also tailor-made for the study of the general n- and T-dependence of physical quantities for series of nuclei, although its applications in this regard are somewhat limited by the fact that it is tied closely to the seniority scheme. Seniority is in general not a good quantum number in nuclei where both neutrons and protons are filling the same shells. Nevertheless some observed simple n, T dependent effects in light- and intermediate-weight nuclei may perhaps be understood at least qualitatively by a very simple application of the quasi-spin formalism if admixtures of high seniorities are relatively unimportant to the understanding of such effects. So-called isobaric mass formulae and Coulomb energy systematics may possibly fall into this category. In a systematic study of the energies of isobaric analogue states, for example, Jänecke [15]) found a simple $T(T+1)$ dependence for the energies of isobaric analogue states of odd-mass nuclei and a similar $T(T+1)$ dependence for the energies of even nuclei supplemented by a strong pairing effect which favours (lowers the energy of) the even T states of nuclei with $A = 4k$, (k is an integer) and the odd T states of nuclei with $A = (4k+2)$. Similarly, the isovector and tensor coefficients of the Coulomb energy (diagonal matrix elements of the Coulomb interaction) show systematic n, T dependent effects [16,17]).

Although the application of the quasi-spin formalism is straightforward in principle, it is complicated in practice by the fact that the group chain $R(5) \supset SU(2) \times U(1)$ is not a canonical one corresponding to a mathematically natural group decomposition. The scheme in which the physically relevant labels n, T and M_T are good quantum numbers does not completely specify the states of the irreducible representations of R(5) without the introduction of a fourth operator which commutes with $\boldsymbol{T}^2$, T_0 and the number operator. Such an operator has the disadvantage that its eigenvalues are not related to the irreducible representation labels of a subgroup of the decomposition. A mathematically natural and complete labelling scheme could be based on the group chain $R(5) \supset R(4) \equiv R(3) \times R(3)$, where the two commuting R(3) groups are the separate neutron and proton quasi-spin groups and together furnish the four commuting operators needed to completely specify the states. Although the generalized Wigner coefficients have been calculated in this scheme [18]), these do not give the needed n, T dependent factors for nuclear matrix elements directly since $\boldsymbol{T}^2$ is not diagonal in this scheme. By calculating transformation coefficients [refs. [13,18])] from this mathematically natural scheme to the physically relevant one, however, it is possible to calculate the needed n, T dependent factors, at least numerically. Although this technique is laborious it has been used to give tables for some of the simpler of these coefficients [11,13]).

Since it is the aim of this investigation to give general algebraic expressions for the n, T dependent factors for all types of nuclear matrix elements and all seniorities of actual interest in shell-model calculations, a more direct method will be employed. Fortunately the R(5) irreducible representations of actual interest in shell-model calculations essentially fall into two simple classes. (i) The first includes those seniorities and reduced isospins for which the quantum numbers n, T and M_T are sufficient to completely label the states of a given irreducible representation. For these a given value of T can occur only once for a given nucleon number, (T-multiplicity $= 1$). In this class are all states with reduced isospin $t = 0$, $t = \frac{1}{2}$ and $t = \Omega - \frac{1}{2}v$ ($\Omega = j + \frac{1}{2}$ for simple configurations and $\Omega = \sum_a (j_a + \frac{1}{2})$ for mixed configurations). Also in this class are the states with $t = 1$ and $n - v = 4k - 2T$; for example the ($n = 4k$, T odd) and ($n = 4k+2$, T even) states with $v = 2$. (ii) The second class includes all those states for which a specific value of T can occur at most twice for a given nucleon number n (T-multiplicity $= 2$). This class includes the remaining states with $t = 1$ (those with $n - v = 4k + 2 - 2T$) and all states with $t = \frac{3}{2}$ and $t = \Omega - \frac{1}{2}v - 1$. For these two simple classes of states, it is possible to give general algebraic expressions for the R(5) Wigner coefficients needed for nuclear matrix elements. The expressions are particularly simple for states of class (i). The calculation of the R(5) Wigner coefficients for class (i) states will be presented in this paper, while the calculations for the more complicated class (ii) states will be presented in a subsequent publication.

In order to establish the notation, a brief review of the five-dimensional quasi-spin formalism is given in sect. 2. The operators of physical interest are classified as to their irreducible tensor character under R(5) in sect. 3. The method of calculating R(5) Wigner coefficients and their properties are discussed in sect. 4. Applications to matrix elements of physical quantities for states of the configuration j^n are given in sect. 5, while tables of R(5) Wigner coefficients for class (i) states are collected in an appendix.

2. Review of the five-dimensional quasi-spin formalism. Definitions and notations.

In order to establish the notation, a brief review of the five-dimensional quasi-spin formalism will be given. [The notation will follow that of ref. [11]).]

The classification scheme of conventional spectroscopy is based on groups generated by infinitesimal operators with conserve nucleon number. The quasi-spin groups on the other hand are generated by operators which include pair creation and annihilation operators, in particular the operators which create or annihilate pairs of nucleons coupled to $J = 0$ and $T = 1$. The five-dimensional quasi-spin group for configurations j^n is generated by the ten infinitesimal operators

$$A^+(M_T) = \tfrac{1}{2}\sum_m \sum_{m_t(m_t')} \langle \tfrac{1}{2}m_t \tfrac{1}{2}m_t' | 1M_T\rangle (-1)^{j-m} a^+_{jmm_t} a^+_{j-mm'_t},$$
$$A(M_T) = \tfrac{1}{2}\sum_m \sum_{m_t(m'_t)} \langle \tfrac{1}{2}m_t \tfrac{1}{2}m_t' | 1M_T\rangle (-1)^{j-m} a_{j-mm'_t} a_{jmm_t},$$
$$T_\pm = \sum_m a^+_{jm\pm\frac{1}{2}} a_{jm\mp\frac{1}{2}}; \qquad T_0 = \tfrac{1}{2}\sum_m (a^+_{jm\frac{1}{2}} a_{jm\frac{1}{2}} - a^+_{jm-\frac{1}{2}} a_{jm-\frac{1}{2}}), \; [\tfrac{1}{2}N_{\text{op}} - (j+\tfrac{1}{2})], \tag{1}$$

where

$$N_{\text{op}} = \sum_{m,\, m_t} a^+_{jmm_t} a_{jmm_t}.$$

They are built from conventional single-nucleon creation (and annihilation) operators $a^+_{jmm_t}$ (and a_{jmm_t}). The generalization to mixed configurations merely requires a sum over all possible j as well as m and a replacement of the weight factor $(j+\frac{1}{2})$ by $\Omega = \sum_a (j_a+\frac{1}{2})$. Except for a normalization factor $(j+\frac{1}{2})^{-\frac{1}{2}}$, the operators $A^+(M_T)$ and $(A(M_T))$ are pair creation (annihilation) operators for nucleon pairs coupled to $J = 0$, $T = 1$ and M_T. They and the components of the isospin operator $\boldsymbol{T}$ and a tenth operator $H_1 = N_{\text{op}} - (j+\frac{1}{2})$ are the generators of a group R(5). The connection with

TABLE 1

The five-dimensional quasi-spin operators and the infinitesimal operators of R_5

Quasi-spin operators [a])	Five-dimensional angular momentum operators [b])	R(5) infinitesimal operators in standard form [c])	R(5) irreducible tensor components [d]) $T^{(11)}_{H_1 T M_T}$
$\frac{1}{2}N_{\text{op}} - (j+\frac{1}{2})$	J_{12}	H_1	$+T^{(11)}_{000}$
T_0	J_{34}	H_2	$+T^{(11)}_{010}$
$A^+(1)$	$\frac{1}{2}[(J_{14}+J_{23})+i(J_{24}+J_{31})]$	E_{11}	$+T^{(11)}_{111}$
$A(1)$	$\frac{1}{2}[(J_{14}+J_{23})-i(J_{24}+J_{31})]$	E_{-1-1}	$-T^{(11)}_{-11-1}$
$A^+(-1)$	$\frac{1}{2}[(J_{14}-J_{23})+i(J_{24}-J_{31})]$	$-E_{1-1}$	$+T^{(11)}_{11-1}$
$A(-1)$	$\frac{1}{2}[(J_{14}-J_{23})-i(J_{24}-J_{31})]$	$-E_{-11}$	$-T^{(11)}_{-111}$
$A^+(0)$	$\frac{1}{\sqrt{2}}(J_{52}+iJ_{15})$	E_{10}	$+T^{(11)}_{110}$
$A(0)$	$\frac{1}{\sqrt{2}}(J_{52}-iJ_{15})$	E_{-10}	$+T^{(11)}_{-110}$
T_+	$(J_{45}+iJ_{53})$	$\sqrt{2}\,E_{01}$	$-\sqrt{2}T^{(11)}_{011}$
T_-	$(J_{45}-iJ_{53})$	$\sqrt{2}\,E_{0-1}$	$+\sqrt{2}T^{(11)}_{01-1}$

[a]) The operators as defined in eqs. (1).
[b]) The five-dimensional angular momentum operators satisfy the same commutation relations as the operators $J_{mn} = -i(x_m\partial/\partial x_n - x_n\partial/\partial x_m)$; $m, n = 1, \ldots, 5$; but no restriction to five-dimensional "orbital" angular momentum is implied. The vector $\boldsymbol{T}$ has been chosen to span the 3, 4, 5 subspace.
[c]) The R(5) infinitesimal operators in standard form for root diagrams of Cartan's symmetry B_2. The operators E_{ab} step up (down) the quantum numbers $(\frac{1}{2}n-j-\frac{1}{2})$ and M_T by a and b units, respectively. The operators E_{ab} are the same as those defined in ref. [10]).
[d]) The phases of the R(5) irreducible tensor components follow from the commutation relations, eqs. (7).

angular momentum operators in an abstract five-dimensional space is illustrated in table 1. The two commuting operators of the rank 2 group are $H_1 = N_{\mathrm{op}}-(j+\frac{1}{2})$ and $H_2 = T_0$ so that the weights are labelled by n and M_T. The eigenvalues H_1 have a simple symmetry property under particle-hole conjugation $H_1 \to -H_1$ as $n \to 4j+2-n$. The remaining infinitesimal operators are organized into standard step-up (down) operators E_{ab} in table 1. The irreducible representations of R(5) are labelled by $(\omega_1 \omega_2)$, the highest weights $(H_{1\ \mathrm{eigen}}, H_{2\ \mathrm{eigen}})$ of the representation. For states with seniority v and reduced isospin t, the largest eigenvalue of H_1 is $\frac{1}{2}n_{\max}-(j+\frac{1}{2})$ with $n_{\max} = 4j+2-v$. The state with $4j+2-v$ nucleons (v holes) has unique isospin t. The largest eigenvalue of $H_2 = T_0$ in this state is thus t. This leads to the identification of the R(5) quantum numbers $(\omega_1 \omega_2)$

$$\omega_1 = j+\tfrac{1}{2}-\tfrac{1}{2}v, \qquad \omega_2 = t, \tag{2}$$

so that the irreducible representations of R(5) are labelled by seniority v and reduced isospin t. A complete labelling scheme for the states of a given irreducible representation of R(5) in general requires four quantum numbers. The physics dictates the choice n, T and M_T for three of these. In general these must be supplemented by a fourth label β. The states for a simple configuration j^n are thus specified by †

$$|\{v, t\}\beta n T M_T; \alpha J M_J\rangle, \tag{3}$$

where the quantum numbers α, J and M_J refer to the decomposition $\mathrm{Sp}(2j+1) \supset \supset \mathrm{R}(3) \supset \mathrm{R}(2)$. The label α is needed in those cases where the v nucleons free of $J = 0$ coupled pairs can be coupled to total J in more than one independent way. The quantum numbers $\beta n T M_T$ refer to the R(5) branch of the group decomposition. States of seniority v can be built from v nucleons free of $J = 0$ coupled pairs, coupled to reduced isospin t, and $p = \frac{1}{2}(n-v)$ pairs of $J = 0$ coupled nucleons. The p-pairs are coupled to resultant isospin T_p, where T_p is restricted to $p, p-2, p-4, \ldots$, since the p-pair creation operators $\boldsymbol{A}^+$ are commuting isospin 1 operators. The total isospin T is thus the result of the vector coupling $\boldsymbol{T} = \boldsymbol{t}+\boldsymbol{T}_p$, and it would appear that the fourth label could be chosen as T_p. Although a labelling scheme based on T_p does give a complete specification of the states, it does not lead to an orthogonal set of basis states since the label T_p cannot be associated with the eigenvalue of a Hermitian operator (commuting with $\boldsymbol{T}^2$, T_0 and N_{op}). In an $|(\omega_1 \omega_2) T_p n T M_T\rangle$ scheme, states with the same n and T but different T_p are thus not orthogonal to each other. One way to overcome this difficulty is through the construction of a fourth operator with eigenvalues that distinguish states of T-multiplicity > 1 for given n. Such an operator must be an isoscalar and conserve nucleon number. On physical grounds, Flowers and

† To avoid confusion the labels $\{v, t\}$ will always be enclosed by curly brackets, while the R(5) quantum numbers $(\omega_1\omega_2) = (j+\frac{1}{2}-\frac{1}{2}v, t)$ will always be enclosed by round parentheses.

Szpikowski [19]) have suggested the operator

$$(A^+ \cdot A^+)(A \cdot A) = (\sum_{M_T} (-1)^{1-M_T} A^+(M_T)A^+(-M_T))(\sum_{M'_T} (-1)^{1-M'_T} A(M'_T)A(-M'_T)). \tag{4}$$

Another possible candidate might be the operator

$$T \cdot \mathscr{T} = \sum_{M_T} (-1)^{1-M_T} T_{M_T} \mathscr{T}_{-M_T},$$

where

$$\mathscr{T}_{M_T} = \sum_{M'_T M''_T} \langle 1M'_T 1M''_T | 1M_T \rangle \frac{(-1)^{1-M_T''}}{\sqrt{2}} [A^+(M'_T)A(-M''_T)+A(-M''_T)A^+(M'_T)]. \tag{5}$$

Neither of the operators of eqs. (4) or (5) is invariant under complex conjugation. Under conjugation $(A^+ \cdot A^+)(A \cdot A) \rightarrow (A \cdot A)(A^+ \cdot A^+)$ while $T \cdot \mathscr{T} \rightarrow -T \cdot \mathscr{T}$. (See appendix 1. In the notation of appendix 1 $(A^+ \cdot A^+)(A \cdot A) = 0_{20}0_{-20}$. The commutator $[(A \cdot A), (A^+ \cdot A^+)]$ is, except for trivial additional factors, equal to $4T \cdot \mathscr{T}$; see table 8 of appendix 1.) If the fourth operator is made invariant under complex conjugation, its eigenfunctions can have simple symmetry properties under conjugation. Since conjugation takes states with H_1 into states with $-H_1(n \rightarrow 4j+2-n)$ this is an important requirement since the physical properties of particle and hole states are simply related. In place of the operators of eqs. (4) and (5), the fourth operator might be chosen as

$$a\{(A^+ \cdot A^+)(A \cdot A)+(A \cdot A)(A^+ \cdot A^+)\}+bH_1(T \cdot \mathscr{T}), \tag{6}$$

which has the necessary symmetry property under conjugation provided a and b are arbitrary constants (including the possibilities $b = 0$ or $a = 0$) or functions even in H_1 and T_0. No attempt has been made to find the best possible values for a and b for the general irreducible representation of R(5) since the algebraic structure of the eigenvalues of (6) is very complicated in the general case. In practice the problem of the fourth operator can essentially be avoided since the R(5) irreducible representations of actual interest for shell-model calculations are relatively simple.

The n, T structure of the general irreducible representation $(\omega_1 \omega_2) = (j+\frac{1}{2}-\frac{1}{2}v, t)$ has been studied by several methods [11,12,20]). In the irreducible representation $(\omega_1 t)$, the allowed values of H_1 and T are given by the possible angular momentum couplings $\boldsymbol{T} = \boldsymbol{T}_p + \boldsymbol{t}$ where:

(i) T_p has the possible values $T_p = p', p'-2, p'-4, \ldots$ for $H_1 = \pm|\omega_1 - p'|$, $p' = 0, 1, 2, \ldots (\leqq \omega_1)$. The allowed states are subject to the additional restrictions.

(ii) $T \leqq \omega_1$; and, if the possible couplings of $\boldsymbol{T}_p + \boldsymbol{t}$ lead to a state of specific T more than once.

(iii) A state with $T = \omega_1 - m$ $(m = 0, 1, 2, \ldots)$ occurs at most q-times where

$q = \min(m+1, \omega_1 - t + 1)$. (See, for example, ref. [11], table 2.) The possible H_1 and T values for the simple representations $(\frac{1}{2}\,\frac{1}{2})$, (10), (11), (20), (22) are given in table 2).

TABLE 2

The H_1, T-structure of some simple representations

$(\frac{1}{2}\frac{1}{2})$

H_1	T
$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	$\frac{1}{2}$

(10)

H_1	T
1	0
0	1
-1	0

(11)

H_1	T
1	1
0	0 1
-1	1

(20)

H_1	T
2	0
1	1
0	0 2
-1	1
-2	0

(22)

H_1	T
2	2
1	1 2
0	0 1 2
-1	1 2
-2	2

The n, T structure is very simple for the irreducible representations $(\omega_1 0)$, $(\omega_1, \frac{1}{2})$ and (tt). In these representations the states are completely specified by the labels n, T and M_T since a specific value of T for a given nucleon number n can occur at most once (T-multiplicity $= 1$). In addition, states of the irreducible representation $(\omega_1 1)$ with $\omega_1 - H_1 - T = 2k$ or $(n-v) = 4k - 2T$ have T-multiplicity $= 1$. Similarly the states of the irreducible representations $(\omega_1 \frac{3}{2})$ and $(t+1, t)$, as well as the remaining states of the irreducible representations $(\omega_1 1)$ (those with $n - v = 4k+2-2T$) belong to a class with T-multiplicity $= 2$ at most. In all these cases, the two independent states with the same values of H_1 and T can be distinguished by their symmetry property under conjugation. The two states $|\beta H_1 T M_T\rangle$ labelled by different values of β can be built such that they have the symmetry property plus or minus, respectively, under the conjugation operation which transforms the state into a corresponding state $|\beta^c, -H_1, T, -M_T\rangle$, so that the quantum number β can be replaced by a symmetry label in these simple cases. Although this symmetry property does not lead to a *unique* labelling of the double states of these representations,† it can be supplemented by a requirement of simplicity to lead to an explicit and tractable construction of these states ††. It is thus possible to calculate general formulae for the needed Wigner coefficients involving the irreducible representations $(\omega_1 0)$, $(\omega_1 \frac{1}{2})$, $(\omega_1 1)$, $(\omega_1 \frac{3}{2})$, (tt), $(t+1, t)$. This includes all possible $\{v, t\}$ values for simple configurations with $j \leq \frac{9}{2}$ or for mixed

† In ref. [11]), the claim was made that use of the symmetry label leads to a unique specification of the states with T-multiplicity $= 2$. With the exception of the $H_1 = 0$ states of the representation $(\omega_1 1)$, this is not true. In particular, the choice made in ref. [11]) for the plus and minus states of the representation $(\omega_1 1)$ is an unnecessarily cumbersome one and will be replaced by a simpler choice which makes it possible to give tractable algebraic expressions for the R(5) Wigner coefficients involving the representation $(\omega_1 1)$.

†† Details for the states with T-multiplicity $= 2$ will be presented in a subsequent paper [27]).

configurations with $\Omega = \sum(j+\frac{1}{2}) \leqq 5$. For a mixed configuration with $\Omega = 6$; $j = \frac{1}{2}, \frac{3}{2}$ and $\frac{5}{2}$, for example, only the single R(5) representation $(\omega_1\omega_2) = (42)$, corresponding to an overall seniority and reduced isospin $\{v, t\} = \{4, 2\}$ falls outside the two simple classes included in the above list. Such exceptional cases can easily be treated numerically. With these rare exceptions therefore it is possible to give general algebraic expressions for the R(5) Wigner coefficients needed for a detailed application of the five-dimensional quasi-spin formalism to shell-model calculations.

3. The R(5) irreducible tensor character of simple operators

All physical operators can be classified as to their irreducible tensor character with respect to three-dimensional physical space, isospin space and in addition also as to their irreducible tensor character with respect to five-dimensional, quasi-spin space. An R(5) irreducible tensor operator can be denoted by $T^{(\omega_1\omega_2)}_{\beta H_1 TM_T}$. Since the operators of actual interest in shell-model calculations belong to the simple irreducible representations of table 2 for which the irreducible tensor components are completely specified without the label β, this label will in general be omitted. The R(5) irreducible tensor operators $T^{(\omega_1\omega_2)}_{\beta H_1 TM_T}$ can be defined through their commutation relations with the infinitesimal operators of the group

$$\begin{aligned}
[H_1, T^{(\omega_1\omega_2)}_{\beta H_1 TM_T}] &= (\tfrac{1}{2}n-j-\tfrac{1}{2})T^{(\omega_1\omega_2)}_{\beta H_1 TM_T},\\
[T_0, T^{(\omega_1\omega_2)}_{\beta H_1 TM_T}] &= M_T T^{(\omega_1\omega_2)}_{\beta H_1 TM_T},\\
[E_{ab}, T^{(\omega_1\omega_2)}_{\beta H_1 TM_T}] &= \sum_{T'\beta'} \langle(\omega_1\omega_2)\beta'(H_1+a)T'(M_T+b)|E_{ab}|(\omega_1\omega_2)\beta H_1 TM_T\rangle\\
&\qquad \times T^{(\omega_1\omega_2)}_{\beta'(H_1+a)T'(M_T+b)}. \quad (7)
\end{aligned}$$

The matrix elements of a component of such an irreducible tensor operator are given through a generalized Wigner-Eckart theorem

$$\begin{aligned}
&\langle(\omega_1\omega_2)\beta H_1 TM_T|T^{(\omega''_1\omega''_2)}_{\beta''H''_1T''M''_T}|(\omega'_1\omega'_2)\beta'H'_1T'M'_T\rangle\\
&\quad = \sum_\rho \langle(\omega'_1\omega'_2)\beta'H'_1T'M'_T;(\omega''_1\omega''_2)\beta''H''_1T''M''_T|(\omega_1\omega_2)\beta H_1TM_T\rangle_\rho\\
&\qquad\qquad \times \langle(\omega_1\omega_2)||T^{(\omega''_1\omega''_2)}||(\omega'_1\omega'_2)\rangle_\rho, \quad (8)
\end{aligned}$$

where the reduced or double-barred matrix elements are independent of quantum numbers of type β, H_1, T and M_T. The dependence on these quantum numbers is carried by the first factor, an R(5) Wigner coefficient. The R(5) Wigner coefficients are the elements of the matrix which reduces the Kronecker product of two irreducible representations of R(5). They are defined by

$$\begin{aligned}
&|[(\omega'_1\omega'_2)(\omega''_1\omega''_2)](\omega_1\omega_2)\rho;\beta H_1TM_T\rangle\\
&\quad = \sum_{\substack{\beta'H'_1T'M'_T\\ \beta''(H''_1)T''(M''_T)}} |(\omega'_1\omega'_2)\beta'H'_1T'M'_T\rangle|(\omega''_1\omega''_2)\beta''H''_1T''M''_T\rangle\\
&\quad \times \langle(\omega'_1\omega'_2)\beta'H'_1T'M'_T;(\omega''_1\omega''_2)\beta''H''_1T''M''_T|(\omega_1\omega_2)\beta H_1TM_T\rangle_\rho. \quad (9)
\end{aligned}$$

Since the isospin group is a subgroup of R(5), the R(5) Wigner coefficients can be factored into an ordinary isospin angular momentum Wigner coefficient and a reduced R(5)/R(3) coefficient or isoscalar factor, to be denoted by a double bar

$$\begin{aligned}&\langle(\omega_1'\,\omega_2')\beta' H_1'\,T'M_T';\,(\omega_1''\omega_2'')\beta'' H_1''T''M_T''|(\omega_1\,\omega_2)\beta H_1\,TM_T\rangle_\rho\\&\quad = \langle T'M_T'T''M_T''|TM_T\rangle\langle(\omega_1'\,\omega_2')\beta' H_1'\,T';\,(\omega_1''\omega_2'')\beta'' H_1''T''||(\omega_1\,\omega_2)\beta H_1\,T\rangle_\rho,\end{aligned} \tag{10}$$

where the double-barred coefficient is completely independent of the quantum numbers M_T. The index ρ in eqs. (8)-(10) and the sum over ρ in eq. (8) are not needed for simply reducible products, such as $(\omega_1\,\omega_2)\times(\frac{1}{2}\frac{1}{2}) = (\omega_1+\frac{1}{2},\,\omega_2+\frac{1}{2})+(\omega_1+\frac{1}{2},\,\omega_2-\frac{1}{2})+(\omega_1-\frac{1}{2},\,\omega_2+\frac{1}{2})+(\omega_1-\frac{1}{2},\,\omega_2-\frac{1}{2})$, for example. They are needed only in those cases where a representation $(\omega_1\,\omega_2)$ can occur more than once in the Kronecker product $(\omega_1'\,\omega_2')\times(\omega_1''\,\omega_2'')$. In the cases where this multiplicity problem arises, the Wigner-Eckart theorem serves partly to define the new quantum numbers ρ, since the decomposition into reduced matrix elements and R(5) Wigner coefficients is determined by the choice of the quantum numbers ρ. The product $(\omega_1\,\omega_2)\times(11)$, for example, contains the representation $(\omega_1\,\omega_2)$ itself with a multiplicity of 2 (with the exception of the special cases $(\omega_1\,\omega_2) = (\omega_1\,0)$ or (tt) for which the multiplicity is 1). Since the infinitesimal generators of the group transform according to the ten-dimensional representation (11), the matrix elements of the infinitesimal generators can be used to define the label ρ in this special case. The matrix elements of the infinitesimal generators are chosen to be proportional to the R(5) Wigner coefficients with $\rho = 1$, that is

$$\begin{aligned}&\langle(\omega_1\,\omega_2)\beta'(H_1+a)T'(M_T+b)|E_{ab}|(\omega_1\,\omega_2)\beta H_1\,TM_T\rangle\\&\quad = (-1)^s\langle(\omega_1\,\omega_2)\beta H_1\,TM_T;\,(11)a1b|(\omega_1\,\omega_2)\beta'(H_1+a)T'(M_T+b)\rangle_{\rho=1}\\&\qquad\qquad\times\langle(\omega_1\,\omega_2)||E||(\omega_1\,\omega_2)\rangle_{\rho=1},\end{aligned} \tag{11}$$

with

$$\begin{aligned}&\langle(\omega_1\,\omega_2)||E||(\omega_1\,\omega_2)\rangle_{\rho=2} = 0 \qquad \text{(by definition)},\\&\langle(\omega_1\,\omega_2)||E||(\omega_1\,\omega_2)\rangle_{\rho=1} = [(\omega_1(\omega_1+3)+\omega_2(\omega_2+1)]^{\frac{1}{2}}.\end{aligned} \tag{12}$$

The phase factor $(-1)^s$ has the values

$$\begin{aligned}(-1)^s &= -1 \qquad \text{for } b=-1, \quad a=\pm 1\\ &\qquad\qquad\ \text{for } b=+1, \quad a=0,\\ (-1)^s &= +1 \qquad \text{for all other cases.}\end{aligned} \tag{13}$$

The phase relations between the E_{ab} and standard R(5) irreducible tensor components, as defined through eqs. (7), follow from the commutation relations of the E_{ab}. (See table 1; $E_{ab} = (-1)^s T^{(11)}_{a1b}$.)

All operators of physical interest in a shell-model calculation involving the simple configuration j^n can be built from the single-nucleon creation and annihilation operators $a^+_{jmm_t}$, a_{jmm_t}. (The generalization to mixed configurations is straightforward.) The R(5) irredubile tensor character of all operators is thus based on the tensor character of the single-nucleon creation and annihilation operators. For fixed values of j and m, the four operators $a^+_{jm\pm\frac{1}{2}}$ and $a_{j,-m,\pm\frac{1}{2}}$ form a basis for the four-dimensional irreducible representation $(\frac{1}{2}\frac{1}{2})$. The phase relations between these operators and standard R(5) irreducible tensor components follow from eqs. (7) and are given by

$$\begin{aligned} a^+_{jmm_t} &= T^{(\frac{1}{2}\frac{1}{2});j}_{\frac{1}{2}\frac{1}{2}m_t;m}, \\ (-1)^{j-m+\frac{1}{2}-m_t} a_{j,-m,-m_t} &= T^{(\frac{1}{2}\frac{1}{2});j}_{-\frac{1}{2}\frac{1}{2}m_t;m}, \end{aligned} \tag{14}$$

where the tensors $T^{(\omega_1\omega_2);j}_{H_1TM_T;m}$ are classified both as to their irreducible character under R(5), by the labels $(\omega_1\omega_2)H_1TM_T$ and as to their spherical tensor character by the angular momentum quantum numbers j and m. More complicated operators can be built from these by successive application of a composition law. Operators built from two single-nucleon creation or annihilation operators can be classified under R(5) by the build-up process

$$\begin{aligned} T^{(\omega_1\omega_2);J}_{H_1TM_T;M} &= \sum_{\substack{m_1m_{t_1}h_1\\(m_2m_{t_2}h_2)}} \langle jm_1\, jm_2|JM\rangle\langle\tfrac{1}{2}m_{t_1}\tfrac{1}{2}m_{t_2}|TM_T\rangle\langle(\tfrac{1}{2}\tfrac{1}{2})h_1\tfrac{1}{2};(\tfrac{1}{2}\tfrac{1}{2})h_2\tfrac{1}{2}||(\omega_1\,\omega_2)H_1\,T\rangle \\ &\quad\times T^{(\frac{1}{2}\frac{1}{2});j}_{h_1\frac{1}{2}m_{t_1};m_1} T^{(\frac{1}{2}\frac{1}{2});j}_{h_2\frac{1}{2}m_{t_2};m_2}, \end{aligned} \tag{15}$$

where the needed R(5) Wigner coefficients are tabulated in table 3a. The Kronecker product $(\frac{1}{2}\frac{1}{2})\times(\frac{1}{2}\frac{1}{2}) = (00)+(10)+(11)$ contains the ten-dimensional (regular) representation (11), the five-dimensional (vector) representation (10) and the one-dimensional (scalar) representation (00). Operators built from products of two single-nucleon operators thus transform according to these representations. These operators include the pair creation operators

$$\mathscr{A}^+(j^2;JM;TM_T) = \sum_{m_1m_{t_1}} \langle jm_1\, jm_2|JM\rangle\langle\tfrac{1}{2}m_{t_1}\tfrac{1}{2}m_{t_2}|TM_T\rangle a^+_{jm_1m_{t_1}} a^+_{jm_2m_{t_2}} \tag{16}$$

and the pair annihilation operators

$$\mathscr{A}(j^2;JM;TM_T) = [\mathscr{A}^+(j^2;JM;TM_T)]^+. \tag{17}$$

In addition these include all one-body operators, which can be expressed in terms of the elementary multipole operators

$$\begin{aligned} U(j^2;JM;TM_T) &= \sum_{m_1m_{t_1}} \langle jm_1\, jm_2|JM\rangle \\ &\quad\times\langle\tfrac{1}{2}m_{t_1}\tfrac{1}{2}m_{t_2}|TM_T\rangle a^+_{jm_1m_{t_1}}(-1)^{j-m_2+\frac{1}{2}-m_{t_2}} a_{j,-m_2,-m_{t_2}}. \end{aligned} \tag{18}$$

TABLE 3a

R(5) Wigner coefficients $\langle(\frac{1}{2}\frac{1}{2})H'_1T';\ (\frac{1}{2}\frac{1}{2})H''_1T''\|(\omega_1\omega_2)H_1T\rangle$

$H'_1T';\ H''_1T''$ \ $(\omega_1\omega_2)$ H_1T	(11) 11	(11) −11	(11) 01	(10) 01	(10) 10	(10) −10	(11) 00	(00) 00
$\frac{1}{2}\frac{1}{2};\ \frac{1}{2}\frac{1}{2}$	1				1			
$-\frac{1}{2}\frac{1}{2};\ -\frac{1}{2}\frac{1}{2}$		1				-1		
$\frac{1}{2}\frac{1}{2};\ -\frac{1}{2}\frac{1}{2}$			$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$			$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$-\frac{1}{2}\frac{1}{2};\ \frac{1}{2}\frac{1}{2}$			$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$			$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

TABLE 3b

R(5) Wigner coefficients $\langle(11)H'_1T';\ (11)H''_1T''\|(\omega_1\omega_2)H_1T\rangle$

$H'_1T';\ H''_1T''$ \ $(\omega_1\omega_2)$ H_1T	(00) 00	(11) 00	(20) 00	(22) 00	(10) 01	(11) 01	(22) 01	(20) 02	(22) 02
11; −11	$\sqrt{\frac{3}{10}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{6}}$
−11; 11	$\sqrt{\frac{3}{10}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{6}}$
00; 00	$-\frac{1}{\sqrt{10}}$	0	$-\sqrt{\frac{2}{5}}$	$\frac{1}{\sqrt{2}}$					
01; 01	$\sqrt{\frac{3}{10}}$	0	$-\frac{4}{\sqrt{30}}$	$-\frac{1}{\sqrt{6}}$	0	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	$\sqrt{\frac{2}{3}}$
00; 01					$\frac{1}{\sqrt{6}}$	0	$\frac{1}{\sqrt{3}}$		
01; 00					$\frac{1}{\sqrt{6}}$	0	$\frac{1}{\sqrt{3}}$		

TABLE 3c

R(5) Wigner coefficients $\langle(10)H'_1T';\ (10)H''_1T''\|(\omega_1\omega_2)H_1T\rangle$

$H'_1T';\ H''_1T''$ \ $(\omega_1\omega_2)$ H_1T	(00) 00	(11) 00	(20) 00
10; −10	$\frac{1}{\sqrt{5}}$	$-\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{10}}$
−10; 10	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{10}}$
01; 01	$\sqrt{\frac{3}{5}}$	0	$-\sqrt{\frac{2}{5}}$

Except for a trivial multiplicative factor, the elementary multipole operators are the unit tensor operators $\mathscr{U}^{JT}_{MM_T}$ introduced by Racah [21])

$$\mathscr{U}^{JT}_{MM_T} = \frac{U(j^2;JM;TM_T)}{[(2J+1)(2T+1)]^{\frac{1}{2}}} = \sum_{\substack{m_1 m t_1 \\ (m_2 m_{t_2})}} \frac{\langle jm_2JM|jm_1\rangle\langle\frac{1}{2}m_{t_2}TM_T|\frac{1}{2}m_{t_1}\rangle}{[(2j+1)2]^{\frac{1}{2}}}\, a^{+}_{jm_1m_{t_1}} a_{jm_2m_{t_2}}. \tag{19}$$

The R(5) tensor character of the pair creation, annihilation and elementary multipole operators is given in table 4. Elementary multipole operators of odd rank (J odd) are R(5) scalars (00) and R(5) vectors (10) for isospin $T = 0$ and 1, respectively; while the elementary multipole operators of even rank (J even) transform according to the ten-dimensional representation (11) for both $T = 0$ and 1.

TABLE 4

R(5) irreducible tensor character of the pair-creation, annihilation operators and the elementary one-body multipole operators

Operator [a]	Tensor character [b] $T^{(\omega_1\omega_2);J}_{H_1TM_T;M}$ $(\omega_1\,\omega_2)$	H_1	T	M_T	J	M
$\mathscr{A}^+(j^2; JM; T=1\ M_T)$	(11)	1	1	M_T	even	M
$\mathscr{A}^+(j^2; JM; 00)$	(10)	1	0	0	odd	M
$-(-1)^{J-M+1-M_T}\mathscr{A}(j^2; JM; T=1\ M_T)$	(11)	-1	1	$-M_T$	even	$-M$
$-(-)^{J-M}\mathscr{A}(j^2; JM; 00)$	(10)	-1	0	0	odd	$-M$
$\sqrt{2}U(j^2; JM; T=1\ M_T)$	(11)	0	1	M_T	even	M
$\sqrt{2}U(j^2; JM; 00)$	(11)	0	0	0	even	M
$-\sqrt{2}U(j^2; JM; T=1\ M_T)$	(10)	0	1	M_T	odd	M
$\sqrt{2}U(j^2; JM; 00)$	(00)	0	0	0	odd	M

[a]) The operators are defined by eqs. (16)-(18).
[b]) The R(5) tensors are constructed through the composition law, eq. (15).

Of the more complicated operators, the two-body interaction

$$V = \sum_{i<j} V_{ij} \tag{20}$$

is the most important in shell-model calculations. For a simple configuration based on a single j, it can be written in the form

$$V = \tfrac{1}{2}\sum_{JT}\sum_{MM_T} V_{JT}\mathscr{A}^+(j^2; JM; TM_T)\mathscr{A}(j^2; JM; TM_T), \tag{21}$$

where V_{JT} is the two-particle matrix element

$$V_{JT} = \langle j^2JMTM_T|V_{12}|j^2JMTM_T\rangle. \tag{22}$$

Pairs coupled to $T = 1$, (even J), and $T = 0$ (odd J) transform according to the representations (11) and (10), respectively. The full two-body interaction thus contains R(5) irreducible tensor operators which arise from the products

$$(11)\times(11) = [(22)+(20)+(10)+(00)]+\{(21)+(11)\}, \tag{23}$$

$$(10)\times(10) = [(20)+(00)]+\{(11)\}, \tag{24}$$

which contain two 35-dimensional representations, (22) and (21), the 14-dimensional representation (20), the ten-dimensional representation (11), the five-dimensional representation (10) and the one-dimensional representation (00). The first terms in these Kronecker products (enclosed by square brackets) correspond to a symmetric coupling of the two identical representations, while the last terms (enclosed by curly brackets) correspond to the antisymmetric coupling of the two identical representations. It is advantageous to split the two-body interaction into symmetrically and antisymmetrically coupled pairs

$$\begin{aligned} V &= V^{(\mathrm{s})}+V^{(\mathrm{a})} \\ &= \tfrac{1}{4}\sum_{JT}\sum_{MM_T} V_{JT}[\mathscr{A}^+\mathscr{A}+\mathscr{A}\mathscr{A}^+]+\tfrac{1}{4}\sum_{JT}\sum_{MM_T} V_{JT}[\mathscr{A}^+\mathscr{A}-\mathscr{A}\mathscr{A}^+]. \end{aligned} \tag{25}$$

The antisymmetrically coupled part reduces via the anticommutation properties of a^+ and a to an operator of one-body form with R(5) irreducible tensor character (11). For an isoscalar (charge-independent), two-body interaction it reduces to the trivial operator

$$V^{(\mathrm{a})} = \sum_{JT} V_{JT}\frac{(2J+1)(2T+1)}{(2j+1)}H_1 = \sum_{JT} V_{JT}\frac{(2J+1)(2T+1)}{(2j+1)}(\tfrac{1}{2}n-j-\tfrac{1}{2}). \tag{26}$$

The symmetrically coupled part contains only the representations (22), (20), (10) and (00). Since only the representations (22), (20) and (00) contain a $T = 0$ state among the nucleon-number conserving-components (see the $H_1 = 0$ rows of the representations in table 2), an isoscalar two-body interaction contains only R(5) irreducible tensors of type (22), (20) and (00). The interaction can be written in terms of the basic two-body tensors

$$\begin{aligned} [T(J^2(\omega_1\omega_2)^2)]^{(\omega'_1\omega'_2);0}_{000;0} &= \sum_{MM_T}\sum_{H_1T}\langle JMJ-M|00\rangle\langle TM_T T-M_T|00\rangle \\ &\quad\times\langle(\omega_1\omega_2)H_1T;(\omega_1\omega_2)-H_1T\|(\omega'_1\omega'_2)00\rangle T^{(\omega_1\omega_2);J}_{H_1TM_T;M}T^{(\omega_1\omega_2);J}_{-H_1T-M_T;-M}, \end{aligned} \tag{27}$$

with $(\omega_1\omega_2) = (10)$ and (11) for $J =$ odd and $J =$ even, respectively. The R(5) Wigner coefficients needed for eqs. (27) are given in tables 3b and c. In terms of these two-body tensors, the symmetrically coupled part of the general isoscalar, two-body interaction can then be written

$$\begin{aligned} V^{(\mathrm{s})} = &-\tfrac{1}{4}\sum_{J\,\mathrm{even}} V_{JT=1}(2J+1)^{\frac{1}{2}} \\ &\times\{\sqrt{2}[T(J^2(11)^2)]^{(22);0}_{000;0}+\sqrt{\tfrac{2}{5}}[T(J^2(11)^2)]^{(20);0}_{000;0}+3\sqrt{\tfrac{2}{5}}[T(J^2(11)^2)]^{(00);0}_{000;0}\} \\ &-\tfrac{1}{4}\sum_{J\,\mathrm{odd}} V_{JT=0}(2J+1)^{\frac{1}{2}}\{\sqrt{\tfrac{6}{5}}[T(J^2(10)^2)]^{(20);0}_{000;}+2\sqrt{\tfrac{1}{5}}[T(J^2(10)^2)]^{(00);0}_{000;0}\}. \end{aligned} \tag{28}$$

Besides the charge-independent, two-body, nuclear interactions of the above form, the Coulomb interaction is of particular importance in shell-model calculations. Decomposing it into isoscalar, isovector and isotensor parts it can be written

$$V_{\text{Coul}} = \sum_{i<j} \frac{e^2}{r_{ij}} \{\tfrac{1}{3}(\tfrac{3}{4}+\boldsymbol{t}_i \cdot \boldsymbol{t}_j) - \tfrac{1}{2}(t_{z_i}+t_{z_j}) + \tfrac{1}{3}(3t_{z_i}t_{z_j} - \boldsymbol{t}_i \cdot \boldsymbol{t}_j)\}. \tag{29}$$

Since the Coulomb interaction acts only on pairs coupled to $T = 1$, it can be decomposed into R(5) irreducible tensors of type $[T(J^2(11)^2)]^{(\omega_1\omega_2);\,0}_{0T0;\,0}$ with $T = 0$, 1 and 2 for the isoscalar, vector and tensor parts, respectively. The nucleon number conserving tensor components include the value $T = 1$ only for the representations (10) and (22) and the value $T = 2$ only for the representations (22) and (20) (see the $H_1 = 0$ rows of table 2). An isovector interaction can thus be built only from the representations (10) and (22), while an isotensor interaction can contain only the representations (22) and (20). The Coulomb interaction can be split into non-trivial and trivial terms as before

$$V_{\text{Coul}} = V^{(\text{s})}_{\text{Coul}} + V^{(\text{a})}_{\text{Coul}}.$$

The trivial antisymmetrically coupled term now reduces to

$$V^{(\text{a})}_{\text{Coul}} = \sum_{J\text{ even}} V^{\text{Coul}}_{J1} \frac{(2J+1)3}{(2j+1)} [H_1 - T_0] = \sum_{J\text{ even}} V^{\text{Coul}}_{J1} \frac{(2J+1)3}{(2j+1)} [\tfrac{1}{2}n - j - \tfrac{1}{2} - M_T]. \tag{30}$$

The non-trivial symmetrically coupled term can be written in terms of the tensor operators $[T(J^2, (11)^2)]^{(\omega_1\omega_2);\,0}_{0T0;\,0}$ to be abbreviated by $T^{(\omega_1\omega_2)}_{0T0}$ as

$$\begin{aligned} V^{(\text{s})}_{\text{Coul}} = -\tfrac{1}{4} \sum_{J\text{ even}} V^{\text{Coul}}_{J1}(2J+1)^{\frac{1}{2}} \{ & [\sqrt{2}T^{(22)}_{000} + \sqrt{\tfrac{2}{5}}T^{(20)}_{000} + 3\sqrt{\tfrac{2}{5}}T^{(00)}_{000}] \\ & + [\sqrt{6}T^{(10)}_{010} - \sqrt{3}T^{(22)}_{010}] + [T^{(22)}_{020} + \sqrt{2}T^{(20)}_{020}]\}, \end{aligned} \tag{31}$$

where the two-particle matrix element V^{Coul}_{J1} is defined by

$$V^{\text{Coul}}_{J1} = \langle j^2 J1 | \frac{e^2}{3r_{12}} | j^2 J1 \rangle. \tag{32}$$

Insofar as the R(5) irreducible tensor character is concerned isovector and isotensor components of a more general, charge-dependent interaction will have the same form as the $T = 1$ and $T = 2$ components of the Coulomb interaction.

4. The R(5) Wigner coefficients

Since the R(5) irreducible tensor character of the operators of interest in shell-model calculations include the representations (00), ($\frac{1}{2}\frac{1}{2}$), (10), (11), (20) and (22), application of the Wigner-Eckart theorem requires knowledge of the R(5) Wigner coefficients for Kronecker products in which one of the representations is a member of this set. Although this includes the 35-dimensional representation (22) and the 14-dimensional representation (20), only Wigner coefficients diagonal in both H_1

(nucleon number) and T are needed for these two representations, so that it is feasible to calculate the needed Wigner coefficients.

The R(5) Wigner coefficients are defined through eqs. (9) and (10). Since the isospin angular momentum Wigner coefficients are well known, only the reduced R(5)/R(3) Wigner coefficients which are denoted by a double bar, need be calculated. Their general properties will be discussed first. (From now on the term R(5) Wigner coefficient will refer to these reduced (double-barred) coefficients unless otherwise stated.)

4.1. ORTHOGONALITY

Since both the full R(5) Wigner coefficients and the isospin R(3) Wigner coefficients form orthogonal matrices, the reduced Wigner coefficients are orthogonal also. In particular, for fixed values of H_1 and T

$$\sum_{\substack{\beta' H'_1 T' \\ \beta'' H''_1 T''}} \langle(\omega'_1\,\omega'_2)\beta' H'_1\,T'\,;(\omega''_1\omega''_2)\beta'' H''_1 T''\,||(\omega_1\,\omega_2)\beta H_1\,T\rangle_\rho$$
$$\times\langle(\omega'_1\,\omega'_2)\beta' H'_1\,T';(\omega''_1\omega''_2)\beta'' H''_1 T''||(\bar{\omega}_1\,\bar{\omega}_2)\bar{\beta} H_1\,T\rangle_{\bar{\rho}} = \delta_{\omega_1\bar{\omega}_1}\delta_{\omega_2\bar{\omega}_2}\delta_{\beta\bar{\beta}}\delta_{\rho\bar{\rho}}, \quad (33)$$

and, with T fixed,

$$\sum_{(\omega_1\omega_2)}\sum_{\beta\rho}\langle(\omega'_1\,\omega'_2)\beta' H'_1\,T'\,;(\omega''_1\omega''_2)\beta'' H''_1 T''||(\omega_1\,\omega_2)\beta H_1\,T\rangle_\rho$$
$$\times\langle(\omega'_1\,\omega'_2)\bar{\beta}'\bar{H}'_1\,\bar{T}';(\omega''_1\omega''_2)\bar{\beta}''\bar{H}''_1\bar{T}''||(\omega_1\,\omega_2)\beta H_1\,T\rangle_\rho$$
$$= \delta_{\beta'\bar{\beta}'}\delta_{\beta''\bar{\beta}''}\delta_{H'_1\bar{H}'_1}\delta_{H''_1\bar{H}''_1}\delta_{T'\bar{T}'}\delta_{T''\bar{T}''}. \quad (34)$$

For fixed values of $(\omega'_1\,\omega'_2)$, $(\omega''_1\,\omega''_2)$, H_1 and T, therefore, the R(5) Wigner coefficients form orthogonal matrices. The rows of these matrices are labelled by the values of $\beta' H'_1 T'$; $\beta'' H''_1 T''$ consistent with H_1 and T, whereas the columns are labelled by the possible values of $(\omega_1\omega_2)$, ρ and β. The states of the uncoupled representation $|(\omega'_1\,\omega'_2)\beta' H'_1 T' M'_T\rangle|(\omega''_1\,\omega''_2)\beta'' H''_1 T'' M''_T\rangle$ are completely specified by the eigenvalues of 12 commuting operators, the quadratic and quartic Casimir invariants which specify the irreducible representation labels and the operators H_1, $\boldsymbol{T}^2$ and T_0, plus the "fourth operator" for both the single primed and double primed representations. The states of the coupled representation $|[(\omega'_1\,\omega'_2)(\omega''_1\,\omega''_2)](\omega_1\,\omega_2)\rho;\beta H_1\,TM_T]$ should therefore in general require a set of 12 commuting operators for their complete specification. In the general case, the labels ρ are thus completely specified only by the eigenvalues of two operators. These operators must lie outside † the group R(5). Since only Wigner coefficients for very simple Kronecker products are needed in the applications to nuclear problems, no attempt has been made to find a general solution to this multiplicity problem. The products $(\omega_1\,\omega_2)\times(\frac{1}{2}\frac{1}{2})$ and $(\omega_1\,\omega_2)\times(10)$ are simply reducible, and the label ρ is not needed at all in these cases. For the product $(\omega_1\,\omega_2)\times(11)$ only the product representation $(\omega_1\,\omega_2)$ itself has a multiplicity of 2,

† For a general discussion of this type of problem see refs. [22, 23]).

and in this case the matrix elements of the infinitesimal operators of the group serve to distinguish the two states with different ρ (see eqs. (11) and (12)]. In the few other cases where the label ρ is needed, the several independent coupled states are chosen somewhat arbitrarily (see subsect. 4.4) and labelled $\rho = 1, 2, 3, \ldots$.

4.2. PHASE CONVENTION

The overall phase of the R(5) Wigner coefficients is fixed by a generalized Condon and Shortley phase convention. The coefficients can be chosen to be real, and the "leading" coefficient connecting the state of highest weight $H_1' = \omega_1', T' = \omega_2' = t'$ to the state of highest weight $H_1 = \omega_1$, $T = \omega_2 = t$ is chosen to be positive. Specifically

$$\langle(\omega_1' \omega_2')H_1' = \omega_1', T' = \omega_2'; (\omega_1'' \omega_2'')\tilde{\beta}'' \tilde{H}_1'' \tilde{T}'' \| (\omega_1 \omega_2) H_1 = \omega_1 T = \omega_2\rangle > 0. \quad (35)$$

If more than one value of T'' is possible, the leading coefficient ($T'' = \tilde{T}''$), is defined as that with the largest possible value of T'' consistent with $T' = \omega_2' = t'$ and $T = \omega_2 = t$. For the simple representations $(\frac{1}{2}\frac{1}{2})$, (10), (11), (20) and (22) no further specification of the label β'' is required.

4.3. SYMMETRY PROPERTIES

The group R(5) is self-adjoint. If the set of matrices $\mathscr{D}$ for the elements of R(5) form an irreducible representation of R(5), the complex conjugates of these matrices $\mathscr{D}^*$ form an equivalent irreducible representation. The basis vectors of an irreducible representation and their conjugates are thus simply related. The conjugation operator K has the following simple properties (see table 1)

$$KJ_{ij}K^{-1} = -J_{ij}, \qquad i, j = 1, \ldots 5,$$
$$KE_{ab}K^{-1} = -E_{-a,-b}, \qquad KH_1K^{-1} = -H_1, \qquad KT_0K^{-1} = -T_0. \quad (36)$$

For states with T-multiplicity $= 1$ for which the quantum number β is not needed it follows from the last two eqs. of (36) that

$$K|(\omega_1 t)H_1 TM_T\rangle = (-1)^{\eta(\omega_1, t)+v+T-M_T}|(\omega_1 t) - H_1, T, -M_T\rangle, \quad (37)$$

where the phase factor $(-1)^{\eta(\omega_1, t)+v+T-M_T}$ has been chosen such that the (T, M_T)-dependent factor carries the usual angular momentum phase conventions associated with the isospin group. The (ω_1, t)-dependent factor η could in principle be chosen arbitrarily but must in practice be chosen to be consistent with the phase convention of eq. (35). The factor v carries the intrinsic R(5) dependence of the phase. To establish the phase factor the basis states and their conjugates are constructed explicitly in appendix 1 for the irreducible representations $(\omega_1 0)$, $(\omega_1 \frac{1}{2})$ and (tt) and the states with $n-v = 4k-2T$ of $(\omega_1 1)$. The latter † are denoted by the quantum number $\beta = 0$.

† These are identical with the $\kappa = 0$ states of ref. [11]). The label κ of ref. [11]) has been replaced by β since the states $\kappa = 1$ and 2 of the irreducible representation $(\omega_1 1)$ can be replaced by simpler states to be denoted by $\beta = 1$ and $\beta = 2$.

The results (see appendix 1) can be summarized as follows:

$$\begin{aligned} \nu &= 0 \text{ for all states of } (\omega_1 0) \text{ and } (\omega_1 \tfrac{1}{2}),\\ \nu &= 0 \text{ for the } \beta = 0 \text{ states of } (\omega_1 1),\\ \nu &= t - T \text{ for all states of } (tt). \end{aligned} \tag{38}$$

For states with T-multiplicity > 1, the fourth quantum number β will always be chosen such that

$$K|(\omega_1 t)\beta H_1 T M_T\rangle = (-1)^{\eta(\omega_1, t)+\nu+T-M_T}|(\omega_1 t)\beta^c, -H_1, T, -M_T\rangle, \tag{39}$$

where either $\beta^c \equiv \beta$ (irreducible representations with integral t) or β^c and β are in simple 1 : 1 correspondence (irreducible representations with $\frac{1}{2}$-integral t). The phase factor ν is now dependent on β. For irreducible representations with states of T-multiplicity > 1, the phase factor ν thus serves partly to define the quantum number β. [For a more arbitrary choice of the fourth quantum number β, the complex conjugate of a state with a specific value of β could in general be a linear combination of states with all possible values of β^c. This would have been the case if β were chosen as the eigenvalue of the operator defined by eq. (4) which is not invariant under the conjugation operator K. If β is associated with the eigenvalues of an operator of the general form of eq. (6), however, the basis vectors will have the symmetry property of eq. (39)].

Using the conjugation relations, eq. (37) or (39) and standard techniques [see for example the discussion of the symmetry properties of SU_3 Wigner coefficients given by de Swart [24])], symmetry relations between the R(5) Wigner coefficients can be established. In particular, the full R(5) Wigner coefficients satisfy the symmetry relation † (interchange of representations 1 and 3)

$$\begin{aligned} &\langle(\omega_{1_1} t_1)\beta_1(H_1)_1 T_1 M_{T_1}; (\omega_{1_2} t_2)\beta_2(H_1)_2 T_2 M_{T_2}|(\omega_{1_3} t_3)\beta_3(H_1)_3 T_3 M_{T_3}\rangle_{\rho_{12,3}}\\ &= \left[\frac{\dim(\omega_{1_3} t_3)}{\dim(\omega_{1_1} t_1)}\right]^{\frac{1}{2}} (-1)^{\zeta+\nu_2+T_2-M_{T_2}}\\ &\quad\times\langle(\omega_{1_3} t_3)\beta_3(H_1)_3 T_3 M_{T_3}; (\omega_{1_2} t_2)\beta_2^c - (H_1)_2 T_2 - M_{T_2}|(\omega_{1_1} t_1)\beta_1(H_1)_1 T_1 M_{T_1}\rangle_{\rho_{32,1}}, \end{aligned} \tag{40}$$

where $\dim(\omega_1 t)$ stands for the dimension of the irreducible representation $(\omega_1 t)$

$$\dim(\omega_1 t) = \tfrac{1}{6}(2\omega_1+3)(2t+1)(\omega_1+t+2)(\omega_1-t+1).$$

The phase factor $(-1)^\zeta$ is a function of the irreducible representation labels $(\omega_{1_i} t_i)$ and can be determined from the phase convention, eq. (35), by applying eq. (40) to

† In principle the symmetry relation (40) implies a "proper" choice for the label ρ. This problem is not met for the Wigner coefficients needed in this investigation, since the label ρ is actually needed only in the special cases where $(\omega_{1_1} t_1) = (\omega_{1_3} t_3)$ so that no distinction need be made between labels of type $\rho_{12,3}$ and $\rho_{32,1}$.

the leading Wigner coefficient connecting the states of highest weight. In particular, by setting both $(H_1)_1 = \omega_{1_1}, T_1 = M_{T_1} = t_1$, and $(H_1)_3 = \omega_{1_3}$, $T_3 = M_{T_3} = t_3$, and $T_2 = \tilde{T}_2$, (where $\tilde{T}_2$ is the largest possible value of T_2 consistent with $\tilde{H}_2 = \omega_{1_3} - \omega_{1_1}$, $T_{1_1} = t_1$, and $T_3 = t_3$), the phase factor ζ can be determined to be

$$(-1)^{\zeta} = (-1)^{t_3 - t_1 - \tilde{\nu}_2 - \tilde{T}_2}. \tag{41}$$

[The phase factor $\tilde{\nu}_2$ is that for the state with $(\tilde{H}_1)_2$, $T_2 = \tilde{T}_2$, $\beta_2 = \tilde{\beta}_2$. For the simple representations needed for shell-model calculations, the phase factor $\tilde{\nu}_2$ is either equal to zero or is determined by the value $\tilde{T}_2$, see eq. (38)]. Combining the symmetry relation (40) with the analogous one for the ordinary isospin Wigner coefficient, the corresponding symmetry relation for the double-barred R(5) Wigner coefficients becomes

$$\begin{aligned}
&\langle(\omega_{1_1} t_1)\beta_1(H_1)_1 T_1; (\omega_{1_2} t_2)\beta_2(H_1)_2 T_2 \| (\omega_{1_3} t_3)\beta_3(H_1)_3 T_3\rangle_\rho \\
&\quad = (-1)^{t_3 - t_1 - T_3 + T_1 + T_2 - \tilde{T}_2 + \nu_2 - \tilde{\nu}_2} \left[\frac{\dim(\omega_{1_3} t_3)(2T_1+1)}{\dim(\omega_{1_1} t_1)(2T_3+1)}\right]^{\frac{1}{2}} \\
&\quad \times \langle(\omega_{1_3} t_3)\beta_3(H_1)_3 T_3; (\omega_{1_2} t_2)\beta_2^c - (H_1)_2 T_2 \| (\omega_{1_1} t_1)\beta_1(H_1)_1 T_1\rangle_\rho .
\end{aligned} \tag{42}$$

In the special case $(\omega_{1_2} t_2) = (\frac{1}{2}\frac{1}{2})$ the phase factor of eq. (42) reduces to the simple value $(-1)^{t_3 - t_1 - T_3 + T_1}$ in agreement with eq. (63) of ref. [12]).

A further symmetry relation for the full Wigner coefficients again follows from the conjugation relations

$$\begin{aligned}
&\langle(\omega_{1_1} t_1)\beta_1(H_1)_1 T_1 M_{T_1}; (\omega_{1_2} t_2)\beta_2(H_1)_2 T_2 M_{T_2} | (\omega_{1_3} t_3)\beta_3(H_1)_3 T_3 M_{T_3}\rangle_\rho \\
&\quad = (-1)^{\xi + \nu_1 + \nu_2 - \nu_3 + T_1 + T_2 - T_3} \\
&\quad \times \langle(\omega_{1_1} t_1)\beta_1^c - (H_1)_1 T_1 - M_{T_1}; (\omega_{1_2} t_2)\beta_2^c - (H_1)_2 T_2 - M_{T_2} | (\omega_{1_3} t_3)\beta_3^c - (H_1)_3 T_3 \\
&\qquad - M_{T_3}\rangle_\rho ,
\end{aligned} \tag{43}$$

where the phase factor ξ can again be a function only of the irreducible representation quantum numbers $(\omega_{1_i} t_i)$. The phase factor ξ has not been evaluated for the most general product $(\omega_{1_1} t_1) \times (\omega_{1_2} t_2) \to (\omega_{1_3} t_3)$ where the whole phase problem may be complicated by the multiplicity problem and the choice of ρ. For Wigner coefficients which involve only irreducible representations of type $(\omega_1 0)$, $(\omega_1 \frac{1}{2})$, (tt) and the $\beta = 0$ states of $(\omega_1 1)$, the phase factor ξ has the simple value

$$(-1)^{\xi} = (-1)^{\omega_{1_1} - t_1 + \omega_{1_2} - t_2 - \omega_{1_3} + t_3}. \tag{44}$$

If the Wigner coefficients include the representations $(\omega_1 \frac{3}{2})$, $(t+1, t)$ and the remaining states of $(\omega_1 1)$, the phase factor ξ can be made to have the same value by a proper choice of the fourth quantum number β since the phase factor ν can in these cases serve partly to define the label β. The phase factor (44) thus serves in all cases of actual interest in this investigation. Combining the symmetry relation (43) with the analogous one for the ordinary isospin Wigner coefficients, the corresponding symmetry relation for the double-barred R(5) Wigner coefficients becomes

$$\begin{aligned}&\langle(\omega_{1_1}t_1)\beta_1(H_1)_1T_1;(\omega_{1_2}t_2)\beta_2(H_1)_2T_2||(\omega_{1_3}t_3)\beta_3(H_1)_3T_3\rangle_\rho\\&\quad=(-1)^{\omega_{1_1}-t_1+\omega_{1_2}-t_2-\omega_{1_3}+t_3+v_1+v_2-v_3}\\&\quad\times\langle(\omega_{1_1}t_1)\beta_1^c-(H_1)_1T_1;(\omega_{1_2}t_2)\beta_2^c-(H_1)_2T_2||(\omega_{1_3}t_3)\beta_3^c-(H_1)_3T_3\rangle_\rho.\end{aligned}\tag{45}$$

The phase factor for this relation differs from that given by Ginocchio [12]), eq. (58).

4.4. CALCULATION OF THE R(5) WIGNER COEFFICIENTS

The calculation of the R(5) Wigner coefficients begins with the calculation of the matrix elements of the infinitesimal operators of the group. For the irreducible representations $(\omega_1 0)$, $(\omega_1 \frac{1}{2})$ and $(\omega_1 1)$ these have been calculated through the use of transformation coefficients to the separate neutron-proton quasi-spin scheme (see ref. [11]), tables 3-5); the elements actually tabulated are the reduced R(5) Wigner coefficients with $\rho = 1$. For the representations $(\omega_1 0)$ and $(\omega_1 \frac{1}{2})$ they have also been calculated by a different technique by Szpikowski [20]). They can also be calculated most directly from the explicit construction of the states with $M_T = T$ given in appendix 1 by operating on these states with the infinitesimal operators $E_{\pm 11}$, $E_{\pm 10}$, $E_{\pm 1,-1}$ in turn. For the irreducible representation (tt), the matrix elements of the infinitesimal operators have been calculated by this technique. They are expressed in terms of reduced R(5) Wigner coefficients and tabulated in appendix 2 (table 11e). Results for the irreducible representations $(\omega_1 \frac{3}{2})$, $(t+1, t)$ and the $\beta = 1$ and 2 states of $(\omega_1 1)$ will be tabulated in a subsequent publication.

With these tabulations of the matrix elements of the infinitesimal operators, the simpler R(5) Wigner coefficients can be calculated by standard recursion techniques. By operating with an operator $E_{ab} = E_{ab}(1)+E_{ab}(2)$ on a state of a coupled system built from systems 1 and 2, the single primed and double primed systems of eq. (9), a recursion relation for the full R(5) Wigner coefficients is obtained

$$\begin{aligned}&\sum_{\bar\beta\bar T}\langle\bar\beta(H_1+a)\bar T(M_T+b)|E_{ab}|\beta H_1TM_T\rangle\\&\qquad\langle\beta'H_1'T'M_T';\beta''H_1''T''M_T''|\bar\beta(H_1+a)\bar T(M_T+b)\rangle\\&\quad=\sum_{\bar\beta'\bar T'}\langle\beta'H_1'T'M_T'|E_{ab}|\bar\beta'(H_1'-a)\bar T'(M_T'-b)\rangle\\&\qquad\times\langle\bar\beta'(H_1'-a)\bar T'(M_T'-b);\beta''H_1''T''M_T''|\beta H_1TM_T\rangle\\&\qquad+\sum_{\bar\beta''\bar T''}\langle\beta''H_1''T''M_T''|E_{ab}|\bar\beta''(H_1''-a)\bar T''(M_T''-b)\rangle\\&\qquad\times\langle\beta'H_1'T'M_T';\bar\beta''(H_1''-a)\bar T''(M_T''-b)|\beta H_1TM_T\rangle.\end{aligned}\tag{46}$$

In this relation the irreducible representation labels $(\omega_1'\omega_2')$, $(\omega_1''\omega_2'')$ and $(\omega_1\omega_2)$ have been omitted for brevity. Although this recusion relation may contain a formidable number of terms for the most general coupling $(\omega_1'\omega_2')\times(\omega''\omega_2'') \to (\omega_1\omega_2)$, it becomes manageable if one of the representations is a simple one such as $(\omega_1''\omega_2'') = (\frac{1}{2}\frac{1}{2})$. The R(5) Wigner coefficients for the product $(\omega_1\omega_2)\times(\frac{1}{2}\frac{1}{2})$ can therefore be calculated by recursion techniques. The R(5) Wigner coefficients involving more complicated representations are then calculated from these by means of a build-up process. For example, Wigner coefficients for the products $(\omega_1\omega_2)\times(10)$ and $(\omega_1\omega_2)\times(11)$ can be expressed in terms of the simpler coefficients for the product $(\omega_1\omega_2)\times(\frac{1}{2}\frac{1}{2})$. Such a build-up process can be based on a recoupling transformation for a coupled system built from the states of three irreducible representations. Two possible ways of coupling the three representations $(\omega_1 t)_i$ with $i = 1, 2, 3$, to a resultant state of the representation $(\omega_1 t)$ are illustrated in fig. 1 by diagrams of the type

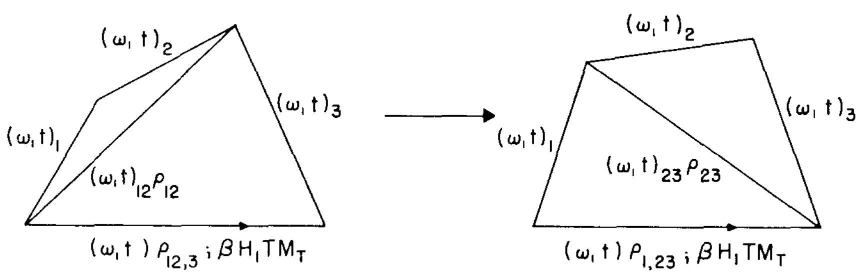

Fig. 1. Coupling and recoupling of three R(5) irreducible representations.

introduced by French [25]), adapted to R(5). The two coupled systems illustrated by fig. 1 are connected by a unitary transformation whose matrix elements are the generalized R(5) Racah coefficients or U-coefficients

$$
\begin{aligned}
&|[\{[(\omega_1 t)_1(\omega_1 t)_2](\omega_1 t)_{12}\rho_{12}\}(\omega_1 t)_3](\omega_1 t)\rho_{12,3};\beta H_1 TM_T\rangle \\
&\quad = \sum_{(\omega_1 t)_{23}}\ \sum_{\rho_{23},\,\rho_{1,23}} |[(\omega_1 t)_1\{[(\omega_1 t)_2(\omega_1 t)_3](\omega_1 t)_{23}\rho_{23}\}](\omega_1 t)\rho_{1,23};\beta H_1 TM_T\rangle \\
&\quad \times U\begin{pmatrix}(\omega_1 t)_1(\omega_1 t)_2;(\omega_1 t)_{12}\rho_{12}\rho_{12,3} \\ (\omega_1 t)_3(\omega_1 t);(\omega_1 t)_{23}\rho_{23}\rho_{1,23}\end{pmatrix},
\end{aligned}
\tag{47}
$$

where the R(5) U-coefficients are the generalization of the recoupling coefficients of ordinary angular momentum theory in their unitary form, although the notation is based on a generalization of the notation for the 6-j symbol. The U-coefficients are independent of $\beta H_1 TM_T$ and are real. They satisfy the orthogonality relations

$$
\sum_\alpha U\begin{pmatrix}\cdot\ \cdot\ \alpha \\ \cdot\ \cdot\ \mu\end{pmatrix} U\begin{pmatrix}\cdot\ \cdot\ \alpha \\ \cdot\ \cdot\ \mu'\end{pmatrix} = \delta_{\mu\mu'}, \qquad \sum_\mu U\begin{pmatrix}\cdot\ \cdot\ \alpha \\ \cdot\ \cdot\ \mu\end{pmatrix} U\begin{pmatrix}\cdot\ \cdot\ \alpha' \\ \cdot\ \cdot\ \mu\end{pmatrix} = \delta_{\alpha\alpha'}, \tag{48}
$$

where α is a shorthand notation for $(\omega_1 t)_{12}$, ρ_{12}, $\rho_{12,3}$ and μ is a shorthand notation for $(\omega_1 t)_{23}$, ρ_{23}, $\rho_{1,23}$. They can be related to the R(5) Wigner coefficients by

$$U\begin{pmatrix}(\omega_1 t)_1(\omega_1 t)_2; (\omega_1 t)_{12}\rho_{12}\rho_{12,3}\\ (\omega_1 t)_3(\omega_1 t); (\omega_1 t)_{23}\rho_{23}\rho_{1,23}\end{pmatrix}$$
$$= \sum_{\substack{\varepsilon_1\varepsilon_2\varepsilon_3\\ \varepsilon_{12}\varepsilon_{23}}} \langle(\omega_1 t)_1\varepsilon_1; (\omega_1 t)_2\varepsilon_2||(\omega_1 t)_{12}\varepsilon_{12}\rangle_{\rho_{12}}\langle(\omega_1 t)_{12}\varepsilon_{12}; (\omega_1 t)_3\varepsilon_3||(\omega_1 t)\varepsilon\rangle_{\rho_{12,3}}$$
$$\times\langle(\omega_1 t)_2\varepsilon_2; (\omega_1 t)_3\varepsilon_3||(\omega_1 t)_{23}\varepsilon_{23}\rangle_{\rho_{23}}\langle(\omega_1 t)_1\varepsilon_1; (\omega_1 t)_{23}\varepsilon_{23}||(\omega_1 t)\varepsilon\rangle_{\rho_{1,23}}$$
$$\times U(T_1 T_2 T T_3; T_{12} T_{23}). \tag{49}$$

The shorthand notation $\varepsilon_i = \beta_i H_{1_i} T_i$ has been used in eq. (49). The sums over M_{T_i} have been performed and expressed in terms of an ordinary isospin angular momentum U-coefficient (unitary or Jahn form of the Racah coefficient). Another very useful relationship between the R(5) U-coefficients and the R(5) Wigner coefficients follows from eq. (47) and the orthogonality of the Wigner coefficients

$$\sum_{\rho_{1,23}} \langle(\omega_1 t)_1\varepsilon_1; (\omega_1 t)_{23}\varepsilon_{23}||(\omega_1 t)\varepsilon\rangle_{\rho_{1,23}} U\begin{pmatrix}(\omega_1 t)_1(\omega_1 t)_2; (\omega_1 t)_{12}\rho_{12}\rho_{12,3}\\ (\omega_1 t)_3(\omega_1 t); (\omega_1 t)_{23}\rho_{23}\rho_{1,23}\end{pmatrix}$$
$$= \sum_{\varepsilon_2\varepsilon_3\varepsilon_{12}} \langle(\omega_1 t)_1\varepsilon_1; (\omega_1 t)_2\varepsilon_2||(\omega_1 t)_{12}\varepsilon_{12}\rangle_{\rho_{12}}\langle(\omega_1 t)_{12}\varepsilon_{12}; (\omega_1 t)_3\varepsilon_3||(\omega_1 t)\varepsilon\rangle_{\rho_{12,3}}$$
$$\times\langle(\omega_1 t)_2\varepsilon_2; (\omega_1 t)_3\varepsilon_3||(\omega_1 t)_{23}\varepsilon_{23}\rangle_{\rho_{23}} U(T_1 T_2 T T_3; T_{12} T_{23}). \tag{50}$$

This is the relation to be used as the basis for the building-up process in the calculation of the Wigner coefficients. Eq. (50) is valid only if the quantum numbers ρ have been chosen such that coupled states with different values of ρ are orthogonal to each other. In particular,

$$\langle[\{[(\omega_1 t)_1(\omega_1 t)_2](\omega_1 t)_{12}\rho_{12}\}(\omega_1 t)_3]$$
$$\times(\omega_1 t)\rho_{12,3}; \beta H_1 T M_T|[\{[(\omega_1 t)_1(\omega_1 t)_2](\omega_1' t')_{12}\rho_{12}'\}(\omega_1 t)_3]$$
$$\times(\omega_1' t')\rho_{12,3}'; \beta' H_1' T' M_T'\rangle$$
$$= \delta_{(\omega_1 t)_{12}(\omega'_1 t')_{12}}\delta_{(\omega_1 t),(\omega'_1 t')}\delta_{\rho_{12}\rho'_{12}}\delta_{\rho_{12,3}\rho'_{12,3}}\delta_{\beta\beta'}\delta_{H_1 H'_1}\delta_{TT'}\delta_{M_T M'_T}. \tag{51}$$

For the representations of interest in shell-model calculations, most of the indices ρ are not needed. For example, if the representations $(\omega_1 t)_2$ and $(\omega_1 t)_3$ are both identified with the representation $(\frac{1}{2}\frac{1}{2})$, the labels ρ_{12}, ρ_{23} and $\rho_{12,3}$ are not needed. In this case $(\omega_1 t)_{23} = (00)$, (10), or (11); and the fourth label $\rho_{1,23}$ is needed only in the one special case, i.e. $(\omega_1 t)_{23} = (11)$ and $(\omega_1 t) = (\omega_1 t)_1$. Except for this case, therefore, the sums over $\rho_{1,23}$ disappear from the left-hand side of eq. (50), and this relation can be used to calculate the R(5) Wigner coefficients for the products $(\omega_1 t)\times(10)$ and $(\omega_1 t)\times(11)$. Eq. (50) gives the R(5) Wigner coefficients to within a common factor, the U-coefficient of the left-hand side of the equation. This U-coefficient can be considered as a normalization factor and can be determined from the normalization condition and the phase conventions for the R(5) Wigner coefficients, eqs. (33) and (35); or alternately, in cases where not all values of $\beta H_1 T$ are of interest, it can be determined by calculating the R(5) Wigner coefficients through recursion techniques for some special values of $\beta H_1 T$ such as those for the highest-weight state. In the special case $(\omega_1 t)\times(11) \to (\omega_1 t)$, for which the label $\rho_{1,23}$ is needed, one set of the R(5) Wigner coefficients (those with $\rho = 1$) are known through the matrix elements of the infinitesimal operators, so that eq. (50) can be used to calculate the R(5) Wigner coefficients with $\rho = 2$.

The R(5) Wigner coefficients for the products $(\omega_1 t)\times(20)$ and $(\omega_1 t)\times(22)$ can be calculated through eq. (50) by setting $(\omega_1 t)_2 = (\omega_1 t)_3 = (11)$; or for the products $(\omega_1 t)\times(20)$ by setting $(\omega_1 t)_2 = (\omega_1 t)_3 = (10)$. In these cases some of the labels ρ are needed. For example the product $(\omega_1 t)\times(22)$ in general contains the representation $(\omega_1 t)$ with a multiplicity of 3 (although in the special cases $(\omega_1 t) = (\omega_1 0)$ or (tt) the multiplicity is only 1, whereas for $(\omega_1 t) = (\omega_1 \frac{1}{2})$ the multiplicity is 2). In cases where the label ρ is needed it has been defined rather arbitrarily, by assigning special values to some of the U-coefficients in eq. (50). This is best illustrated by a specific example. There are three independent sets of Wigner coefficients $\langle(\omega_1 1)\varepsilon'; (22)\varepsilon''||(\omega_1 1)\varepsilon\rangle_\rho$ corresponding to the possible values $\rho = 1, 2, 3$. These have been calculated for $\varepsilon'' = H_1''T'' = 00$, 01 and 02 through the system of three equations

$$\sum_\rho \langle(\omega_1 1)\varepsilon_1; (22)\varepsilon_{23}||(\omega_1 1)\varepsilon\rangle_\rho\, U\begin{pmatrix}(\omega_1 1)(11); (\omega_1' t')- - \\ (11)(\omega_1 1); (22)-\rho\end{pmatrix}$$
$$= \sum_{\varepsilon_2\varepsilon_3\varepsilon_{12}} \langle(\omega_1 1)\varepsilon_1; (11)\varepsilon_2||(\omega_1' t')\varepsilon_{12}\rangle\langle(\omega_1' t')\varepsilon_{12}; (11)\varepsilon_3||(\omega_1 1)\varepsilon\rangle$$
$$\times \langle(11)\varepsilon_2; (11)\varepsilon_3||(22)\varepsilon_{23}\rangle U(T_1 T_2 T T_3; T_{12} T_{23}), \tag{52}$$

with $(\omega_1' t') = (\omega_1 0)$, $(\omega_1+1, 0)$, and $(\omega_1-1, 0)$. In this case the three intermediate couplings $(\omega_1 1)\times(11)\to(\omega_1' 0)$; $(\omega_1' 0)\times(11)\to(\omega_1 1)$ and $(11)\times(11)\to(22)$; with $\omega_1' = \omega_1$, $\omega_1 \pm 1$; needed for the right-hand side of eqs. (52), all have multiplicity 1. Labels ρ_{12}, $\rho_{12,3}$ and ρ_{23} are thus not needed for these intermediate couplings. In all similar cases where the label $\rho_{1,23}$ has actually been needed for the final coupling it has been possible to find a sufficient number of intermediate couplings with a multiplicity of 1. They furnish a sufficient number of equations to solve for the independent sets of R(5) Wigner coefficients with different values of $\rho_{1,23}$ for the final coupling †. In the above example, the three equations obtained from (52) by setting $(\omega_1' t') = (\omega_1 0)$, $(\omega_1+1, 0)$, and $(\omega_1-1, 0)$ form a system of three independent equations in the three unknowns $\langle(\omega_1 1)\varepsilon_1; (22)\varepsilon_{23}||(\omega_1 1)\varepsilon\rangle_\rho$ with $\rho = 1, 2, 3$. In this specific case the labels ρ have been defined such that the two U-coefficients with $(\omega_1' t') = (\omega_1 0)$ and

† In this investigation the labels ρ have been needed only in relatively simple cases. It is interesting to speculate whether it is possible to find a sufficient number of intermediate couplings with a multiplicity of 1 in the general case.

$\rho = 2$ and 3 are equal to zero, and the single U-coefficient with $(\omega_1' t') = (\omega_1 - 1, 0)$ and $\rho = 3$ is also equal to zero. This choice for the three independent states facilitates the calculation of the R(5) Wigner coefficients (especially since the values of the U-coefficients are not known initially). Eq. (52) with $(\omega_1' t') = (\omega_1 0)$ now gives the R(5) Wigner coefficients with $\rho = 1$ at once. The single, non-zero U-coefficient with $(\omega_1' t') = (\omega_1 0)$ follows from the normalization of the $\rho = 1$ Wigner coefficients. Next, eq. (52) with $(\omega_1' t') = (\omega_1 - 1, 0)$ can be used to calculate the R(5) Wigner coefficients with $\rho = 2$. The two, non-zero U-coefficients with $(\omega_1' t') = (\omega_1 - 1, 0)$ now follow from the normalization of the $\rho = 2$ Wigner coefficients and the orthogonality of the Wigner coefficients with $\rho = 1$ and 2 and so forth.

In the relatively small number of cases where the label ρ is needed in this investigation, it has been defined in a similar way. This is of course a very arbitrary choice. In an actual application of the Wigner-Eckart theorem, however, the R(5) Wigner coefficients with different values of ρ may not be needed. Instead it will be sufficient to know the ρ sums such as

$$\sum_\rho \langle(\omega_1 1)\beta' H_1' T'; (22)H_1'' T''||(\omega_1 1)\beta H_1 T\rangle_\rho U \begin{pmatrix} (\omega_1 1)(11); (\omega_1' t') - - \\ (11)(\omega_1 1); (22) - \rho \end{pmatrix} \tag{53}$$

for different values of $(\omega_1' t')$. If the three reduced matrix elements $\langle(\omega_1 1)||T^{(22)}||(\omega_1 1)\rangle_\rho$ of a tensor operator $T^{(22)}$ are expressed in terms of a new set of three reduced matrix elements $\overline{\langle(\omega_1 1)||T^{(22)}||(\omega_1 1)\rangle_\alpha}$ defined by the system of three equations

$$\langle(\omega_1 1)||T^{(22)}||(\omega_1 1)\rangle_\rho = \sum_\alpha U \begin{pmatrix} (\omega_1 1)(11); (\omega_1' t')_\alpha - - \\ (11)(\omega_1 1); (22) - \rho \end{pmatrix} \overline{\langle(\omega_1 1)||T^{(22)}||(\omega_1 1)\rangle_\alpha}, \tag{54}$$

with $(\omega_1' t')_\alpha = (\omega_1 0), (\omega_1 + 1, 0), (\omega_1 - 1, 0)$ for $\alpha = 1, 2, 3$, respectively, the Wigner-Eckart theorem can be expressed in terms of the $\overline{\langle(\omega_1 1)||T^{(22)}||(\omega_1 1)\rangle_\alpha}$

$$\begin{aligned} &\langle(\omega_1 1)\beta H_1 T M_T|T^{(22)}_{H''_1 T'' M''_T}|(\omega_1 1)\beta' H_1' T' M_T'\rangle \\ &\quad = \sum_\rho \langle(\omega_1 1)\beta' H_1' T' M_T'; (22)H_1'' T'' M_T''|(\omega_1 1)\beta H_1 T M_T\rangle_\rho \langle(\omega_1 1)||T^{(22)}||(\omega_1 1)\rangle_\rho \\ &\quad = \sum_\alpha \Bigg\{ \sum_\rho \langle(\omega_1 1)\beta' H_1' T' M_T'; (22)H_1'' T'' M_T''|(\omega_1 1)\beta H_1 T M_T\rangle_\rho \\ &\qquad \times U \begin{pmatrix} (\omega_1 1)(11); (\omega_1' t')_\alpha - - \\ (11)(\omega_1 1); (22) - \rho \end{pmatrix} \Bigg\} \overline{\langle(\omega_1 1)||T^{(22)}||(\omega_1 1)\rangle_\alpha}. \end{aligned} \tag{55}$$

The last is a convenient form. The $\beta H_1 T M_T$ dependence of the matrix elements now appears only in the ρ sums, (enclosed by the curly bracket), and these are the $\beta H_1 T M_T$ dependent factors which are most easily calculated through the build-up process. In all those cases where a high multiplicity appears it is therefore more convenient to tabulate the ρ sums [such as those of eqs. (53)] rather than the R(5) Wigner coefficients for specific values of ρ. If needed, the latter can be calculated from the tabulated U-coefficients for the specific choices of ρ which have been made.

4.5. TABULATIONS OF R(5) WIGNER COEFFICIENT

The R(5) Wigner coefficients to be tabulated are those for the couplings $(\omega_1' t')\times(\omega_1'' t'') \rightarrow (\omega_1 t)$, where (i) $(\omega_1'' t'') = (\frac{1}{2}\frac{1}{2})$, (10), (11), (20) and (22) and (ii) $(\omega_1' t')$ and $(\omega_1 t)$ include the representations $(\omega_1 0)$, $(\omega_1 \frac{1}{2})$, (tt) and the $\beta = 0$ states of $(\omega_1 1)$.

The coefficients $\langle(\omega_1 t);(11)||(\omega_1 t\rangle_{\rho=1}$ with $t = 0, \frac{1}{2}$ and 1 are tabulated in ref. [11]) (tables 3-5); [the $\beta = 0$ states are identical with the $\kappa = 0$ states of ref. [11])]. The remaining coefficients from the above list are tabulated in appendix 2. In particular, the coefficients for the couplings $(\omega_1 \frac{1}{2})\times(\frac{1}{2}\frac{1}{2}) \rightarrow (\omega_1 \pm\frac{1}{2} 0)$; $(\omega_1 0)\times(\frac{1}{2}\frac{1}{2}) \rightarrow (\omega_1 \pm\frac{1}{2}\frac{1}{2})$; $(\omega_1 \frac{1}{2})\times(\frac{1}{2}\frac{1}{2}) \rightarrow (\omega_1 \pm\frac{1}{2}\, 1)\beta = 0$ states; $(\omega_1 1)\beta = 0$ states $\times(\frac{1}{2}\frac{1}{2}) \rightarrow (\omega_1 \pm\frac{1}{2}\frac{1}{2})$, and $(tt)\times(\frac{1}{2}\frac{1}{2}) \rightarrow (t\pm\frac{1}{2}, t\pm\frac{1}{2})$ are given in tables 9a-e. Some of these have previously been calculated by Ginocchio [12]). Since some of Ginocchio's coefficients differ from the present ones not only in overall phase but also in relative phase, they are tabulated again in appendix 2. The coefficients for the product $(\omega_1 0)\times(10)$ are given in table 8 of ref. [11]). Unfortunately these coefficients also differ in phase from the present conventions, but only overall phase factors are involved. To be in agreement with the present cases all coefficients of table 8, ref. [11]), in the second row (labelled by $H_1' T'$; $H''T'' = H_1 + 1T$; -10) and in the second column (labelled by $(\omega_1 \omega_2) = (j-\frac{1}{2}0)$) must be multiplied by -1. (They are also given again in table 10a of appendix 2.) The coefficients for the couplings $(\omega_1 \frac{1}{2})\times(10) \rightarrow (\omega_1' \frac{1}{2})$; $(\omega_1 1)\times(10) \rightarrow (\omega_1 1)\beta = 0$ states; and $(tt)\times(10) \rightarrow (tt)$ are tabulated in tables 10b-d. Finally, the coefficients for the couplings $(\omega_1 \frac{1}{2})\times(11) \rightarrow (\omega_1 \frac{1}{2})\rho = 2$; $(\omega_1 \frac{1}{2})\times(11) \rightarrow (\omega_1 \pm\frac{1}{2} 1)\beta = 0$ states; $(\omega_1 0)\times(11) \rightarrow (\omega_1 1)\beta = 0$ states; $(\omega_1 1)\times(11) \rightarrow (\omega_1 1)\rho = 2, \beta = 0$ states are given in tables 11a-e.

In the case of the products $(\omega_1 t)\times(20)$ and $(\omega_1 t)\times(22)$, only R(5) Wigner coefficients diagonal in H_1 and T are needed in the applications to shell-model calculations. These are tabulated for the couplings

$(\omega_1 0)\times(20) \rightarrow (\omega_1 0)$; $(\omega_1 0)\times(22) \rightarrow (\omega_1 0)$; $(\omega_1 \frac{1}{2})\times(20) \rightarrow (\omega_1 \frac{1}{2})$;
$(\omega_1 \frac{1}{2})\times(22) \rightarrow (\omega_1 \frac{1}{2})\rho = 1, 2$; $(\omega_1 1)\times(20) \rightarrow (\omega_1 1)\rho = 1, 2, \beta = 0$ states;
$(\omega_1 1)\times(22) \rightarrow (\omega_1 1)\rho$ sums, $\beta = 0$ states only; and $(tt)\times(20) \rightarrow (tt)$;
$(tt)\times(22) \rightarrow (tt)$; in tables 12-15. Finally some of the U-coefficients which are a by-product of the method of calculation are given in tables 16.

5. Applications

5.1. COEFFICIENTS OF FRACTIONAL PARENTAGE

Reduction formulae for one- and two-nucleon fractional parentage coefficients can be obtained through their simple relationship to the matrix elements of the single-nucleon and nucleon-pair creation operators. The precise relationship between the c.f.p. and the reduced matrix elements of the single-nucleon and nucleon-pair creation operators are [26])

$$\langle j^{n-1}\{v't'\}\beta'T', \alpha'J'|\}j^n\{v,t\}\beta T, \alpha J\rangle = \frac{\langle j^n\{v,t\}\beta T, \alpha J||a^+||j^{n-1}\{v't'\}\beta'T', \alpha'J'\rangle}{[n(2J+1)(2T+1)]^{\frac{1}{2}}}, \tag{56}$$

$$\langle j^{n-2}\{v't'\}\beta'T', \alpha'J'; j^2T''J''|\}j^n\{v,t\}\beta T, \alpha J\rangle = \frac{\langle j^n\{v,t\}\beta T, \alpha J||\mathscr{A}^+(J''T'')||j^{n-2}\{v't'\}\beta'T', \alpha'J'\rangle}{[n(n-1)(2J+1)(2T+1)]^{\frac{1}{2}}}. \tag{57}$$

The pair creation operators $\mathscr{A}^+$ are defined by eqs. (16). The quantum numbers α and J refer to the decomposition of the symplectic group through the chain $Sp(2j+1) \supset R(3)$ just as β, T and n refer to the decomposition of R(5). By using a generalized Wigner-Eckart-theorem in both spaces c.f.p. can be written in terms of generalized Wigner coefficients for both R(5) and $Sp(2j+1)$

$$\langle j^{n-1}\{v't'\}\beta'T', \alpha'J'|\}j^n\{vt\}\beta T, \alpha J\rangle = \frac{1}{\sqrt{n}}\langle(\omega_1' t')\beta'(H_1-\tfrac{1}{2})T'; (\tfrac{1}{2}\tfrac{1}{2})\tfrac{1}{2}\tfrac{1}{2}||(\omega_1 t)\beta(H_1 = \tfrac{1}{2}n-j-\tfrac{1}{2})T\rangle\langle\{v't'\}\alpha'J'; j||\{v,t\}\alpha J\rangle, \tag{58}$$

$$\langle j^{n-2}\{v't'\}\beta'T', \alpha'J'; j^2T''J''|\}j^n\{vt\}\beta T, \alpha J\rangle = \frac{1}{\sqrt{n(n-1)}}\langle(\omega_1' t')\beta'(H_1-1)T'; (\omega_1'' t'')1T''||(\omega_1 t)\beta(H_1 = \tfrac{1}{2}n-j-\tfrac{1}{2})T\rangle \times\{\langle v't'\}\alpha'J'; J''||\{v,t\}\alpha J\rangle. \tag{59}$$

The first factor in these relations is the reduced R(5) Wigner coefficient as defined in this work. In eq. (59), $(\omega_1'' t'') = (10)$ and (11) for $T'' = 0$ and 1, respectively, (see table 4), so that $(\omega_1'' t'') = (1T'')$. The second factor is completely independent of the quantum numbers β, T and n and is made to carry all of the dependence on the quantum numbers α and J.

Particle-hole relationships for the c.f.p. follow from the symmetry relations, eqs. (42) and (45) for the R(5) Wigner coefficients and the corresponding symmetry relations for the coefficients of the $Sp(2j+1) \supset R(3)$ chain. It will be assumed that the phases of the latter have been chosen such that the c.f.p. satisfy the particle-hole relationship [21,25,26])

$$\frac{\langle j^{n-1}\{v't'\}\beta'T', \alpha'J'|\}j^n\{vt\}\beta T\alpha J\rangle}{\langle j^{4j+2-n}\{vt\}\beta T, \alpha J|\}j^{4j+2-n+1}\{v't'\}\beta'T', \alpha'J'\rangle} = (-1)^{J-J'-j+T-T'-\frac{1}{2}}\left[\frac{(4j+2-n+1)(2J'+1)(2T'+1)}{n(2J+1)(2T+1)}\right]^{\frac{1}{2}}. \tag{60}$$

This leads to the further particle-hole relation

$$\frac{\langle j^{n-2}\{v't'\}\beta'T',\alpha'J';j^2T''J''|\}j^n\{vt\}\beta T,\alpha J\rangle}{\langle j^{4j+2-n}\{vt\}\beta T,\alpha J;j^2T''J''|\}j^{4j+4-n}\{v't'\}\beta'T',\alpha'J'\rangle}$$
$$= (-1)^{J-J'+T-T'}\left[\frac{(4j+4-n)(4j+3-n)(2J'+1)(2T'+1)}{n(n-1)(2J+1)(2T+1)}\right]^{\frac{1}{2}}. \quad (61)$$

Through the use of the R(5) Wigner coefficients it is now possible to give reduction formulae whereby the c.f.p. for arbitrary n can be expressed in terms of those containing the smallest possible number of particles (such as $n = v$). For one-nucleon c.f.p. two reduction formulae can be written [12]).

Case 1. $v' = v-1$

$$\frac{\langle j^{n-1}\{v-1,t'\}\beta'T',\alpha'J'|\}j^n\{vt\}\beta T,\alpha J\rangle}{\langle j^{v-1}\{v-1,t'\}t',\alpha'J'|\}j^v\{vt\}t,\alpha J\rangle} = (-1)^{T'-T-t'+t-\varphi}$$
$$\times\left[\frac{v(2T'+1)(2t+1)}{n(2T+1)(2t'+1)}\right]^{\frac{1}{2}}\langle(\omega_1 t)\beta H_1 T;(\tfrac{1}{2}\tfrac{1}{2})-\tfrac{1}{2}\tfrac{1}{2}||(\omega_1+\tfrac{1}{2}t')\beta'(H_1-\tfrac{1}{2})T'\rangle, \quad (62)$$

where $H_1 = \frac{1}{2}n-j-\frac{1}{2}$; $\omega_1 = j+\frac{1}{2}-\frac{1}{2}v$ in the R(5) Wigner coefficient. The phase φ is that of eq. (45) with $\nu_2 = 0$, while ν_1 and ν_3 refer to the highest weight states. For the simple representations $(\omega_1 0)$, $(\omega_1 \frac{1}{2})$, (tt), and $(\omega_1 1)$ $\beta = 0$ states, the R(5) Wigner coefficients for eq. (62) are given in appendix 2. In all these cases the phase factor has the simple value $(-1)^{T'-T-t'+t-\varphi} = (-1)^{T'-T-\frac{1}{2}}$. Eq. (62) follows from eq. (58) and the symmetry relations, eq. (42) and (45), together with the special value $\langle(\omega_1 t)\omega_1 t;(\frac{1}{2}\frac{1}{2})+\frac{1}{2}\frac{1}{2}||(\omega_1+\frac{1}{2}t')\omega_1+\frac{1}{2}t'\rangle = +1$.

Case 2. $v' = v+1$

$$\frac{\langle j^{n-1}\{v+1,t'\}\beta'T',\alpha'J'|\}j^n\{vt\}\beta T,\alpha J\rangle}{\langle j^{v}\{vt\}t,\alpha J|\}j^{v+1}\{v+1,t'\}t',\alpha'J'\rangle} = (-1)^{J-J'-j+t-t'-\frac{1}{2}}$$
$$\times\left[\frac{(v+1)(2J'+1)(2t'+1)}{n(2J+1)(2t+1)}\right]^{\frac{1}{2}}\langle(\omega_1-\tfrac{1}{2}t')\beta'(H_1-\tfrac{1}{2})T';(\tfrac{1}{2}\tfrac{1}{2})\tfrac{1}{2}\tfrac{1}{2}||(\omega_1 t)\beta H_1 T\rangle, \quad (63)$$

with $H_1 = \frac{1}{2}n-j-\frac{1}{2}$ and $\omega_1 = j+\frac{1}{2}-\frac{1}{2}v$. Eq. (63) follows from eq. (60), and the special value $\langle(\omega_1-\frac{1}{2}t')\omega_1-\frac{1}{2}t';(\frac{1}{2}\frac{1}{2})\frac{1}{2}\frac{1}{2}||(\omega_1 t)\omega_1 t\rangle = +1$. These two relations have been given by Ginocchio [12]). They are reproduced here since the present phase properties of the R(5) Wigner coefficients differ from those of ref. [12]).

For two-nucleon c.f.p., similar reduction formulae can be written in terms of the R(5) Wigner coefficients. Several cases must be considered. With $H_1 = \frac{1}{2}n-j-\frac{1}{2}$ and $\omega_1 = j+\frac{1}{2}-\frac{1}{2}v$;

Case 1. $v' = v-2$,

$$= \frac{\langle j^{n-2}\{v-2, t'\}\beta' T', \alpha' J'; j^2 T'' J''|\} j^n\{vt\}\beta T, \alpha J\rangle}{\langle j^{v-2}\{v-2, t'\}t', \alpha' J'; j^2 T'' J''|\} j^v\{vt\}t, \alpha J\rangle} = (-1)^{T'-T-t'+t-\varphi}$$

$$\times \left[\frac{v(v-1)(2T'+1)(2t+1)}{n(n-1)(2T+1)(2t'+1)}\right]^{\frac{1}{2}} \langle(\omega_1 t)\beta H_1 T; (1T'')-1T''||(\omega_1+1t')\beta'(H_1-1)T'\rangle. \tag{64}$$

The phase factor φ is again that of eq. (45) with $\nu_2 = 0$, while ν_1 and ν_3 refer to the highest weight values for the representations $(\omega_1 t)$ and $(\omega_1+1\,t')$, respectively. In the derivation of eq. (64), the symmetry relations, eqs. (42) and (45), have been used; also the special value

$$\langle(\omega_1 t)\omega_1 t; (1T'')1T''||(\omega_1+1t')\omega_1+1t')\rangle = +1.$$

Case 2. $v' = v+2$,

$$\frac{\langle j^{n-2}\{v+2, t'\}\beta' T'; \alpha' J'; j^2 T'' J''|\} j^n\{vt\}\beta T, \alpha J\rangle}{\langle j^v\{v, t\}t, \alpha J; j^2 T'' J''|\} j^{v+2}\{v+2, t'\}t', \alpha' J'\rangle} = (-1)^{J-J'+t-t'}$$

$$\times \left[\frac{(v+2)(v+1)(2J'+1)(2t'+1)}{n(n-1)(2J+1)(2t+1)}\right]^{\frac{1}{2}} \langle(\omega_1-1t')\beta'(H_1-1)T'; (1T'')1T''||(\omega_1 t)\beta H_1 T\rangle, \tag{65}$$

where eq. (61) has been used; also the special value

$$\langle(\omega_1-1t')\omega_1-1t'; (1T'')1T''||(\omega_1 t)\omega_1 t\rangle = +1.$$

Case 3. $v' = v$; arbitrary t' for $T'' = 0$, but $t' \neq t$ for $T'' = 1$,

$$\frac{\langle j^{n-2}\{vt'\}\beta' T', \alpha' J'; j^2 T'' J''|\} j^n\{vt\}\beta T, \alpha J\rangle}{\langle j^v\{vt\}t, \alpha J; j^2 T'' J''|\} j^{v+2}\{vt'\}t'+b, \alpha' J'\rangle}$$

$$= (-1)^{J-J'+t-t'-b} \left[\frac{(v+2)(v+1)(2J'+1)(2t'+2b+1)}{n(n-1)(2J+1)(2t+1)}\right]^{\frac{1}{2}}$$

$$\times \frac{\langle(\omega_1 t')\beta'(H_1-1)T'; (1T'')1T''||(\omega_1 t)\beta H_1 T\rangle}{\langle(\omega_1 t')\omega_1-1t'+b; (1T'')1T''||(\omega_1 t)\omega_1 t\rangle}, \tag{66}$$

where b can have the values 0, ± 1 since the $v+2$ nucleon state of seniority v and reduced isospin t' can have total isospin $T' = t'$ or $t' \pm 1$. Since each of these states is single, the label β' is not needed. In this case, the R(5) Wigner coefficient with $H_1 T = \omega_1 t$ does not have the simple value $+1$ and, like the coefficient for arbitrary H_1 and T, must be read off from tables such as those of appendix 2.

Case 4. $v' = v$, $T'' = 1$, $t' = t$. For general v, t, J'', this case is complicated by the multiplicity problem. The product $(\omega_1 t) \times (11)$ in general contains the representation

$(\omega_1 t)$ twice, and the reduction formulae will depend on the two independent R(5) Wigner coefficients with $\rho = 1$ and 2. The exceptions are the representation $(\omega_1 0)$ and (tt) for which the products are simply reducible. In these two cases, eq. (66) holds also for the case $v' = v$, $T'' = 1$, $t' = t$. Finally with $J'' = 0$ the pair-creation operator belongs to the family of infinitesimal operators of the group, so that in this case the reduced matrix elements with $\rho = 2$ are equal to zero. In this case the reduction formula is again given by eq. (66) where the R(5) Wigner coefficients are those with $\rho = 1$. For this special case numerical values of the R(5) Wigner coefficients have been tabulated by Ichimura [13]). In the general case, $v' = v$, $T'' = 1$, $t' = t$, the c.f.p. can be written

$$\begin{aligned}
&\langle j^{n-2}\{vt\}\beta' T', \alpha' J'; j^2 T'' = 1 J'' \neq 0 |\} j^n\{vt\}\beta T, \alpha J\rangle \\
&= \frac{1}{\sqrt{n(n-1)}} \langle(\omega_1 t)\beta' H_1 - 1 T'; (11)11 || (\omega_1 t)\beta H_1 T\rangle_{\rho=1} F_1(v, t, \alpha\alpha' j J' J'' J) \\
&+ \frac{1}{\sqrt{n(n-1)}} \langle(\omega_1 t)\beta' H_1 - 1 T'; (11)11 || (\omega_1 t)\beta H_1 T\rangle_{\rho=2} F_2(v, t, \alpha\alpha' j J' J'' J),
\end{aligned} \tag{67}$$

where the coefficients F_1 and F_2 are independent of β, $H_1(n)$ and T.

One possible reduction formula in this case could be given by

$$\begin{aligned}
&\langle j^{n-2}\{vt\}\beta' T', \alpha' J'; j^2 T'' = 1 J'' \neq 0 |\} j^n\{vt\}\beta T, \alpha J\rangle \\
&= (-1)^{J-J'} \left[\frac{(v+2)(v+1)(2J'+1)}{n(n-1)(2J+1)}\right]^{\frac{1}{2}} \\
&\times \{\langle(\omega_1 t)\beta'(H_1 - 1)T'; (11)11 || (\omega_1 t)\beta H_1 T\rangle_{\rho=1} \\
&\times [\Gamma_2(0)\langle j^v\{vt\}t, \alpha J; j^2 1 J'' |\} j^{v+2}\{vt\}t+1, \alpha' J'\rangle \\
&+ \Gamma_2(1)\langle j^v\{vt\}t, \alpha J; j^2 1 J'' |\} j^{v+2}\{vt\}t, \alpha' J'\rangle] \\
&- \langle(\omega_1 t)\beta'(H_1 - 1)T'; (11)11 || (\omega_1 t)\beta H_1 T\rangle_{\rho=2} \\
&\times [\Gamma_1(0)\langle j^v\{vt\}t, \alpha J; j^2 1 J'' |\} j^{v+2}\{vt\}t+1, \alpha' J'\rangle \\
&+ \Gamma_1(1)\langle j^v\{vt\}t, \alpha J; j^2 1 J'' |\} j^{v+2}\{vt\}t, \alpha' J'\rangle]\},
\end{aligned} \tag{68}$$

where the short-hand notation $\Gamma_\rho(b)$ has been used;

$$\Gamma_\rho(b) = \frac{\langle(\omega_1 t)\omega_1 - 1, t+b; (11)11 || (\omega_1 t)\omega_1 t\rangle_\rho}{\begin{vmatrix} \langle(\omega_1 t)\omega_1 - 1 t; (11)11 || (\omega_1 t)\omega_1 t\rangle_{\rho=1} & \langle(\omega_1 t)\omega_1 - 1 t; (11)11 || (\omega_1 t)\omega_1 t\rangle_{\rho=2} \\ \langle(\omega_1 t)\omega_1 - 1, t+1; (11)11 || (\omega_1 t)\omega_1 t\rangle_{\rho=1} & \langle(\omega_1 t)\omega_1 - 1, t+1; (11)11 || (\omega_1 t)\omega_1 t\rangle_{\rho=2} \end{vmatrix}}. \tag{69}$$

5.2. ONE-BODY OPERATORS

One-body operators of definite rank (spherical tensor character K in physical three-dimensional space and isospin character τ) can be expressed in terms of the elementary multipole operators of eq. (18).

The operator $F^{K;\tau}_{q;\gamma} = \sum_{i=1}^{n} f(i)^{K;\tau}_{q;\gamma}$ can be written

$$F^{K;\tau}_{q;\gamma} = \langle j\tfrac{1}{2}\|f^{K;\tau}\|j\tfrac{1}{2}\rangle \frac{U(j^2;\,Kq;\,\tau\gamma)}{[(2K+1)(2\tau+1)]^{\frac{1}{2}}}, \tag{70}$$

where the double-barred matrix element is the conventional angular momentum reduced matrix element of the one-particle operator in the single-particle state with angular momentum j and isospin $\frac{1}{2}$. The R(5) tensor character of the elementary multipole operators is given in table 4. Operators with K odd and $\tau = 0$ are R(5) scalars. Their matrix elements are therefore diagonal in v and t and independent of nucleon number and isospin. One-body operators with K odd and $\tau = 1$ have R(5) tensor character (10), while those with K even $\tau = 0$ or 1 have R(5) tensor character (11). Their matrix elements can thus be off-diagonal in v and t and have a complicated n, T dependence. The diagonal matrix elements of the one-body operators in states with $v = 1$ ($t = \frac{1}{2}$) are perhaps of greatest interest. The R(5) Wigner coefficients needed to calculate these are tabulated in appendix 2, (tables 10b and 11a) and table 4 of ref. [11]). Although the states of the irreducible representation $(\omega_1\,\frac{1}{2})$ are completely specified by n and T; the R(5) Wigner coefficients do depend on a fourth quantum number, e (or o) for $\omega_1+\frac{1}{2}-H_1-T =$ even (or odd) integer, or $\frac{1}{2}(n+v-1)+T =$ even (or odd), respectively. This dependence makes itself felt only through phase factors of the form $(-1)^{\frac{1}{2}n-T}$. Since the product $(\omega_1\,\frac{1}{2})\times(11)$ contains the representation $(\omega_1\,\frac{1}{2})$ twice, matrix elements of one-body operators with K even are governed by two R(5) reduced matrix elements, eq. (8). They can be determined from the matrix elements of the one-particle and one-hole states. Results for the full matrix elements are shown in table 5. Operators with K even, in particular, lead to a complicated n, T dependence. By using the proper combinations of the $\tau = 0$ and 1 operators, Parikh [ref. [10])] has used such matrix elements to find the n, T dependence of the magnetic dipole and electric quadrupole moments for the seniority 1 states of the configuration j^n. (For the magnetic dipole moment, see also de-Shalit and Talmi [5]), pp. 449 and 536.)

5.3. THE TWO-BODY INTERACTION

The n, T dependence of the general two-body interaction can in principle be determined for any matrix element, diagonal or off-diagonal in v and t, by the techniques outlined in this investigation. Since the two-body interaction includes the relatively complicated R(5) irreducible tensors of type (20) and (22), the calculations are simple only for states involving the simpler irreducible representations of R(5). In particular, since the needed R(5) Wigner coefficients for the representations $(\omega_1 0)$, $(\omega_1\,\frac{1}{2})$ and the $\beta = 0$ states of $(\omega_1\,1)$ are known,(appendix 2), it is possible to extract the n, T

TABLE 5

Diagonal matrix elements of one-body operators in $v = 1$ states

$\langle j^n\{v=1, t=\frac{1}{2}\} jm; TM_T | F^{K;\tau}_{0;0} | j^n\{v=1, t=\frac{1}{2}\} jm; TM_T\rangle$

Rank	Matrix element
K odd; $\tau = 0$	c
K odd; $\tau = 1$	$c\left\{\frac{(2j+1-n)+(-1)^{\frac{1}{2}n-T}(2j+3)(2T+1)}{4T(T+1)(j+1)}\right\}M_T$
K even; $\tau = 0$	$c\left\{\frac{(2j+1-n)(2j+3)}{(4j^2+12j+3)}-\frac{(2j+1)[5(n-2j-1)(2j+3)+(-1)^{\frac{1}{2}n-T}(2T+1)(4j^2+12j+3)]}{2(j+1)(2j-1)(4j^2+12j+3)}\right\}$
K even; $\tau = 1$	$-2c\left\{\frac{(2j+3)}{(4j^2+12j+3)}+\frac{(2j+1)[20(2j+3)T(T+1)-\{(2j+3)-(-1)^{\frac{1}{2}n-T}(n-2j-1)(2T+1)\}(4j^2+12j+3)]}{8T(T+1)(j+1)(2j-1)(4j^2+12j+3)}\right\}M_T$

One particle matrix element $c = \frac{\langle j\frac{1}{2}\|f^{K;\tau}\|j\frac{1}{2}\rangle}{[2(2j+1)]^{\frac{1}{2}}}\langle jmK0|jm\rangle\langle\frac{1}{2}\frac{1}{2}\tau 0|\frac{1}{2}\frac{1}{2}\rangle$.

dependence for the diagonal matrix elements in states of low seniority. The calculations are simple for states with $v = 0$, $v = 1$, and with $v = 2$, $t = 0$, as well as for the ($n = 4k$, T odd) and ($n = 4k+2$, T even) states with $v = 2$, $t = 1$; the latter, the $\beta = 0$ states of $(j-\frac{1}{2}, 1)$. Although this does not include all $v = 2$ states, it includes the most interesting ones, since the major components for the low-lying states of doubly odd nuclei may be expected to be given by the above types.

The general, charge-independent, scalar, two-body interaction has been classified in terms of the basic R(5) irreducible tensors $[T(J^2(1T)^2]^{(\omega_1\omega_2);\,0}_{000;\,0}$ by eqs. (27) and (28) of sect. 3. The reduced matrix elements of the interaction can be defined in terms of the reduced matrix elements of these basic tensors. In particular, if the reduced matrix elements are defined by

$$\begin{aligned}
&\langle (j+\tfrac{1}{2}-\tfrac{1}{2}v,\, t)\alpha J || \mathscr{V}^{(\omega_1\omega_2)}_{\text{even}} || (j+\tfrac{1}{2}-\tfrac{1}{2}v,\, t)\alpha J\rangle_\rho \\
&= -\tfrac{1}{4} \sum_{\text{even } J'} V_{J'T=1}[2J'+1]^{\frac{1}{2}} \langle (j+\tfrac{1}{2}-\tfrac{1}{2}v,\, t)\alpha J || [T(J'^2(11)^2]^{(\omega_1\omega_2);\,0} || (j+\tfrac{1}{2}-\tfrac{1}{2}v,\, t)\alpha J\rangle_\rho, \\
&\langle (j+\tfrac{1}{2}-\tfrac{1}{2}v,\, t)\alpha J || \mathscr{V}^{(\omega_1\omega_2)}_{\text{odd}} || (j+\tfrac{1}{2}-\tfrac{1}{2}v,\, t)\alpha J\rangle_\rho \\
&= -\tfrac{1}{4} \sum_{\text{odd } J'} V_{J'T=0}[2J'+1]^{\frac{1}{2}} \langle (j+\tfrac{1}{2}-\tfrac{1}{2}v,\, t)\alpha J || [T(J'^2(10)^2])^{(\omega_1\omega_2);\,0} || (j+\tfrac{1}{2}-\tfrac{1}{2}v,\, t)\alpha J\rangle_\rho,
\end{aligned} \tag{71}$$

the diagonal matrix elements of a charge independent two-body interaction can be written

$$\begin{aligned}
&\langle j^n\{vt\}\beta TM_T,\, \alpha JM_J| \sum_{i<k}^{n} V_{ik} | j^n\{vt\}\beta TM_T,\, \alpha JM_J\rangle \\
&\quad = \sqrt{2} \sum_\rho \langle (\omega_1 t)\alpha J || \mathscr{V}^{(22)}_{\text{even}} || (\omega_1 t)\alpha J\rangle_\rho \langle (\omega_1 t)\beta H_1 T;\, (22)00 || (\omega_1 t)\beta H_1 T\rangle_\rho \\
&\qquad + \sqrt{\tfrac{2}{5}} \sum_\rho \langle (\omega_1 t)\alpha J || \mathscr{V}^{(20)}_{\text{even}} || (\omega_1 t)\alpha J\rangle_\rho \langle (\omega_1 t)\beta H_1 T;\, (20)00 || (\omega_1 t)\beta H_1 T\rangle_\rho \\
&\qquad + 3\sqrt{\tfrac{2}{5}} \langle (\omega_1 t)\alpha J || \mathscr{V}^{(00)}_{\text{even}} || (\omega_1 t)\alpha J\rangle \cdot 1 + \frac{3H_1}{(2j+1)} \sum_{\text{even } J} V_{J1}(2J+1) \\
&\qquad + \sqrt{\tfrac{6}{5}} \sum_\rho \langle (\omega_1 t)\alpha J || \mathscr{V}^{(20)}_{\text{odd}} || (\omega_1 t)\alpha J\rangle_\rho \langle (\omega_1 t)\beta H_1 T;\, (20)00 || (\omega_1 t)\beta H_1 T\rangle_\rho \\
&\qquad + 2\sqrt{\tfrac{1}{5}} \langle (\omega_1 t)\alpha J || \mathscr{V}^{(00)}_{\text{odd}} || (\omega_1 t)\alpha J\rangle \cdot 1 + \frac{H_1}{(2j+1)} \sum_{\text{odd } J} V_{J0}(2J+1).
\end{aligned} \tag{72}$$

The reduced matrix elements can be calculated from the matrix elements of the interaction in states with $n = v$ and $n = v+2$. The number of such matrix elements needed is equal to the largest number of terms in the ρ sums of eq. (72) which is at most equal to three in the general case. For the simple configurations j^n, the reduced matrix elements have been calculated for states with $v = 0$, $v = 1$, $t = 0$, and $v = 2$, $t = 1$. The

TABLE 6a

Reduced matrix elements for the two-body interaction

$\langle (j+\frac{1}{2}-\frac{1}{2}v, t)||\mathscr{V}^{(\omega_1\omega_2)}_{(r)}||(j+\frac{1}{2}-\frac{1}{2}v, t)\rangle$

for states with $v = 0 (t = 0)$

R(5) Rank $(\omega_1\omega_2)$	(r)	$\langle (j+\frac{1}{2}, 0)\|\mathscr{V}^{(\omega_1\omega_2)}_{(r)}\|(j+\frac{1}{2}, 0)\rangle$
(22)	even J	$\frac{1}{2}\left[\frac{(j+\frac{1}{2})(j+\frac{7}{2})(j+\frac{9}{2})}{2(j-\frac{1}{2})}\right]^{\frac{1}{2}}(2j\bar{V}_{\text{even}}-V_0)$
(20)	even J	$\left[\frac{(j+\frac{1}{2})(j+\frac{7}{2})(j+3)}{10(j+1)}\right]^{\frac{1}{2}}(j\bar{V}_{\text{even}}+V_0)$
(00)	even J	$\frac{(2j+1)}{4\sqrt{10}}(6j\bar{V}_{\text{even}}+V_0)$
(20)	odd J	$\left[\frac{3(j+\frac{1}{2})(j+\frac{7}{2})(j+1)(j+3)}{10}\right]^{\frac{1}{2}}\bar{V}_{\text{odd}}$
(00)	odd J	$\frac{(j+1)(2j+1)}{2\sqrt{5}}\bar{V}_{\text{odd}}$

$\bar{V}_{\text{even}}$ and ($\bar{V}_{\text{odd}}$) are the average two-particle interaction energies for even (and odd) J states, respectively, see eqs. (74) and (75).

TABLE 6b

Reduced matrix elements for the two-body interaction

$\langle (j+\frac{1}{2}-\frac{1}{2}v, t)||\mathscr{V}^{(\omega_1\omega_2)}_{(r)}||(j+\frac{1}{2}-\frac{1}{2}v, t)\rangle$

for states with $v = 1$ $(t = \frac{1}{2})$

R(5) Rank $(\omega_1\omega_2)$ ρ	(r)	$\langle (j\frac{1}{2})\|\mathscr{V}^{(\omega_1\omega_2)}_{(r)}\|(j\frac{1}{2})\rangle_\rho$
(22) $\rho = 1$	even J	$\left[\frac{(2j+7)(j^2+3j+\frac{1}{4})}{8(2j-1)}\right]^{\frac{1}{2}}(2j\bar{V}_{\text{even}}-V_0)$
(22) $\rho = 2$	even J	0
(20)	even J	$\frac{1}{2}\left[\frac{(j+2)(2j-1)(2j+7)}{10(j+1)}\right]^{\frac{1}{2}}(j\bar{V}_{\text{even}}+V_0)$
(00)	even J	$\frac{1}{4\sqrt{10}}[4(3j+1)j\bar{V}_{\text{even}}+(2j-1)V_0]$
(20)	odd J	$\frac{1}{2}\left[\frac{3(j+1)(j+2)(2j-1)(2j+7)}{10}\right]^{\frac{1}{2}}\bar{V}_{\text{odd}}$
(00)	odd J	$\frac{1}{4\sqrt{5}}(j+1)(4j+3)\bar{V}_{\text{odd}}$
(10) [a]	even J	$-\frac{1}{4}\left[\frac{2(j+2)}{(j+1)}\right]^{\frac{1}{2}}(j\bar{V}_{\text{even}}+V_0)$

[a]) Needed only for an isovector interaction.

TABLE 6c

Reduced matrix elements for the two-body interaction
$\langle J_2(j+\frac{1}{2}-\frac{1}{2}v,t)||\mathscr{V}^{(\omega_1\omega_2)}_{(r)}||(j+\frac{1}{2}-\frac{1}{2}v,t),J_2\rangle$
for states with $v=2$, $t=0$

R(5) Rank $(\omega_1\omega_2)$	(r)	$\langle J_2,(j-\frac{1}{2},0||\mathscr{V}^{(\omega_1\omega_2)}_{(r)}||(j-\frac{1}{2},0),J_2\rangle$
(22)	even J	$\frac{1}{2}\left[\frac{(j-\frac{3}{2})(j+\frac{5}{2})(j+\frac{7}{2})}{2(j-\frac{1}{2})}\right]^{\frac{1}{2}}(2j\bar{V}_{\text{even}}-V_0)$
(20)	even J	$\frac{1}{2}\left[\frac{(j+\frac{5}{2})(j+2)}{10(j-\frac{1}{2})j}\right]^{\frac{1}{2}}[(2j-3)j\bar{V}_{\text{even}}+(2j-3)V_0+6j\overline{(V_{\text{even}}U_{J_2})}]$
(00)	even J	$\frac{1}{4\sqrt{10}}[2(6j+1)j\bar{V}_{\text{even}}+(2j-3)V_0-4j\overline{(V_{\text{even}}U_{J_2})}]$
(20)	odd J	$\left[\frac{6(j-\frac{1}{2})(j+\frac{5}{2})j(j+2)}{5}\right]^{\frac{1}{2}}\frac{1}{j(2j-1)}[V_{J_2}+(j-1)(j+1)\bar{V}_{\text{odd}}-\frac{1}{2}(j+1)\overline{(V_{\text{odd}}U_{J_2})}]$
(00)	odd J	$\frac{1}{\sqrt{5}}[V_{J_2}+(j+\frac{1}{4})(j+1)\bar{V}_{\text{odd}}+\frac{3}{4}(j+1)\overline{(V_{\text{odd}}U_{J_2})}]$

The average two-particle interaction energies $\bar{V}_{\text{even}}$, $\bar{V}_{\text{odd}}$, $\overline{(V_{\text{even}}U_{J_2})}$ and $\overline{(V_{\text{odd}}U_{J_2})}$ are defined in eqs. (74), (75), (78) and (79).

results are collected in tables 6a-d. For states with $v=0$ and $v=1$, eq. (72) merely leads to a well-known result, (see de-Shalit and Talmi [5]), p. 456). In those two cases, the interaction energy has the form

$$\frac{n(n-1)}{2}\left\{\frac{(6j+5)j\bar{V}_{\text{even}}-(2j+3)V_0+(2j-1)(j+1)\bar{V}_{\text{odd}}}{4(2j-1)(j+1)}\right\}$$
$$+\{T(T+1)-\tfrac{3}{4}n\}\left\{\frac{(2j+3)j\bar{V}_{\text{even}}-V_0(2j+1)-(2j-1)(j+1)\bar{V}_{\text{odd}}}{2(2j-1)(j+1)}\right\}$$
$$+\left[\frac{n}{2}\right]\frac{2j}{(2j-1)}(V_0-\bar{V}_{\text{even}}). \tag{73}$$

In eq. (73) the function $[\frac{1}{2}n]$ is equal to $\frac{1}{2}n$ for n even ($v=0$) and $\frac{1}{2}(n-1)$ for n odd ($v=1$). The interaction energy has been expressed in terms of the two-particle energy V_0 in the state with $J=0$ and the average two-particle energies in states with even and odd J defined by

$$\bar{V}_{\text{even}}=\frac{\sum_{\text{even }J}V_J(2J+1)}{\sum_{\text{even }J}(2J+1)}=\frac{\sum_{\text{even }J}V_J(2J+1)}{(2j+1)j}, \tag{74}$$

$$\bar{V}_{\text{odd}}=\frac{\sum_{\text{odd }J}V_J(2J+1)}{\sum_{\text{odd }J}(2J+1)}=\frac{\sum_{\text{odd }J}V_J(2J+1)}{(2j+1)(j+1)}. \tag{75}$$

TABLE 6d

Reduced matrix elements for the two-body interaction for states with $v = 2$, $t = 1$

R(5) rank $(\omega_1\omega_2)$	ρ	(r)	$\langle (j-\frac{1}{2}, 1), J_2 \Vert \mathscr{V}^{(\omega_1\omega_2)}_{(r)} \Vert (j-\frac{1}{2}, 1), J_2 \rangle_\rho$ [a]
(20)	$\rho = 1$	even J	$\frac{1}{3}\left[\frac{(j+\frac{1}{2})}{30(j+\frac{3}{2})(2j^2+4j-\frac{7}{2})}\right]^{\frac{1}{2}}\left[-(j+4)(2j-3)(j\bar{V}_{\text{even}}+V_0)+(16j+39)(V_{J_2}+j\overline{(V_{\text{even}}U_{J_2})})\right]$
(20)	$\rho = 2$	even J	$\frac{1}{6}\left[\frac{5(j-\frac{3}{2})(j+\frac{3}{2})(j+\frac{7}{2})(j+2)}{3j(j+\frac{1}{2})(2j^2+4j-\frac{7}{2})}\right]^{\frac{1}{2}}\left[(2j-1)(j\bar{V}_{\text{even}}+V_0)+2(V_{J_2}+j\overline{(V_{\text{even}}U_{J_2})})\right]$
(00)		even J	$\frac{1}{4\sqrt{10}}\left[(2j-3)V_0+2(6j+1)j\bar{V}_{\text{even}}+4V_{J_2}+4j\overline{(V_{\text{even}}U_{J_2})}\right]$
(10)	[b]	even J	$\frac{4[2(j+\frac{1}{2})(j+\frac{3}{2})]^{\frac{1}{2}}}{\sqrt{6}(2j-1)}\left[V_0-3j\bar{V}_{\text{even}}-j\overline{(V_{\text{even}}U_{J_2})}\right]$
(20)	$\rho = 1$	odd J	$\frac{1}{3}\left[\frac{(j+\frac{1}{2})}{10(j+\frac{3}{2})(2j^2+4j-\frac{7}{2})}\right]^{\frac{1}{2}}\left[-(j+4)(2j-3)(j+1)\bar{V}_{\text{odd}}+(16j+39)(j+1)\overline{(V_{\text{odd}}U_{J_2})}\right]$
(20)	$\rho = 2$	odd J	$\frac{1}{6}\left[\frac{5(j-\frac{3}{2})(j+\frac{3}{2})(j+\frac{7}{2})(j+2)}{j(j+\frac{1}{2})(2j^2+4j-\frac{7}{2})}\right]^{\frac{1}{2}}\left[(2j-1)(j+1)\bar{V}_{\text{odd}}+2(j+1)\overline{(V_{\text{odd}}U_{J_2})}\right]$
(00)		odd J	$\frac{1}{2\sqrt{5}}\left[2(j+1)^2\bar{V}_{\text{odd}}-(j+1)\overline{(V_{\text{odd}}U_{J_2})}\right]$
(22)	$(\omega'_1\omega'_2)_\alpha = (\omega_1 0)$ [a]	even J	$-\frac{[3(j-\frac{1}{2})(j+\frac{1}{2})(j+\frac{3}{2})(j+\frac{5}{2})]^{\frac{1}{2}}}{(2j+1)(2j-3)(2j-5)}\left[(4j^2-12j+1)(2j\bar{V}_{\text{even}}-V_0)+8(V_{J_2}-2j\overline{(V_{\text{even}}U_{J_2})})\right]$
(22)	$(\omega'_1\omega'_2)_\alpha = (\omega_1-1, 0)$	even J	$\frac{(2j+3)[3(j+\frac{1}{2})(j+\frac{5}{2})j(j+1)]^{\frac{1}{2}}}{2j(j+1)(2j+1)(2j-3)(2j-5)}\left[(4j^3-12j^2+j+8)(2j\bar{V}_{\text{even}}-V_0)+4(V_{J_2}-2j\overline{(V_{\text{even}}U_{J_2})})\right]$
(22)	$(\omega'_1\omega'_2)_\alpha = (\omega_1+1, 0)$	even J	$\frac{[3(j-\frac{1}{2})(j+\frac{3}{2})(j+1)(j+2)]^{\frac{1}{2}}(2j+1)}{2(j+1)(2j-3)(2j-5)}\left[(2j-7)(2j\bar{V}_{\text{even}}-V_0)+4(V_{J_2}-2j\overline{(V_{\text{even}}U_{J_2})})\right]$

[a]) For $(\omega_1\omega_2) = (22)$ the reduced matrix elements tabulated are those of type $\overline{\langle(\omega_1 1)\Vert T^{(22)}\Vert(\omega_1 1)\rangle_\alpha}$ associated with the ρ sums, (see eq. (54)). The index α refers to the intermediate representation $(\omega'_1\omega'_2)_\alpha$.

[b]) Needed only for an isovector interaction.

Eq. (73) has been derived by de-Shalit and Talmi from the expression for the average interaction energy for levels with the same v, t, n and T. Since states with $v = 0$, (and 1), have only a single possible value of J, the interaction energies in these two special cases follow directly from the expression for the average interaction energy. The derivation of this result by means of the quasi-spin formalism is somewhat more general, however. It shows that the n, T dependence of the interaction energy in any state with $t = 0$ (arbitrary v, α and J) has the same general form as that given by eq. (73). A similar result is obtained for the n, T dependence of the interaction energies in almost all states of the R(5) irreducible representations with T-multiplicities $= 1$ (see tables 12-15 of appendix 2). This result can be summarized as follows. For states with

(i) $t = 0$, $v = 0, 2, 4, \ldots$,

(ii) $t = \frac{1}{2}$, $v = 1$,

(iii) $t = 1$ $v = 2, 4, \ldots$; $(n-v) = 4k-2T$ (k = integer); the $\beta = 0$ states of $(\omega_1 1)$,

(iv) $v = 2j+1-2t$; (R(5) representations (tt)),

the general charge-independent interaction energy (diagonal matrix element), has the form

$$\langle j^n\{vt\}TM_T, \alpha JM_J | \sum_{i<k}^{n} V_{ik} | j^n\{vt\}TM_T, \alpha JM_J \rangle$$
$$= A\tfrac{1}{4}(n-2j-1)^2 + BT(T+1) + C + D\tfrac{1}{2}(n-2j-1), \quad (76)$$

where the coefficients A, B and C are functions of v, t, α and J. The coefficient D has the value

$$D = \frac{1}{(2j+1)} \{3 \sum_{\text{even } J} V_J(2J+1) + \sum_{\text{odd } J} V_J(2J+1)\}.$$

For states with $t = \frac{1}{2}$, $v \geqq 3$, an $n-T$ dependent term of the form

$$E(-1)^{\frac{1}{2}n-T}(n-2j-1)(2T+1) \quad (77)$$

may have to be added to the simple terms of eq. (76). This additional term arises through the R(5) Wigner coefficient for the coupling $(\omega_1 \frac{1}{2}) \times (22) \to (\omega_1 \frac{1}{2})$ with $\rho = 2$ (see table 13). It is absent in states with $v = 1$ for which the corresponding reduced matrix element with $\rho = 2$ has the value zero (see table 6b). For states with $v = 2$, $n = 4k$, T odd; $n = 4k+2$, T even; the coefficients A, B and C of eq. (76) can be evaluated from the reduced matrix elements of tables 6c and d. The results can be expressed in terms of the two particle energies in states with $J = 0$ and $J = J_2$ (with $J_2 = 2, 4, \ldots 2j-1$ for $t = 1$, and $J_2 = 1, 3, \ldots 2j$ for $t = 0$), the average two-particle energies defined by eqs. (74) and (75), and additional weighted two-particle energy averages defined by

TABLE 7

Coefficients of the n, T dependent factors for the interaction energies of eq. (76)

$$A\tfrac{1}{4}(n-2j-1)^2+BT(T+1)+C+D\tfrac{1}{2}(n-2j-1)$$

in states with $v=2$; $n=4k$, T odd; $n=4k+2$, T even

$\{v,t\}$	Coeff.	Value
$\{2,0\}$	A	$\frac{1}{j(2j-1)}\left[V_{J_2}+(j-1)(j+1)\bar{V}_{\text{odd}}-\tfrac{1}{2}(j+1)\overline{(V_{\text{odd}}U_{J_2})}-\tfrac{1}{2}(2j+1)V_0+\tfrac{1}{2}(6j-1)j\bar{V}_{\text{even}}+j\overline{(V_{\text{even}}U_{J_2})}\right]$
	B	$\frac{1}{j(2j-1)}\left[-V_{J_2}-(j-1)(j+1)\bar{V}_{\text{odd}}+\tfrac{1}{2}(j+1)\overline{(V_{\text{odd}}U_{J_2})}-\tfrac{1}{2}(2j-1)V_0+\tfrac{1}{2}(2j+1)j\bar{V}_{\text{even}}-j\overline{(V_{\text{even}}U_{J_2})}\right]$
	C	$\frac{1}{4j}\left[(2j+1)V_{J_2}+(j+1)^2(2j-1)\bar{V}_{\text{odd}}+\tfrac{1}{2}(2j-1)(j+1)\overline{(V_{\text{odd}}U_{J_2})}+\tfrac{1}{2}(2j-1)(2j+1)V_0+\tfrac{1}{2}(2j-1)(6j+1)j\bar{V}_{\text{even}}-(2j-1)j\overline{(V_{\text{even}}U_{J_2})}\right]$
$\{2,1\}$	A	$\frac{2}{3j(2j-3)(2j-5)}\left[(2j+3)(V_{J_2}-2j\overline{(V_{\text{even}}U_{J_2})})+2(2j-1)(j-3)(2j\bar{V}_{\text{even}}-V_0)\right]$ $+\frac{1}{6j(2j+1)}\left[(2j-1)(V_0+j\bar{V}_{\text{even}})+2V_{J_2}+2j\overline{(V_{\text{even}}U_{J_2})}\right]+\frac{(j+1)}{2j(2j+1)}\left[(2j-1)V_{\text{odd}}+2\overline{(V_{\text{odd}}U_{J_2})}\right]$
	B	$\frac{-2}{3j(2j+1)(2j-3)(2j-5)}\left[(4j^2-16j+3)(V_{J_2}-2j\overline{(V_{\text{even}}U_{J_2})})+(4j^3-12j^2+5j-6)(V_0-2j\bar{V}_{\text{even}})\right]$ $-\frac{1}{6j(2j+1)}\left[(2j-1)(V_0+j\bar{V}_{\text{even}})+2V_{J_2}+2j\overline{(V_{\text{even}}U_{J_2})}\right]-\frac{(j+1)}{2j(2j+1)}\left[(2j-1)\bar{V}_{\text{odd}}+2\overline{(V_{\text{odd}}U_{J_2})}\right]$ $-\frac{(2j+1)(2j+3)}{12j(2j-3)(2j-5)}\left[2V_{J_2}-4j\overline{(V_{\text{even}}U_{J_2})}+(2j^2-7j+4)(2j\bar{V}_{\text{even}}-V_0)\right]$
	C	$\frac{1}{12j}\left[(4j^2-5j+\tfrac{1}{2})V_0+(22j^2+4j+\tfrac{1}{2})j\bar{V}_{\text{even}}+(6j-1)(V_{J_2}+j\overline{(V_{\text{even}}U_{J_2})})\right]+\frac{(2j+1)(j+1)}{8j}\left[(2j+1)V_{\text{odd}}-2\overline{(V_{\text{odd}}U_{J_2})}\right]$

$\bar{V}_{\text{even}}$, $\bar{V}_{\text{odd}}$, $\overline{(V_{\text{even}}U_{J_2})}$ and $\overline{(V_{\text{odd}}U_{J_2})}$ are defined in eqs. (74), (75), (78) and (79).

$$\overline{(V_{\text{even}}\,U_{J_2})} = \frac{1}{(2j+1)j} \sum_{\text{even } J} V_J(2J+1)U(JjjJ_2;jj), \tag{78}$$

$$\overline{(V_{\text{odd}}\,U_{J_2})} = \frac{1}{(2j+1)(j+1)} \sum_{\text{odd } J} V_J(2J+1)U(JjjJ_2;jj). \tag{79}$$

The results are collected in table 7. (For the states with $v = 2$, $t = 1$ the coefficients have the factors $(2j-3)$, $(2j-5)$ in the denominator. For $j = \frac{3}{2}$, however, the only states with $n-2 = 4k-2T$ are those with $H_1 = \frac{1}{2}(n-2j-1) = 0$, $T = 1$. For $j = \frac{5}{2}$ the only states with $n-2 = 4k-2T$ are those with $H_1 = \pm 1$, $T = 1$, and $H_1 = 0$, $T = 2$. In these special cases the factors $(2j-3)$ or $(2j-5)$ are cancelled by compensating factors in the numerators of eq. (76) so that the interaction energies are finite for $j = \frac{3}{2}$ or $j = \frac{5}{2}$.)

Eq. (76) is valid not only for simple configurations j^n but gives the n, T dependence of the interaction energies (diagonal matrix elements) also for mixed configurations, if the seniorities and reduced isospins listed are replaced by over-all or multi-level seniorities and reduced isospins, and the factors $(n-2j-1)$ are replaced by $(n-2\Omega)$. In a mixed configuration based on single-particle levels $j_a, j_b \ldots$ with corresponding single-level seniorities and reduced isospins $v_a, v_b, \ldots, t_a, t_b, \ldots$, the over-all or multi-level seniorities v and reduced isospins t are given by the possible R(5) representations in the Kronecker product

$$(j_a+\tfrac{1}{2}-\tfrac{1}{2}v_a,\, t_a)\times(j_b+\tfrac{1}{2}-\tfrac{1}{2}v_b,\, t_b)\times \ldots = \sum_{v,\,t}(\Omega-\tfrac{1}{2}v,\, t).$$

Eq. (76) predicts a very simple T-dependence for the interaction energies of the form $T(T+1)$. Since this seems to account for the observed energy systematics of isobaric analogue states [Jänecke [15]], it may be possible that these energy systematics are governed mainly by the low-seniority ($v \leqq 2$) components of the wave functions. Since seniority is in general not a good quantum number in nuclei where both neutrons and protons are filling the same shell, admixtures of higher seniorities may be relatively unimportant as far as the T-dependence of the energies is concerned. For the $v = 0$ states, it is possible to investigate this point. The Kronecker products of $(\omega_1 0) = (j+\frac{1}{2}-\frac{1}{2}v, 0)$ with (20) and (22) are given by

$$(\omega_1 0)\times(20) = (\omega_1+2,0)+(\omega_1-2,0)+(\omega_1+1,1)+(\omega_1-1,1)+(\omega_1 0)+(\omega_1 2), \tag{80}$$

$$(\omega_1 0)\times(22) = (\omega_1+2,2)+(\omega_1-2,2)+(\omega_1+1,2)+(\omega_1+1,1)+(\omega_1-1,2) + (\omega_1-1,1)+(\omega_1 2)+(\omega_1 1)+(\omega_1 0). \tag{81}$$

Thus R(5) tensors of rank (20) connect $v = 0$ states only to states with $v = 4$, $t = 0$ and $v = 2$, $t = 1$, while R(5) tensors of rank (22) connect the $v = 0$ states only to states with $v = 4$, $t = 2$ and $v = 2$, $t = 1$. For simple configurations j^n with $j \leqq \frac{7}{2}$, only the $v = 4$, $t = 0$ states include a state with $J = 0$. The coupling $(\omega_1 0)\times(20) \rightarrow (\omega_1-2,0)$

may thus be the most important in determining the effects of higher-seniority admixtures. The R(5) Wigner coefficient for this coupling has the form

$$\langle(\omega_1 0)H_1 T;(20)00||(\omega_1-2,0)H_1 T\rangle$$
$$= \left[\frac{5(\omega_1-H_1-T)(\omega_1+H_1-T)(\omega_1+1-H_1+T)(\omega_1+1+H_1+T)}{6(\omega_1+1)(\omega_1+2)(2\omega_1+1)(2\omega_1+3)}\right]^{\frac{1}{2}}. \tag{82}$$

The energy difference between the states with $v = 4$, $t = 0$ and $v = 0$, $t = 0$ is of the form $\mathscr{A}(n)+\mathscr{B}T(T+1)$ given by eq. (76). If admixtures of $v = 4$ states can be treated in perturbation theory, these will lead to correction terms to the $v = 0$ states of the following n, T dependent form:

$$\mathscr{C}\frac{\{[(j+\frac{1}{2})^2-H_1^2][(j+\frac{3}{2})^2-H_1^2]-2T(T+1)[(j+\frac{1}{2})(j+\frac{3}{2})+H_1^2]+T^2(T+1)^2\}}{\mathscr{A}(n)+\mathscr{B}T(T+1)}, \tag{83}$$

with $H_1 = \frac{1}{2}(n-2j-1)$. This does include a term of the form $T^2(T+1)^2$. The general T-dependence of the energies of isobaric analogue states may well be given by a series in powers of $T(T+1)$ dominated by the first term. In the case of mixed configurations, the state of overall $v = 0$ may be connected also to states with $v = 2$, $t = 1$. If admixtures of this type can be treated by perturbation theory, they will lead to n, T dependent contributions to the interaction energy of the same form as those given by eqs. (32) and (33) of ref. [11]). It was seen there that effectively these differ little from the simple $T(T+1)$ dependent form.

Since the Coulomb energy of nuclei shows interesting systematic n, T dependent effects [16,17]), the study of the n, T-dependence of the two-body interaction has been extended to include the isovector and isotensor parts of the Coulomb interaction (diagonal matrix elements). Since the needed R(5) Wigner coefficients are available only for the simpler R(5) representations, only states with seniority $v = 0$, $v = 1$, and the ($n = 4k$, T odd), ($n = 4k+2$, T even) states of $v = 2$ will be studied. Since the seniority scheme may be poor for nuclei where both neutrons and protons are filling the same shell, Coulomb energy formulae based on simple configurations j^n and states of lowest possible seniority may be only a guide to the true n, T dependence of the Coulomb energy. Nevertheless, the observed n, T dependent effects in light and intermediate weight nuclei seem to be explained at least qualitatively by the diagonal matrix elements of the Coulomb interaction in states with $v \leqq 2$. The observed effects have been summarized by Jänecke [16,17]) †.

The isovector coefficient of the Coulomb energy shows a simple linear dependence on n for even nuclei. For odd nuclei the same linear dependence on n is observed but now with a superimposed oscillation which distinguishes nuclei with $A = 4k+3$ and $A = 4k+1$. The amplitude of these oscillations is large enough to be clearly observable for states with $T = \frac{1}{2}$ but decreases with increasing T. The isotensor coefficient of the Coulomb energy on the other hand seems to show no marked dependence on

† See ref. [16]) for a review of this subject and for additional references.

nucleon number n but shows an observable pairing effect which distinguishes nuclei with $A = 4k$ and $4k+2$.

The Coulomb interaction has been decomposed into its R(5) irreducible tensor components in eq. (31). The R(5) reduced matrix elements for the Coulomb interaction can be read off from tables 6, if the two-particle interaction energies are interpreted as

$$V_J = \langle j^2 J | \frac{1}{3} \frac{e^2}{r_{12}} | j^2 J \rangle \tag{84}$$

and restricted to states with J even. The R(5) Wigner coefficients for the isovector and tensor components are given in tables 12-15 and 10. With these the diagonal matrix elements of the Coulomb interaction

$$\langle j^n \{vt\} T M_T, J_2 | \sum_{i<k}^{n} V_{ik}^{\mathrm{Coul}} | j^n \{vt\} T M_T, J_2 \rangle \tag{85}$$

can be evaluated for states with $v \leqq 2$ through a generalization of eq. (72). The isoscalar part of the interaction has the form of eqs. (73) and (76). The isovector and tensor parts of the interaction give the following contributions to the diagonal matrix elements (85) of the Coulomb energy:

(i) For $v = 0$, $(t = 0; J = 0)$

$$-M_T\{3a+3b(n-2j-1)\}+[3M_T^2-T(T+1)]\left\{b+c-c\left[\frac{(n-2j-1)^2-(2j+4)^2}{(2T-1)(2T+3)}\right]\right\}; \tag{86}$$

(ii) for $v = 1$ $(t = \frac{1}{2}; J = j)$

$$-M_T\left\{3a+\left[3b+\frac{3c}{2T(T+1)}\right](n-2j-1)-(-1)^{\frac{1}{2}n-T}\frac{3c(2T+1)(2j+3)}{2T(T+1)}\right\}$$
$$+[3M_T^2-T(T+1)]\left\{b+c-c\left[\frac{(n-2j-1)^2-(2j+3)^2}{4T(T+1)}\right]\right\}; \tag{87}$$

(iii) for $v = 2$, $t = 0$, $J_2 = $ odd

$$-M_T\{3a+3b(n-2j-1)\}+[3M_T^2-T(T+1)]\left\{d-e\left[\frac{(n-2j-1)^2-(2j+2)^2}{(2T-1)(2T+3)}\right]\right\}; \tag{88}$$

(iv) for $v = 2$, $t = 1$, $J_2 = $ even; states with $n = 4k$, T odd; or $n = 4k+2$, T even

$$-M_T\left\{3a+\left[3(b+b')+\frac{f}{2T(T+1)}\right](n-2j-1)\right\}$$
$$+[3M_T^2-T(T+1)]\left\{d'+g\frac{(n-2j-1)^2}{(2T-1)(2T+3)}\right.$$
$$\left.+h\frac{(n-2j-1)^2}{4T(T+1)}+\frac{k}{(2T-1)(2T+3)}+\frac{l}{4T(T+1)}\right\}. \tag{89}$$

The coefficients in eqs. (86)–(89) are given by

$$a = j\overline{V}_{\text{even}} + \text{core contributions},$$

$$b = \frac{2j\overline{V}_{\text{even}} - V_0}{2(2j-1)},$$

$$c = \frac{j}{4(j+1)(2j-1)}(V_0 - \overline{V}_{\text{even}}),$$

$$d = \frac{(4j-1)j\overline{V}_{\text{even}} - (j+1)V_0 + 2j\overline{(V_{\text{even}}U_{J_2})}}{4j(2j-1)},$$

$$e = \frac{(j-1)V_0 - j\overline{V}_{\text{even}} + 2j\overline{(V_{\text{even}}U_{J_2})}}{4j(2j-1)},$$

$$b' = \frac{4[2V_0 - 4j\overline{V}_{\text{even}} + (2j-1)(V_{J_2} - 2j\overline{(V_{\text{even}}U_{J_2})})]}{(2j-1)(2j+1)(2j-3)(2j-5)},$$

$$f = \frac{(2j-9)[V_0 - 2j\overline{V}_{\text{even}}] - (2j+3)[V_{J_2} - 2j\overline{(V_{\text{even}}U_{J_2})}]}{(2j-3)(2j-5)} + \frac{4}{(2j-1)}[3j\overline{V}_{\text{even}} - V_0 + j\overline{(V_{\text{even}}U_{J_2})}],$$

$$d' = \frac{[(2j-1)(V_0 + j\overline{V}_{\text{even}}) + 2V_{J_2} + 2j\overline{(V_{\text{even}}U_{J_2})}]}{12j(2j+1)} + \frac{[(20j^3 - 60j^2 - 11j + 24)(2j\overline{V}_{\text{even}} - V_0) + 4(4j^2 + 2j + 3)(V_{J_2} - 2j\overline{(V_{\text{even}}U_{J_2})})]}{12j(2j+1)(2j-3)(2j-5)},$$

$$g = \frac{[(2j-1)(V_0 + j\overline{V}_{\text{even}}) + 2V_{J_2} + 2j\overline{(V_{\text{even}}U_{J_2})}]}{4j(2j+1)} - \frac{[(4j^3 - 12j^2 - 7j + 12)(2j\overline{V}_{\text{even}} - V_0) + 2(4j^2 - 4j + 3)(V_{J_2} - 2j\overline{(V_{\text{even}}U_{J_2})})]}{4j(2j+1)(2j-3)(2j-5)},$$

$$h = -\frac{[(2j-1)(V_0 + j\overline{V}_{\text{even}}) + 2V_{J_2} + 2j\overline{(V_{\text{even}}U_{J_2})}]}{3j(2j+1)} + \frac{[2(2j-1)(j-3)(2j\overline{V}_{\text{even}} - V_0) + (2j+3)(V_{J_2} - 2j\overline{(V_{\text{even}}U_{J_2})})]}{6j(2j-3)(2j-5)},$$

$$k = -(2j+2)^2 g,$$

$$l = \frac{[(4j^2 + 4j - 1)(V_0 + j\overline{V}_{\text{even}}) + 2V_{J_2} + 2j\overline{(V_{\text{even}}U_{J_2})}]}{3j} - (2j+1)(2j+3)\frac{[2(2j^2 - 7j + 1)(2j\overline{V}_{\text{even}} - V_0) + (6j+1)(V_{J_2} - 2j\overline{(V_{\text{even}}U_{J_2})})]}{6j(2j-3)(2j-5)}. \quad (90)$$

The two-particle energy averages are given by eqs. (74) and (78) with V_J given by eq. (84). The Coulomb interaction between nucleons in partly filled shells and the nucleons in the closed shells of the core can make a contribution only to the coefficient a.

In states with $v = 0$ and $v = 2$, $t = 0$ the isovector coefficient of the Coulomb energy (coefficient of the $-M_T$ term) has a simple linear n-dependence given by the coefficient b. In states with $v = 1$ and $v = 2$, $t = 1$ additional small n, T dependent terms are predicted. However, their coefficients (c, b' and f) are small compared with b. The largest of these is the oscillatory term for states with $v = 1$ which, through its dependence on the factor $(-1)^{\frac{1}{2}n-T}$, gives a contribution of opposite sign to states with $n = 4k+3$ and $n = 4k+1$, respectively. The magnitude of this term decreases with increasing T. Several small n, T dependent terms are predicted for the isotensor coefficients of the Coulomb energy (coefficient of the $[3M_T^2-T(T+1)]$ term). However, the coefficients of these terms (c, e, g, h, k and l) are small compared with b, d and d'. The major contribution to the isotensor coefficient of the Coulomb energy should thus be expected to be independent of n and T. The large coefficients are different in states with $v = 2$ and $v = 0$ so that pairing effects can be expected for the isotensor part of the Coulomb energy. All these effects are in essential agreement with the experimentally observed facts [16]). Although Coulomb energy formulae based on states of good seniority with $v \leqq 2$ cannot be expected to give good quantitative results in nuclei where both neutrons and protons are filling the same shells, all of the observed n, T dependent effects in light and intermediate weight nuclei are explained at least qualitatively by the above formulae.

6. Concluding remarks

Although the applications discussed in this work are somewhat limited, it is hoped that they can be considerably extended when algebraic expressions for the R(5) Wigner coefficients involving the representations $(\omega_1 \frac{3}{2})$, $(t+1, t)$ and the remaining states of $(\omega_1 1)$ are added to the coefficients tabulated in appendix 2. Even when large seniority admixtures are important, shell-model calculations can be facilitated if the n, T dependent factors can be given for all matrix elements of interest in the seniority scheme.

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Appendix A

A.1. EXPLICIT CONSTRUCTION OF STATES WITH T-MULTIPLICITY = 1

In order to study the behaviour of the states $|(\omega_1 t)H_1 TM_T\rangle$ under conjugation, it is convenient to give explicit constructions for these states in terms of lowering and raising operators acting on the state of highest weight or some state of maximal weight.

Since states with $M_T < T$ can be constructed through well-known three-dimensional angular momentum techniques, it is sufficient to construct states with $M_T = T$ and to define lowering or raising operators O_{ab} acting on states with $M_T = T$ through the relation

$$O_{ab}|(\omega_1 t)H_1 TT\rangle = f(a, b)|(\omega_1 t)(H_1+a)(T+b)(T+b)\rangle,$$

where $f(a, b)$ is a normalization factor. Operators O_{ab} are constructed as functions of the infinitesimal operators E_{ab} (standard R_5 notation, see table 1). The following have been found most useful:

$$\begin{aligned}
O_{-1,+1} &= E_{-11}, \qquad O_{+1,+1} = E_{11},\\
O_{-1,-1} &= E_{0-1}^2 E_{-11} - E_{0-1}E_{-10}(2T_0+1) - E_{-1-1}T_0(2T_0+1),\\
O_{+1,-1} &= E_{0-1}^2 E_{11} - E_{0-1}E_{10}(2T_0+1) - E_{1-1}T_0(2T_0+1),\\
O_{-1,0} &= E_{-10}(T_0+1) - E_{0-1}E_{-11},\\
O_{+1,0} &= E_{10}(T_0+1) - E_{0-1}E_{11},\\
O_{-2,0} &= E_{-10}^2 + 2E_{-1-1}E_{-11}, \qquad O_{20} = E_{10}^2 + 2E_{1-1}E_{11}.
\end{aligned} \tag{A.1}$$

TABLE 8

Commutators $[X, Y]$ of some lowering and raising operators acting in the subspace of states with $M_T = T$

$X \backslash Y$	E_{-11}	E_{11}	$O_{-1,-1}$	$O_{+1,-1}$	$O_{-2,0}$	$O_{2,0}$
E_{-11}	0	0	$O_{-2,0}(2T_0+1)$	P	0	$-E_{11}(2H_1-2T_0-1)$
E_{11}		0	Q	$O_{20}(2T_0+1)$	$E_{-11}(2H_1+2T_0+1)$	0
$O_{-1,-1}$			0	0	0	$-O_{+1,-1}(2H_1+2T_0+1)$
$O_{+1,-1}$				0	$O_{-1,-1}(2H_1-2T_0-1)$	0
$O_{-2,0}$					0	R
$O_{2,0}$						0

where

$P = -2\mathscr{T}\cdot \boldsymbol{T}+\frac{1}{2}C_5(2T_0+1)-\frac{1}{2}(H_1^2+3T_0^2+3T_0)(2T_0+1)+H_1(2T_0^2+2T_0+\frac{3}{2})$,

$Q = +2\mathscr{T}\cdot \boldsymbol{T}+\frac{1}{2}C_5(2T_0+1)-\frac{1}{2}(H_1^2+3T_0^2+3T_0)(2T_0+1)-H_1(2T_0^2+2T_0+\frac{3}{2})$,

$R = +4\mathscr{T}\cdot \boldsymbol{T}-2H_1C_5+2H_1(H_1^2+T_0^2)+H_1(2T_0-3)$,

with $C_5 = \omega_1(\omega_1+3)+\omega_2(\omega_2+1)$

and where the components of the isovector $\mathscr{T}$ are given by eq. (5)

$$\begin{aligned}
\mathscr{T}_1 &= -\frac{1}{\sqrt{2}}\mathscr{T}_+ = +(E_{10}E_{-11}-E_{-10}E_{11}),\\
\mathscr{T}_{-1} &= \frac{1}{\sqrt{2}}\mathscr{T}_- = +(E_{10}E_{-1-1}-E_{-10}E_{1-1}),\\
\mathscr{T}_0 &= (E_{11}E_{-1-1}-E_{1-1}E_{-11}-T_0).
\end{aligned}$$

Note in particular that $O_{20} = O^{+}_{-20}$ and that O_{20} is the operator which creates an α-like four particle cluster with $T = 0$ built from two $J = 0$-coupled pairs ($O_{20} = -\sqrt{3}\sum_{M_T} \langle 1M_T 1-M_T|00\rangle A^{+}_{(M_T)}A^{+}_{(-M_T)}$). Some useful commutators involving the lowering and raising operators are listed in table 8.

In the representations with small values of t, ($t = 0, \frac{1}{2}, 1$), it is most convenient to generate the full set of states through lowering and raising operations acting on the maximal state $|(\omega_1 t)H_1 TM_T\rangle = |(\omega_1 t)t\omega_1\omega_1\rangle$, whereas for states with larger t, such as (tt), it is more convenient to use the highest-weight state $|(\omega_1 t)H_1 TM_T\rangle = |(\omega_1 t)\omega_1 tt\rangle$ as starting point. The explicit constructions for states with T-multiplicity $= 1$ are listed below.

A.1.1. *The representation* $(\omega_1 0)$

$$|(\omega_1 0)H_1 = \alpha-\beta, T = M_T = (\omega_1-\alpha-\beta)\rangle = N(\alpha, \beta)O^{\beta}_{-1,-1}O^{\alpha}_{+1,-1} \times |(\omega_1 0)0\omega_1\omega_1\rangle, \quad \text{(A.2a)}$$

where †

$$N(\alpha, \beta) = (-1)^{\alpha+\beta}\left[\frac{2^{2\alpha+2\beta}[(\omega_1-\alpha-\beta)!]^2(2\omega_1+1-2\alpha)!(2\omega_1+1-2\beta)!(2\omega_1+1-2\alpha-2\beta)!}{\alpha!\beta!(\omega_1-\alpha)!(\omega_1-\beta)![(2\omega_1+1)!]^3}\right]^{\frac{1}{2}} = N(\beta, \alpha). \quad \text{(A.2b)}$$

A.1.2. *The representation* $(\omega_1 \frac{1}{2})$.

(i) Type e states ($\omega_1+\frac{1}{2}-H_1-T$ = even integer)

$$|(\omega_1 \tfrac{1}{2})\mathrm{e}H_1 = \alpha-\beta+\tfrac{1}{2}, T = M_T = (\omega_1-\alpha-\beta)\rangle = N(\alpha, \beta)O^{\beta}_{-1,-1}O^{\alpha}_{+1,-1}|(\omega_1\tfrac{1}{2})\tfrac{1}{2}\omega_1\omega_1\rangle, \quad \text{(A.3a)}$$

where †

$$N(\alpha, \beta) = (-1)^{\alpha+\beta}\left[\frac{2^{2\alpha+2\beta-2}(\omega_1+\frac{1}{2}-\alpha-\beta)!(\omega_1-\frac{1}{2}-\alpha-\beta)! \times (2\omega_1+1-2\alpha)!(2\omega_1+3-2\beta)!(2\omega_1+1-2\alpha-2\beta)!}{\alpha!\beta!(\omega_1+\frac{1}{2}-\alpha)!(\omega_1+\frac{3}{2}-\beta)!(2\omega_1+2)!(2\omega_1+1)!(2\omega_1)!}\right]^{\frac{1}{2}}. \quad \text{(A.3b)}$$

(ii) Type o states ($\omega_1+\frac{1}{2}-H_1-T$ = odd integer)

$$|(\omega, \tfrac{1}{2})\mathrm{o}H_1 = \alpha-\beta-\tfrac{1}{2}, T = M_T = (\omega_1-\alpha-\beta)\rangle = \mathcal{N}(\alpha, \beta)O^{\beta}_{-1,-1}O^{\alpha}_{+1,-1}|(\omega_1\tfrac{1}{2})-\tfrac{1}{2}\omega_1\omega_1\rangle, \quad \text{(A.4a)}$$

where

$$\mathcal{N}(\alpha, \beta) = N(\beta, \alpha), \quad \text{(A.4b)}$$

[see eq. (A.3b) for $N(\beta, \alpha)$].

† The phases of the normalization factors are somewhat arbitrary. They have been chosen here to be consistent with ref. [11]).

A.1.3. *The representation* $(\omega_1 1)$. The $\kappa = 0$ states only (states with $\omega_1 - H_1 - T$ = even integer)

$$|(\omega_1 1)\kappa = 0, H_1 = \alpha-\beta, T = M_T = (\omega_1-\alpha-\beta)\rangle$$
$$= N(\alpha, \beta)O^{\beta}_{-1,-1}O^{\alpha}_{+1,-1}|(\omega_1 1)\kappa = 0, 0\omega_1\omega_1\rangle, \quad \text{(A.5a)}$$

where

$$N(\alpha, \beta) = N(\beta, \alpha)$$
$$= (-1)^{\alpha+\beta}\left[\frac{\omega_1 2^{2\alpha+2\beta}(\omega_1+1-\alpha-\beta)!(\omega_1-1-\alpha-\beta)!(2\omega_1+1-2\alpha)!\,(2\omega_1+1-2\beta)!(2\omega_1+1-2\alpha-2\beta)!}{(\omega_1+1)\alpha!\beta!(\omega_1-\alpha)!(\omega_1-\beta)![(2\omega_1+1)!]^3}\right]^{\frac{1}{2}}. \quad \text{(A.5b)}$$

A.1.4. *The representation* (tt).

(i) States with (H_1-T) = even integer.

$$|(tt)H_1 = t-x-2y, T = M_T = t-x\rangle = N(x, y)O^{y}_{-20}O^{x}_{-1,-1}|(tt)ttt\rangle, \quad \text{(A.6a)}$$

where

$$N(x, y) = \left[\frac{2^{2x+2y}(2t+1-x)!(2t+1-2x)!(2t-2x-2y)!}{x!(2y)!(2t)![(2t+1)!]^2}\right]^{\frac{1}{2}}. \quad \text{(A.6b)}$$

(ii) States with (H_1-T) = odd integer.

$$|(tt)H_1 = t-1-x-2y, T = M_T = t-x\rangle = \mathcal{N}(x, y)O^{y}_{-20}O^{x}_{-1,-1}O_{-1,0}|(tt)ttt\rangle, \quad \text{(A.7a)}$$

where

$$\mathcal{N}(x, y) = \left[\frac{2^{2x+2y+1}(2t+1-x)!(2t+1-2x)!(2t-1-2x-2y)!}{x!(2y+1)!(2t)![(2t+1)!]^2(t+1)^2}\right]^{\frac{1}{2}}. \quad \text{(A.7b)}$$

By operating on these states with the infinitesimal operators E_{ab}, matrix elements of the infinitesimal operators can be calculated at once.

A.2. CONJUGATION PROPERTIES

The group R(5) is self-adjoint. The irreducible representations and their conjugates are equivalent and the basis vectors of an irreducible representation and their conjugates are simply related. To establish the phases of this relationship the conjugates of the above states are constructed by operating on them with the conjugation operator K. The conjugation operator K has the following properties (see table 1):

$$KJ_{ij}K^{-1} = -J_{ij}, \qquad i, j = 1, \ldots, 5,$$
$$KE_{ab}K^{-1} = -E_{-a,-b}, \qquad KH_1K^{-1} = -H_1, \qquad KT_0K^{-1} = -T_0. \quad \text{(A.8)}$$

For states with T-multiplicity $= 1$ for which the quantum numbers H_1, T and M_T are sufficient to completely specify the states, it follows from the last two equations of (A.8) that

$$K|(\omega_1 t)H_1 TM_T\rangle = c|(\omega_1 t)-H_1 T-M_T\rangle. \tag{A.9}$$

The phase factor c will be denoted by $(-1)^{\eta+\nu+T-M_T}$ [eq. (37)]. The (TM_T) dependent factor has been chosen to be in agreement with the usual angular momentum phase conventions. The following relations are particularly useful in the construction of the states (A.9)

$$KO_{-20}K^{-1} = O_{20}, \tag{A.10}$$

$$\begin{aligned}(KO_{+1,-1}K^{-1})|(\omega_1 t)H_1 T, -T\rangle &= (KO_{+1,-1}K^{-1})\frac{T_-^{2T}}{(2T)!}|(\omega_1 t)H_1 TT\rangle \\ &= \frac{T_-^{2(T-1)}}{[2(T-1)]!}O_{-1,-1}|(\omega_1 t)H_1 TT\rangle,\end{aligned} \tag{A.11}$$

$$(KO_{-1,-1}K^{-1})|(\omega_1 t)H_1 T, -T\rangle = \frac{T_-^{2(T-1)}}{[2(T-1)]!}O_{+1,-1}|(\omega_1 t)H_1 TT\rangle. \tag{A.12}$$

With these relations the conjugates of the basis vectors for the simple irreducible representations with T-multiplicity $= 1$ can be constructed.

A.2.1. *The representations* $(\omega_1 0)$, $(\omega_1 \frac{1}{2})$ *and the* $\kappa = 0$ *states of* $(\omega_1 1)$. In these three cases, the phase factor under conjugation follows from the relation

$$\begin{aligned}K|(\omega_1 t)H_1 TT\rangle &= K(N(\alpha,\beta)O^{\beta}_{-1,-1}O^{\alpha}_{+1,-1}|(\omega_1 t)\bar{H}_1\omega_1\omega_1\rangle) \\ &= (-1)^{\eta(\omega_1,t)}\frac{T_-^{2(\omega_1-\alpha-\beta)}}{[2(\omega_1-\alpha-\beta)]!}N(\beta,\alpha)O^{\alpha}_{-1,-1}O^{\beta}_{+1,-1}|(\omega_1 t)-\bar{H}_1\omega_1\omega_1\rangle \\ &= +(-1)^{\eta(\omega_1,t)}|(\omega_1 t)-H_1, T, -T\rangle,\end{aligned} \tag{A.13}$$

where eqs. (A.11) and (A.12) have been used. (Note the interchange of α and β.) The values of H_1 for the maximal-weight states have been denoted by $\bar{H}_1$, where $\bar{H}_1 = 0$ for the representations $(\omega_1 0)$ and the $\kappa = 0$ states of $(\omega_1 1)$ and $\bar{H}_1 = \frac{1}{2}$ for the representation $(\omega_1 \frac{1}{2})$. The overall-phase factor in the three cases has been fixed through the conventions

$$\begin{aligned}K|(\omega_1 0)0\omega_1\omega_1\rangle &= (-1)^{\eta(\omega_1)}|(\omega_1 0)0\omega_1, -\omega_1\rangle, \\ K|(\omega_1 \tfrac{1}{2})\tfrac{1}{2}\omega_1\omega_1\rangle &= (-1)^{\eta(\omega_1,\frac{1}{2})}|(\omega_1 \tfrac{1}{2})-\tfrac{1}{2}, \omega_1, -\omega_1\rangle, \\ K|(\omega_1 1)0\omega_1\omega_1\rangle &= (-1)^{\eta(\omega_1,1)}|(\omega_1 1)0\omega_1, -\omega_1\rangle.\end{aligned} \tag{A.14}$$

(Note that the over-all choice of phase is always arbitrary to within an (ω_1, t)-dependent factor.) For the representations $(\omega_1 0)$, $(\omega_1 \frac{1}{2})$ and the $\kappa = 0$ states of $(\omega_1 1)$, therefore, the phase factor $(-1)^v$ of eq. (37) has the value $+1$, $(v = 0)$, for *all* states. Note that the conjugation operation exchanges states of type e and type o in the representation $(\omega_1 \frac{1}{2})$.

A.2.2. *The representation* (tt). The two cases $t =$ integer and $t =$ half integer must be treated separately.

(a) $t =$ integer; $H_1 - T =$ even integer.

Through the use of eqs. (A.10) and (A.12) and the over-all phase convention $K|(tt)ttt\rangle = (-1)^{\eta(t)}|(tt)-t, t, -t\rangle$, the conjugation operation gives

$$\begin{aligned} K|(tt)H_1 TT\rangle &= K(N(x, y)O^y_{-20}O^x_{-1,-1}|(tt)ttt\rangle) \\ &= (-1)^\eta N(x, y)\frac{T_-^{2(t-x)}}{[2(t-x)]!}O^x_{+1,-1}O^y_{20}O^t_{-20}\frac{2^t}{(2t)!}|(tt)ttt\rangle. \end{aligned} \tag{A.15}$$

Through the further use of the commutation relations of table 8

$$\begin{aligned} K|(tt)H_1 TT\rangle &= (-1)^\eta N(x, y)\frac{T_-^{2(t-x)}}{[2(t-x)]!}O^x_{+1,-1}O^{t-y}_{-20}\frac{2^{t-2y}(2y)!}{(2t-2y)!}|(tt)ttt\rangle \\ &= (-1)^\eta(-1)^x N(x, t-x-y)\frac{T_-^{2(t-x)}}{[2(t-x)]!}O^{t-y-x}_{-20}O^x_{-1,-1}|(tt)ttt\rangle \\ &= (-1)^{\eta(t)}(-1)^{t-T}|(tt)-H_1, T, -T\rangle. \end{aligned} \tag{A.16}$$

(b) $t =$ integer; $H_1 - T =$ odd integer.

Using similar techniques

$$\begin{aligned} K|(tt)H_1 TT\rangle &= K(\mathcal{N}(x, y)O^y_{-20}O^x_{-1,-1}O_{-1,0}|(tt)ttt\rangle) \\ &= (-1)^\eta(-1)^x\mathcal{N}(x, t-1-x-y)\frac{T_-^{2(t-x)}}{[2(t-x)]!}O^{t-1-x-y}_{-20}O^x_{-1,-1}O_{-1,0}|(tt)ttt\rangle \\ &= (-1)^{\eta(t)}(-1)^{t-T}|(tt)-H_1, T, -T\rangle. \end{aligned} \tag{A.17}$$

(c) $t = \frac{1}{2}$ integer.

For $H_1 - T =$ even integer, using eqs. (A.10) and (A.12)

$$\begin{aligned} K|(tt)H_1 TT\rangle &= K(N(x, y)O^x_{-1,-1}O^y_{-20}|(tt)ttt\rangle) \\ &= (-1)^\eta N(x, y)\frac{T_-^{2(t-x)}}{[2(t-x)]!}O^x_{+1,-1}O^y_{20}O^{t-\frac{1}{2}}_{-20}O_{-1,0}\frac{2^t}{(2t)!(t+1)}|(tt)ttt\rangle. \end{aligned} \tag{A.18}$$

Through the further use of the commutation relations of table 8

$$K|(tt)H_1 TT\rangle$$

$$= (-1)^{\eta}(-1)^{x}\mathcal{N}(x, t-\tfrac{1}{2}-x-y)\frac{T_{-}^{2(t-x)}}{[2(t-x)]!}O_{-1,-1}^{x}O_{-\frac{1}{2}0}^{t-\frac{1}{2}-y-x}O_{-1,0}|(tt)ttt\rangle$$

$$= (-1)^{\eta(t)}(-1)^{t-T}|(tt)-H_1, T, -T\rangle. \tag{A.19}$$

For (H_1-T) = odd integer, an identical result is obtained. For all values of t, therefore, the conjugation properties of the states of the irreducible representation (tt) are given by

$$K|(tt)H_1 TT\rangle = (-1)^{\eta(t)}(-1)^{t-T}|(tt)-H_1, T, -T\rangle,$$

so that the phase factor $(-1)^{\nu}$ of eq. (37) has the value $(-1)^{t-T}$.

Appendix B

TABLES OF R(5) WIGNER AND RACAH COEFFICIENTS

The R(5) Wigner coefficients are tabulated for the couplings $(\omega_1' t')\times(\omega_1'' t'') \to (\omega_1 t)$ with $(\omega_1'' t'') = (\frac{1}{2}\frac{1}{2})$: tables 9, $(\omega_1'' t'') = (10)$: tables 10, $(\omega_1'' t'') = (11)$ tables 11; $(\omega_1'' t'') = (20)$ and (22), Wigner coefficients diagonal in $H_1 T$, tables 12-15. The representations $(\omega_1 t')$ and $(\omega_1 t)$ are restricted to the special cases $(\omega_1 0)$, $(\omega_1 \frac{1}{2})$, (tt) and the $\beta = 0$ states of $(\omega_1 1)$.
The following notation is used:

$$\omega_1 = j+\tfrac{1}{2}-\tfrac{1}{2}v = \Omega-\tfrac{1}{2}v,$$
$$H_1 = \tfrac{1}{2}n-j-\tfrac{1}{2} = \tfrac{1}{2}n-\Omega.$$

The states of the representations $(\omega_1 \frac{1}{2})$ are completely specified by H_1 and T. However, the R(5) Wigner coefficients involving the representation $(\omega_1 \frac{1}{2})$ do depend on the fourth quantum numbers β, where

$$\beta = \mathrm{e} \text{ for } \omega_1+\tfrac{1}{2}-H_1-T = \text{even integer, or } \tfrac{1}{2}(n+v-1)+T = \text{even},$$
$$\beta = \mathrm{o} \text{ for } \omega_1+\tfrac{1}{2}-H_1-T = \text{odd integer, or } \tfrac{1}{2}(n+v-1)+T = \text{odd}.$$

TABLE 9a

The R(5) Wigner coefficients

$\langle(\omega_1 0)H'_1T';\,;(\frac{1}{2}\frac{1}{2})H''_1T''||(\bar{\omega}_1\frac{1}{2})\beta H_1T\rangle$

$H'_1T';\ H''_1T''$	β	$(\bar{\omega}_1\frac{1}{2}) = (\omega_1+\frac{1}{2},\frac{1}{2})$	β	$(\bar{\omega}_1\frac{1}{2}) = (\omega_1-\frac{1}{2},\frac{1}{2})$
$H_1-\frac{1}{2}\ T-\frac{1}{2};\ \frac{1}{2}\frac{1}{2}$	e	$\left[\frac{\omega_1+2+H_1+T}{2\omega_1+3}\right]^{\frac{1}{2}}$	o	$-\left[\frac{\omega_1+1-H_1-T}{2\omega_1+3}\right]^{\frac{1}{2}}$
$H_1+\frac{1}{2}\ T+\frac{1}{2};\ -\frac{1}{2}\frac{1}{2}$	e	$-\left[\frac{\omega_1+1-H_1-T}{2\omega_1+3}\right]^{\frac{1}{2}}$	o	$-\left[\frac{\omega_1+2+H_1+T}{2\omega_1+3}\right]^{\frac{1}{2}}$
$H_1-\frac{1}{2}\ T+\frac{1}{2};\ \frac{1}{2}\frac{1}{2}$	o	$-\left[\frac{\omega_1+1+H_1-T}{2\omega_1+3}\right]^{\frac{1}{2}}$	e	$\left[\frac{\omega_1+2-H_1+T}{2\omega_1+3}\right]^{\frac{1}{2}}$
$H_1+\frac{1}{2}\ T-\frac{1}{2};\ -\frac{1}{2}\frac{1}{2}$	o	$\left[\frac{\omega_1+2-H_1+T}{2\omega_1+3}\right]^{\frac{1}{2}}$	e	$\left[\frac{\omega_1+1+H_1-T}{2\omega_1+3}\right]^{\frac{1}{2}}$

TABLE 9b

$\langle(\omega_1\frac{1}{2})\beta' H'_1T';\ (\frac{1}{2}\frac{1}{2})H''_1T''||(\bar{\omega}_1 0)H_1T\rangle$

$H'_1T';\ H''_1T''$	β'	$(\bar{\omega}_1 0) = (\omega_1+\frac{1}{2}, 0)$	β'	$(\bar{\omega}_1 0) = (\omega_1-\frac{1}{2}, 0)$
$H_1-\frac{1}{2}\ T+\frac{1}{2};\ \frac{1}{2}\frac{1}{2}$	e	$\left[\frac{(T+1)(\omega_1+\frac{1}{2}+H_1-T)}{(2T+1)(2\omega_1+1)}\right]^{\frac{1}{2}}$	o	$\left[\frac{(T+1)(\omega_1+\frac{5}{2}-H_1+T)}{(2T+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$
$H_1-\frac{1}{2}\ T-\frac{1}{2};\ \frac{1}{2}\frac{1}{2}$	o	$\left[\frac{T(\omega_1+\frac{3}{2}+H_1+T)}{(2T+1)(2\omega_1+1)}\right]^{\frac{1}{2}}$	e	$\left[\frac{T(\omega_1+\frac{3}{2}-H_1-T)}{(2T+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$
$H_1+\frac{1}{2}\ T+\frac{1}{2};\ -\frac{1}{2}\frac{1}{2}$	o	$-\left[\frac{(T+1)(\omega_1+\frac{1}{2}-H_1-T)}{(2T+1)(2\omega_1+1)}\right]^{\frac{1}{2}}$	e	$\left[\frac{(T+1)(\omega_1+\frac{5}{2}+H_1+T)}{(2T+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$
$H_1+\frac{1}{2}\ T-\frac{1}{2};\ -\frac{1}{2}\frac{1}{2}$	e	$-\left[\frac{T(\omega_1+\frac{3}{2}-H_1+T)}{(2T+1)(2\omega_1+1)}\right]^{\frac{1}{2}}$	o	$\left[\frac{T(\omega_1+\frac{3}{2}+H_1-T)}{(2T+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$

The coefficients of tables 9a and b have previously been calculated by Ginocchio [12]). They are tabulated here since Ginocchio's coefficients differ from the present ones in relative phase.

TABLE 9c

$\langle(\omega'_1\frac{1}{2})\beta' H'_1T';\ (\frac{1}{2}\frac{1}{2})H''_1T''||(\omega_1 1)\beta = 0,\ H_1T\rangle$

$H'_1\quad T';\quad H''_1T''$	β'	$(\omega'_1\frac{1}{2}) = (\omega_1-\frac{1}{2},\frac{1}{2})$	β'	$(\omega'_1\frac{1}{2}) = (\omega_1+\frac{1}{2},\frac{1}{2})$
$H_1-\frac{1}{2}\ T-\frac{1}{2};\ \ \frac{1}{2}\ \frac{1}{2}$	o	$\left[\frac{(T+1)(\omega_1+1+H_1+T)}{2(\omega_1+1)(2T+1)}\right]^{\frac{1}{2}}$	e	$-\left[\frac{(T+1)(\omega_1+2-H_1-T)}{2(\omega_1+2)(2T+1)}\right]^{\frac{1}{2}}$
$H_1-\frac{1}{2}\ T+\frac{1}{2};\ \ \frac{1}{2}\ \frac{1}{2}$	e	$-\left[\frac{T(\omega_1+H_1-T)}{2(\omega_1+1)(2T+1)}\right]^{\frac{1}{2}}$	o	$\left[\frac{T(\omega_1+3-H_1+T)}{2(\omega_1+2)(2T+1)}\right]^{\frac{1}{2}}$
$H_1+\frac{1}{2}\ T-\frac{1}{2};\ -\frac{1}{2}\ \frac{1}{2}$	e	$\left[\frac{(T+1)(\omega_1+1-H_1+T)}{2(\omega_1+1)(2T+1)}\right]^{\frac{1}{2}}$	o	$\left[\frac{(T+1)(\omega_1+2+H_1-T)}{2(\omega_1+2)(2T+1)}\right]^{\frac{1}{2}}$
$H_1+\frac{1}{2}\ T+\frac{1}{2};\ -\frac{1}{2}\ \frac{1}{2}$	o	$-\left[\frac{T(\omega_1-H_1-T)}{2(\omega_1+1)(2T+1)}\right]^{\frac{1}{2}}$	e	$-\left[\frac{T(\omega_1+3+H_1+T)}{2(\omega_1+2)(2T+1)}\right]^{\frac{1}{2}}$

Table 9d

$\langle(\omega_1 1)\beta' = 0H'_1T';\ (\frac{1}{2}\frac{1}{2})H''_1T''||(\omega_1\frac{1}{2})\beta H_1T\rangle$

$\beta'=0\ H'_1T';\ H''_1T''$	β	$(\bar{\omega}_1\frac{1}{2}) = (\omega_1-\frac{1}{2}, \frac{1}{2})$	β	$(\bar{\omega}_1\frac{1}{2}) = (\omega_1+\frac{1}{2}, \frac{1}{2})$
$0\ H_1-\frac{1}{2}\ T-\frac{1}{2};\ \frac{1}{2}\ \frac{1}{2}$	o	$\left[\frac{(2T-1)(\omega_1+1-H_1-T)(\omega_1+2)}{3(2T+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	e	$\left[\frac{(2T-1)(\omega_1+2+H_1+T)(\omega_1+1)}{3(2T+1)\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$
$0\ H_1-\frac{1}{2}\ T+\frac{1}{2};\ \frac{1}{2}\ \frac{1}{2}$	e	$\left[\frac{(2T+3)(\omega_1+2-H_1+T)(\omega_1+2)}{3(2T+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	o	$\left[\frac{(2T+3)(\omega_1+1+H_1-T)(\omega_1+1)}{3(2T+1)\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$
$0\ H_1+\frac{1}{2}\ T-\frac{1}{2};\ -\frac{1}{2}\ \frac{1}{2}$	e	$\left[\frac{(2T-1)(\omega_1+1+H_1-T)(\omega_1+2)}{3(2T+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	o	$-\left[\frac{(2T-1)(\omega_1+2-H_1+T)(\omega_1+1)}{3(2T+1)\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$
$0\ H_1+\frac{1}{2}\ T+\frac{1}{2};\ -\frac{1}{2}\ \frac{1}{2}$	o	$\left[\frac{(2T+3)(\omega_1+2+H_1+T)(\omega_1+2)}{3(2T+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	e	$-\left[\frac{(2T+3)(\omega_1+1-H_1-T)(\omega_1+1)}{3(2T+1)\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$

Table 9e

$\langle(tt)H'_1T';\ (\frac{1}{2}\frac{1}{2})H''_1T''||(\bar{t}\bar{t})H_1T\rangle$

$H'_1\ \ T';$	$H''_1\ \ T''$	$(\bar{t}\bar{t}) = (t-\frac{1}{2}, t-\frac{1}{2})$	$(\bar{t}\bar{t}) = (t+\frac{1}{2}, t+\frac{1}{2})$
$H_1-\frac{1}{2}\ T-\frac{1}{2};$	$\frac{1}{2}\ \frac{1}{2}$	$-\left[\frac{(t-T+\frac{1}{2})(H_1+T)}{(2t+3)(2T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(t+T+\frac{3}{2})(T+H_1)}{(2t+1)(2T+1)}\right]^{\frac{1}{2}}$
$H_1-\frac{1}{2}\ T+\frac{1}{2};$	$\frac{1}{2}\ \frac{1}{2}$	$\left[\frac{(t+T+\frac{3}{2})(T-H_1+1)}{(2t+3)(2T+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(t-T+\frac{1}{2})(T-H_1+1)}{(2t+1)(2T+1)}\right]^{\frac{1}{2}}$
$H_1+\frac{1}{2}\ T-\frac{1}{2};$	$-\frac{1}{2}\ \frac{1}{2}$	$\left[\frac{(t-T+\frac{1}{2})(T-H_1)}{(2t+3)(2T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(t+T+\frac{3}{2})(T-H_1)}{(2t+1)(2T+1)}\right]^{\frac{1}{2}}$
$H_1+\frac{1}{2}\ T+\frac{1}{2};$	$-\frac{1}{2}\ \frac{1}{2}$	$\left[\frac{(t+T+\frac{3}{2})(T+H_1+1)}{(2t+3)(2T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(t-T+\frac{1}{2})(T+H_1+1)}{(2t+1)(2T+1)}\right]^{\frac{1}{2}}$

The coefficients in table 9e have previously been calculated by Ginocchio [12]). The present ones agree in both phase and magnitude with those of ref. [12]).

Table 10a

$\langle(\omega_1 0)H'_1T';\ (10)H''_1T''||(\bar{\omega}_1 0)H_1T\rangle$

$H'_1\ \ T';$	$H''_1\ T''$	$(\bar{\omega}_1 0) = (\omega_1+1, 0)$	$(\bar{\omega}_1 0) = (\omega_1-1, 0)$
$H_1-1\ T;$	$1\ 0$	$\left[\frac{(\omega_1+1+H_1-T)(\omega_1+2+H_1+T)}{2(\omega_1+1)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\omega_1+1-H_1-T)(\omega_1+2-H_1+T)}{2(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$
$H_1+1\ T;$	$-1\ 0$	$\left[\frac{(\omega_1+1-H_1-T)(\omega_1+2-H_1+T)}{2(\omega_1+1)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\omega_1+1+H_1-T)(\omega_1+2+H_1+T)}{2(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$
$H_1\ T-1;$	$0\ 1$	$\left[\frac{T(\omega_1+2+H_1+T)(\omega_1+2-H_1+T)}{(2T+1)(\omega_1+1)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{T(\omega_1+1+H_1-T)(\omega_1+1-H_1-T)}{(2T+1)(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$
$H_1\ T+1;$	$0\ 1$	$-\left[\frac{(T+1)(\omega_1+1+H_1-T)(\omega_1+1-H_1-T)}{(2T+1)(\omega_1+1)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(T+1)(\omega_1+2+H_1+T)(\omega_1+2-H_1+T)}{(2T+1)(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$

These coefficients agree with those given in table 8, ref. [11]); provided the coefficients in row 2 and column 2 of table 8, ref. [11]) are multiplied by an overall phase factor of -1.

TABLE 10b

$\langle(\omega_1\frac{1}{2})\beta' H'_1 T';\,(10)H''_1 T''||(\bar{\omega}_1\frac{1}{2})\beta H_1 T\rangle$

$\beta' H'_1 T';\ H''_1 T''\beta$	$(\bar{\omega}_1\frac{1}{2}) = (\omega_1-1,\frac{1}{2})$	β	$(\bar{\omega}_1\frac{1}{2}) = (\omega_1+1,\frac{1}{2})$
e H_1-1 T; 1 0 e	$\left[\frac{(\omega_1+\frac{3}{2}-H_1-T)(\omega_1+\frac{3}{2}-H_1+T)}{2(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	e	$\left[\frac{(\omega_1+\frac{1}{2}+H_1-T)(\omega_1+\frac{5}{2}+H_1+T)}{2(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
e H_1 $T+1$; 0 1 e	$\left[\frac{(2T+3)(\omega_1+\frac{3}{2}-H_1+T)(\omega_1+\frac{5}{2}+H_1+T)}{4(T+1)(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	e	$-\left[\frac{(2T+3)(\omega_1+\frac{1}{2}+H_1-T)(\omega_1+\frac{3}{2}-H_1-T)}{4(T+1)(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
o H_1 T; 0 1 e	$-\left[\frac{(\omega_1+\frac{1}{2}+H_1-T)(\omega_1+\frac{3}{2}-H_1+T)}{4T(T+1)(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	e	$\left[\frac{(\omega_1+\frac{5}{2}+H_1+T)(\omega_1+\frac{3}{2}-H_1-T)}{4T(T+1)(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
e H_1 $T-1$; 0 1 e	$-\left[\frac{(2T-1)(\omega_1+\frac{1}{2}+H_1-T)(\omega_1+\frac{3}{2}-H_1-T)}{4T(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	e	$\left[\frac{(2T-1)(\omega_1+\frac{5}{2}+H_1+T)(\omega_1+\frac{3}{2}-H_1+T)}{4T(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
e H_1+1 T; -1 0 e	$\left[\frac{(\omega_1+\frac{1}{2}+H_1-T)(\omega_1+\frac{5}{2}+H_1+T)}{2(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	e	$\left[\frac{(\omega_1+\frac{3}{2}-H_1+T)(\omega_1+\frac{3}{2}-H_1-T)}{2(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$

Coefficients with $\beta = $ o can be obtained from the above through the symmetry relation:

$$\langle(\omega_1\tfrac{1}{2})^{\mathrm{o}}_{\mathrm{e}}H'_1 T';(10)H''_1 T''||(\bar{\omega}_1\tfrac{1}{2})\mathrm{o}H_1 T\rangle = (-)^{\omega_1+1-\bar{\omega}_1}\langle(\omega_1\tfrac{1}{2})^{\mathrm{e}}_{\mathrm{o}}-H'_1 T';(10)-H''_1 T''||(\bar{\omega}_1\tfrac{1}{2})\mathrm{e}-H_1 T\rangle.$$

TABLE 10b (continued)

$\langle(\omega_1\frac{1}{2})\beta' H'_1 T';(10)H''_1 T''||(\bar{\omega}_1\frac{1}{2})\beta H_1 T\rangle$

β' H'_1 T'; $H''_1 T''$	β	$(\bar{\omega}_1\frac{1}{2}) = (\omega_1\frac{1}{2})$
o H_1-1 T; 1 0	e	$-\left[\frac{(\omega_1+\frac{3}{2}+H_1+T)(\omega_1+\frac{5}{2}-H_1+T)}{6(\omega_1+1)(\omega_1+2)}\right]^{\frac{1}{2}}$
o H_1 $T+1$; 0 1	e	$\frac{1}{2}\left[\frac{(2T+3)(\omega_1+\frac{1}{2}-H_1-T)(\omega_1+\frac{5}{2}-H_1+T)}{3(T+1)(\omega_1+1)(\omega_1+2)}\right]^{\frac{1}{2}}$
e H_1 T ; 0 1	e	$\frac{\{H_1+(2T+1)(\omega_1+\frac{3}{2})\}}{2[3T(T+1)(\omega_1+1)(\omega_1+2)]^{\frac{1}{2}}}$
o H_1 $T-1$; 0 1	e	$\frac{1}{2}\left[\frac{(2T-1)(\omega_1+\frac{3}{2}+H_1+T)(\omega_1+\frac{3}{2}+H_1-T)}{3T(\omega_1+1)(\omega_1+2)}\right]^{\frac{1}{2}}$
o H_1+1 T; -1 0	e	$\left[\frac{(\omega_1+\frac{3}{2}+H_1-T)(\omega_1+\frac{1}{2}-H_1-T)}{6(\omega_1+1)(\omega_1+2)}\right]^{\frac{1}{2}}$

Coefficients with $\beta = $ o can be obtained from the above through symmetry relation:

$$\langle(\omega_1\tfrac{1}{2})^{\mathrm{e}}_{\mathrm{o}}H'_1 T';(10)H''_1 T''||(\bar{\omega}_1\tfrac{1}{2})\mathrm{o}H_1 T\rangle$$
$$= (-1)^{\omega_1+1-\bar{\omega}_1}\langle(\omega_1\tfrac{1}{2})^{\mathrm{o}}_{\mathrm{e}}-H'_1 T';(10)-H''_1 T''||(\bar{\omega}_1\tfrac{1}{2})\mathrm{e}-H_1 T\rangle.$$

TABLE 10c

$\langle(\omega_1 1)\beta' = 0H'_1 T'; (10)H''_1 T''||(\bar{\omega}_1 1)\beta = 0 H_1 T\rangle$

β' H'_1 T'; H''_1 T''	$(\bar{\omega}_1 1) = (\omega_1+1, 1)$	$(\bar{\omega}_1 1) = (\omega_1-1, 1)$
0 H_1-1 T; 1 0	$\left[\frac{(\omega_1+2+H_1+T)(\omega_1+1+H_1-T)(\omega_1+1)}{2\omega_1(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\omega_1+1-H_1-T)(\omega_1+2-H_1+T)(\omega_1+2)}{2(\omega_1+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$
0 H_1+1 T; -1 0	$\left[\frac{(\omega_1+2-H_1+T)(\omega_1+1-H_1-T)(\omega_1+1)}{2\omega_1(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\omega_1+1+H_1-T)(\omega_1+2+H_1+T)(\omega_1+2)}{2(\omega_1+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$
0 H_1 $T+1$; 0 1	$-\left[\frac{T(T+2)(\omega_1+1-H_1-T)(\omega_1+1+H_1-T)(\omega_1+1)}{(T+1)(2T+1)\omega_1(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{T(T+2)(\omega_1+2+H_1+T)(\omega_1+2-H_1+T)(\omega_1+2)}{(T+1)(2T+1)(\omega_1+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$
0 H_1 $T-1$; 0 1	$\left[\frac{(T-1)(T+1)(\omega_1+2-H_1+T)(\omega_1+2+H_1+T)(\omega_1+1)}{T(2T+1)\omega_1(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{(T-1)(T+1)(\omega_1+1+H_1-T)(\omega_1+1-H_1-T)(\omega_1+2)}{T(2T+1)(\omega_1+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$
$\sum_{\beta'=1,2}\left\langle\begin{matrix}(\omega_1 1) & (1 0)\\ \beta' H_1 T; & 0 1\end{matrix}\middle\|\begin{matrix}(\bar{\omega}_1 1)\\ \beta=0 H_1 T\end{matrix}\right\rangle^2 =$	$\left\{\frac{(\omega_1+1)(\omega_1+2)^2-(\omega_1+1)H_1^2-(\omega_1+2)T(T+1)}{T(T+1)\omega_1(\omega_1+2)(2\omega_1+3)}\right\}$	$\left\{\frac{(\omega_1+2)(\omega_1+1)^2-(\omega_1+2)H_1^2-(\omega_1+1)T(T+1)}{T(T+1)(\omega_1+1)(\omega_1+3)(2\omega_1+3)}\right\}$

Also

$$\langle(\omega_1 1)\beta' = 0H_1 T;(10)01||(\omega_1 0)H_1 T\rangle = \left[\frac{(\omega_1+1)(\omega_1+2)}{3\omega_1(\omega_1+3)}\right]^{\frac{1}{2}},$$

$$\langle(\omega_1 0)H_1 T; (10)01||(\omega_1 1)\beta = 0H_1 T\rangle = -1,$$

$$\langle(\omega_1 1)\beta' = 0 H_1 T; (10)01||(\omega_1 1)\beta = 0 H_1 T\rangle = \frac{H_1}{[2T(T+1)(\omega_1+1)(\omega_1+2)]^{\frac{1}{2}}}.$$

TABLE 10d

$\langle (tt)H'_1T'; (10)H''_1T''\|(tt)H_1T\rangle$

$H'_1 \quad T' \quad ; H''_1 \; T''$	$(\bar{t}\bar{t}) = (tt)$
$H_1-1 \; T \quad ; \quad 1 \; 0$	$-\left[\dfrac{(T+H_1)(T-H_1+1)}{2t(t+2)}\right]^{\frac{1}{2}}$
$H_1 \quad T+1; \quad 0 \; 1$	$-\left[\dfrac{(t-T)(t+T+2)(T+H_1+1)(T-H_1+1)}{t(t+2)(T+1)(2T+1)}\right]^{\frac{1}{2}}$
$H_1 \quad T \quad ; \quad 0 \; 1$	$\dfrac{H_1(t+1)}{[t(t+2)T(T+1)]^{\frac{1}{2}}}$
$H_1 \quad T-1; \quad 0 \; 1$	$\left[\dfrac{(t-T+1)(t+T+1)(T+H_1)(T-H_1)}{t(t+2)T(2T+1)}\right]^{\frac{1}{2}}$
$H_1+1 \; T \quad ; -1 \; 0$	$\left[\dfrac{(T-H_1)(T+H_1+1)}{2t(t+2)}\right]^{\frac{1}{2}}$

TABLE 11a

$\langle(\omega_1\tfrac{1}{2})\beta' H'_1 T';\,(11)H''_1 T''\|(\bar\omega_1\tfrac{1}{2})\beta H_1 T\rangle_\rho$

β' H'_1 T' ;	H''_1 T''	β	$(\bar\omega_1\tfrac{1}{2})=(\omega_1-1,\tfrac{1}{2})$	$(\bar\omega_1\tfrac{1}{2})=(\omega_1+1,\tfrac{1}{2})$
o H_1-1 $T+1$;	1 1	e	$\left[\frac{(2T+3)(\omega_1+\frac{7}{2}-H_1+T)(\omega_1+\frac{3}{2}-H_1+T)}{6(T+1)(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	0
e H_1-1 T ;	1 1	e	$\left[\frac{T(\omega_1+\frac{3}{2}-H_1-T)(\omega_1+\frac{3}{2}-H_1+T)}{6(T+1)(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	$\left[\frac{(T+1)(\omega_1+\frac{1}{2}+H_1-T)(\omega_1+\frac{5}{2}+H_1+T)}{6T(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
o H_1-1 $T-1$;	1 1	e	0	$\left[\frac{(2T-1)(\omega_1+\frac{1}{2}+H_1+T)(\omega_1+\frac{5}{2}+H_1+T)}{6T(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
e H_1 $T+1$;	0 1	e	$\left[\frac{(2T+3)(\omega_1+\frac{5}{2}+H_1+T)(\omega_1+\frac{3}{2}-H_1+T)}{3\cdot 4(T+1)(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	$-\left[\frac{(2T+3)(\omega_1+\frac{1}{2}+H_1-T)(\omega_1+\frac{3}{2}-H_1-T)}{3\cdot 4(T+1)(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
o H_1 T ;	0 1	e	$\frac{(2T+1)}{2}\left[\frac{(\omega_1+\frac{3}{2}-H_1+T)(\omega_1+\frac{1}{2}+H_1-T)}{3T(T+1)(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	$-\frac{(2T+1)}{2}\left[\frac{(\omega_1+\frac{3}{2}-H_1-T)(\omega_1+\frac{5}{2}+H_1+T)}{3T(T+1)(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
e H_1 $T-1$;	0 1	e	$\left[\frac{(2T-1)(\omega_1+\frac{3}{2}-H_1-T)(\omega_1+\frac{1}{2}+H_1-T)}{3\cdot 4T(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	$-\left[\frac{(2T-1)(\omega_1+\frac{3}{2}-H_1+T)(\omega_1+\frac{5}{2}+H_1+T)}{3\cdot 4T(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
o H_1 T ;	0 0	e	$-\left[\frac{(\omega_1+\frac{3}{2}-H_1+T)(\omega_1+\frac{1}{2}+H_1-T)}{3(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\omega_1+\frac{3}{2}-H_1-T)(\omega_1+\frac{5}{2}+H_1+T)}{3(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
o H_1+1 $T+1$;	−1 1	e	0	$\left[\frac{(2T+3)(\omega_1-\frac{1}{2}-H_1-T)(\omega_1+\frac{3}{2}-H_1-T)}{6(T+1)(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
e H_1+1 T ;	−1 1	e	$\left[\frac{(T+1)(\omega_1+\frac{1}{2}+H_1-T)(\omega_1+\frac{5}{2}+H_1+T)}{6T(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	$\left[\frac{T(\omega_1+\frac{3}{2}-H_1+T)(\omega_1+\frac{3}{2}-H_1-T)}{6(T+1)(\omega_1+2)(2\omega_1+1)}\right]^{\frac{1}{2}}$
o H_1+1 $T-1$;	−1 1	e	$\left[\frac{(2T-1)(\omega_1+\frac{1}{2}+H_1+T)(\omega_1+\frac{5}{2}+H_1-T)}{6T(\omega_1+1)(2\omega_1+5)}\right]^{\frac{1}{2}}$	0

Coefficients with $\beta=$ o can be obtained from the above by the symmetry relation:

$$\langle(\omega_1\tfrac{1}{2})^{\mathrm{o}}_{\mathrm{e}} H'_1 T';\,(11)H''_1 T''\|(\bar\omega_1\tfrac{1}{2})\,\mathrm{o}\,H_1 T\rangle_\rho$$

$$=(-1)^{\omega_1-\bar\omega_1+\nu''}\langle(\omega_1\tfrac{1}{2})^{\mathrm{e}}_{\mathrm{o}} -H'_1 T';\,(11)-H''_1 T''\|(\bar\omega_1\tfrac{1}{2})\,\mathrm{e}\,-H_1 T\rangle_\rho$$

with $\nu''=1$ for $H''_1 T''=00$ and $\nu''=0$ for all other values of $H''_1 T''$.

TABLE 11a (continued)

$\langle(\omega_1 \tfrac{1}{2})\beta' H'_1 T';\ (11)H''_1 T''\|(\bar{\omega}_1 \tfrac{1}{2})\beta H_1 T\rangle_\rho$

β'	$H'_1\ T'$	$;\ H''_1 T''$	β	$(\bar{\omega}_1 \tfrac{1}{2}) = (\omega_1 \tfrac{1}{2});\ \rho = 2$
e	$H_1-1\ \ T+1$	; 1 1	e	$(2\omega_1+9)\left[\frac{(\omega_1+1)(2T+3)(\omega_1+\frac{5}{2}-H_1+T)(\omega_1-\frac{1}{2}+H_1-T)}{3\cdot 8(T+1)(\omega_1+2)(2\omega_1-1)(2\omega_1+7)(\omega_1^2+3\omega_1+\frac{3}{4})}\right]^{\frac{1}{2}}$
o	$H_1-1\ \ T$	; 1 1	e	$\frac{\{T(4\omega_1^2+12\omega_1+3)+(2\omega_1-3)(\omega_1+2)\}[(\omega_1+\frac{5}{2}-H_1+T)(\omega_1+\frac{3}{2}+H_1+T)]^{\frac{1}{2}}}{[3\cdot 8T(T+1)(\omega_1+1)(\omega_1+2)(2\omega_1-1)(2\omega_1+7)(\omega_1^2+3\omega_1+\frac{3}{4})]^{\frac{1}{2}}}$
e	$H_1-1\ \ T-1$	; 1 1	e	$(2\omega_1-3)\left[\frac{(\omega_1+2)(2T-1)(\omega_1+\frac{5}{2}-H_1-T)(\omega_1+\frac{3}{2}+H_1+T)}{3\cdot 8T(\omega_1+1)(2\omega_1-1)(2\omega_1+7)(\omega_1^2+3\omega_1+\frac{3}{4})}\right]^{\frac{1}{2}}$
o	$H_1\ \ T+1$	; 0 1	e	$-\left[\frac{(2T+3)(\omega_1+\frac{5}{2}-H_1+T)(\omega_1+\frac{1}{2}-H_1-T)(\omega_1^2+3\omega_1+\frac{3}{4})}{3(T+1)(\omega_1+1)(\omega_1+2)(2\omega_1-1)(2\omega_1+7)}\right]^{\frac{1}{2}}$
e	$H_1\ \ T$	; 0 1	e	$\frac{\{(\omega_1^2+3\omega_1+\frac{3}{4})[(2\omega_1+3)+2H_1(2T+1)]-5(2\omega_1+3)T(T+1)\}}{2[3T(T+1)(\omega_1+1)(\omega_1+2)(2\omega_1-1)(2\omega_1+7)(\omega_1^2+3\omega_1+\frac{3}{4})]^{\frac{1}{2}}}$
o	$H_1\ \ T-1$	; 0 1	e	$\left[\frac{(2T-1)(\omega_1+\frac{3}{2}+H_1-T)(\omega_1+\frac{3}{2}+H_1+T)(\omega_1^2+3\omega_1+\frac{3}{4})}{3T(\omega_1+1)(\omega_1+2)(2\omega_1-1)(2\omega_1+7)}\right]^{\frac{1}{2}}$
e	$H_1\ \ T$	; 0 0	e	$\frac{\{2(\omega_1^2+3\omega_1+\frac{3}{4})(2T+1)-5H_1(2\omega_1+3)\}}{2[3(\omega_1+1)(\omega_1+2)(2\omega_1-1)(2\omega_1+7)(\omega_1^2+3\omega_1+\frac{3}{4})]^{\frac{1}{2}}}$
e	$H_1+1\ \ T+1$	; −1 1	e	$-(2\omega_1-3)\left[\frac{(\omega_1+2)(2T+3)(\omega_1+\frac{1}{2}-H_1-T)(\omega_1+\frac{7}{2}+H_1+T)}{3\cdot 8(T+1)(\omega_1+1)(2\omega_1-1)(2\omega_1+7)(\omega_1^2+3\omega_1+\frac{3}{4})}\right]^{\frac{1}{2}}$
o	$H_1+1\ \ T$	; −1 1	e	$\frac{\{(2\omega_1-3)(\omega_1+2)-(T+1)(4\omega_1^2+12\omega_1+3)\}[(\omega_1+\frac{1}{2}-H_1-T)(\omega_1+\frac{3}{2}+H_1-T)]^{\frac{1}{2}}}{[3\cdot 8T(T+1)(\omega_1+1)(\omega_1+2)(2\omega_1-1)(2\omega_1+7)(\omega_1^2+3\omega_1+\frac{3}{4})]^{\frac{1}{2}}}$
e	$H_1+1\ \ T-1$	; −1 1	e	$-(2\omega_1+9)\left[\frac{(\omega_1+1)(2T-1)(\omega_1+\frac{1}{2}-H_1+T)(\omega_1+\frac{3}{2}+H_1-T)}{3\cdot 8T(\omega_1+2)(2\omega_1-1)(2\omega_1+7)(\omega_1^2+3\omega_1+\frac{3}{4})}\right]^{\frac{1}{2}}$

Coefficients with $\beta = \mathrm{o}$ can be obtained from the above by the symmetry relation:

$$\langle(\omega_1 \tfrac{1}{2})^{\mathrm{o}}_{\mathrm{e}} H'_1 T';(1\,1)H''_1 T''\|(\bar{\omega}_1 \tfrac{1}{2})\,\mathrm{o}\, H_1 T\rangle_\rho$$

$$= (-)^{\omega_1-\bar{\omega}_1+\nu''}\langle(\omega_1 \tfrac{1}{2})^{\mathrm{e}}_{\mathrm{o}} -H'_1 T';(1\,1)-H''_1 T''\|(\bar{\omega}_1 \tfrac{1}{2})\mathrm{e}-H_1 T\rangle_\rho$$

with $\nu'' = 1$ for $H''_1 T'' = 00$ and $\nu'' = 0$ for all other values of $H''_1 T''$.

TABLE 11b

$\langle(\omega_1 0)H'_1 T'; (11)H''_1 T''\|(\bar{\omega}_1 1)\beta = 0 H_1 T\rangle$

$H'_1\ T'$; $H''_1\ T''$	$(\bar{\omega}_1\, 1) = (\omega_1 - 1, 1)$	$(\bar{\omega}_1\, 1) = (\omega_1 + 1, 1)$
$H_1 - 1\ T$; $1\ 1$	$\left[\frac{(\omega_1+2-H_1+T)(\omega_1+1-H_1-T)}{2(\omega_1+1)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\omega_1+1+H_1-T)(\omega_1+2+H_1+T)}{2(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$
$H_1\ T+1$; $0\ 1$	$\left[\frac{T(\omega_1+2-H_1+T)(\omega_1+2+H_1+T)}{(2T+1)(\omega_1+1)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{T(\omega_1+1-H_1-T)(\omega_1+1+H_1-T)}{(2T+1)(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$
$H_1\ T-1$; $0\ 1$	$\left[\frac{(T+1)(\omega_1+1-H_1-T)(\omega_1+1+H_1-T)}{(2T+1)(\omega_1+1)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{(T+1)(\omega_1+2-H_1+T)(\omega_1+2+H_1+T)}{(2T+1)(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$
$H_1 + 1\ T$; $-1\ 1$	$\left[\frac{(\omega_1+2+H_1+T)(\omega_1+1+H_1-T)}{2(\omega_1+1)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\omega_1+1-H_1-T)(\omega_1+2-H_1+T)}{2(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$

$H'_1\ T'$; $H''_1\ T''$	$(\bar{\omega}_1 1) = (\omega_1 1)$
$H_1 - 1\ T+1$; $1\ 1$	$-\left[\frac{T(\omega_1+H_1-T)(\omega_1+3-H_1+T)}{2(2T+1)(\omega_1+1)(\omega_1+2)}\right]^{\frac{1}{2}}$
$H_1 - 1\ T-1$; $1\ 1$	$-\left[\frac{(T+1)(\omega_1+1+H_1+T)(\omega_1+2-H_1-T)}{2(2T+1)(\omega_1+1)(\omega_1+2)}\right]^{\frac{1}{2}}$
$H_1\ T$; $0\ 1$	$\frac{-H_1}{[(\omega_1+1)(\omega_1+2)]^{\frac{1}{2}}}$
$H_1\ T$; $0\ 0$	$-\left[\frac{T(T+1)}{(\omega_1+1)(\omega_1+2)}\right]^{\frac{1}{2}}$
$H_1 + 1\ T+1$; $-1\ 1$	$\left[\frac{T(\omega_1-H_1-T)(\omega_1+3+H_1+T)}{2(2T+1)(\omega_1+1)(\omega_1+2)}\right]^{\frac{1}{2}}$
$H_1 + 1\ T-1$; $-1\ 1$	$\left[\frac{(T+1)(\omega_1+1-H_1+T)(\omega_1+2+H_1-T)}{2(2T+1)(\omega_1+1)(\omega_1+2)}\right]^{\frac{1}{2}}$

TABLE 11c

$\langle(\omega_1 1)\beta' = 0\, H'_1 T';\, (11)H''_1 T''||(\bar{\omega}_1 0)H_1 T\rangle$

β' H'_1 T' ;	$H''_1 T''$	$(\bar{\omega}_1 0) = (\omega_1 - 1, 0)$	$(\bar{\omega}_1 0) = (\omega_1 + 1, 0)$
0 $H_1 - 1$ T ;	1 1	$-\left[\dfrac{(\omega_1+1-H_1-T)(\omega_1+2-H_1+T)}{6(\omega_1+3)(2\omega_1+3)}\right]$	$-\left[\dfrac{(\omega_1+2+H_1+T)(\omega_1+1+H_1-T)}{6\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$
0 H_1 $T+1$;	0 1	$-\left[\dfrac{(T+2)(\omega_1+2-H_1+T)(\omega_1+2+H_1+T)}{3(2T+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{(T+2)(\omega_1+1-H_1-T)(\omega_1+1+H_1-T)}{3(2T+1)\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$
0 H_1 $T-1$;	0 1	$-\left[\dfrac{(T-1)(\omega_1+1-H_1-T)(\omega_1+1+H_1-T)}{3(2T+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{(T-1)(\omega_1+2-H_1+T)(\omega_1+2+H_1+T)}{3(2T+1)\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$
0 H_1+1 T ;	-1 1	$-\left[\dfrac{(\omega_1+1+H_1-T)(\omega_1+2+H_1+T)}{6(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$-\left[\dfrac{(\omega_1+2-H_1+T)(\omega_1+1-H_1-T)}{6\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$

β' H'_1 T' ;	H''_1 T''	$(\bar{\omega}_1 0) = (\omega_1 0)$
0 H_1-1 $T+1$;	1 1	$\left[\dfrac{(T+2)(\omega_1+3-H_1+T)(\omega_1+H_1-T)}{6(2T+1)\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$
0 H_1-1 $T-1$;	1 1	$\left[\dfrac{(T-1)(\omega_1+2-H_1-T)(\omega_1+1+H_1+T)}{6(2T+1)\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$
0 H_1 T ;	0 1	$\dfrac{H_1}{[3\omega_1(\omega_1+3)]^{\frac{1}{2}}}$
0 H_1 T ;	0 0	$\left[\dfrac{T(T+1)}{3\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$
0 H_1+1 $T+1$;	-1 1	$-\left[\dfrac{(T+2)(\omega_1+3+H_1+T)(\omega_1-H_1-T)}{6(2T+1)\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$
0 H_1+1 $T-1$;	-1 1	$-\left[\dfrac{(T-1)(\omega_1+2+H_1-T)(\omega_1+1-H_1+T)}{6(2T+1)\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$

TABLE 11d

$\langle(\omega_1 1)\beta' = 0\ H'_1T';(11)H''_1T''\|(\bar{\omega}_1 1)\beta = 0\ H_1T\rangle_\rho$

β' H'_1T' ;	H''_1T''	$(\bar{\omega}_1 1) = (\omega_1 1);\ \rho = 2$
$0\ H_1-1\ T+1;$	$1\ 1$	$-\frac{3}{2}\left[\frac{T(T+2)(\omega_1+H_1-T)(\omega_1+3-H_1+T)}{(\omega_1-1)(\omega_1+1)(\omega_1+2)(\omega_1+4)(T+1)(2T+1)}\right]^{\frac{1}{2}}$
$0\ H_1-1\ T-1;$	$1\ 1$	$\frac{3}{2}\left[\frac{(T-1)(T+1)(\omega_1+1+H_1+T)(\omega_1+2-H_1-T)}{(\omega_1-1)(\omega_1+1)(\omega_1+2)(\omega_1+4)T(2T+1)}\right]^{\frac{1}{2}}$
$0\ H_1\ \ T\ ;$	$0\ 1$	$\frac{\{3T(T+1)-(\omega_1+1)(\omega_1+2)\}}{[2T(T+1)(\omega_1-1)(\omega_1+1)(\omega_1+2)(\omega_1+4)]^{\frac{1}{2}}}$
$0\ H_1\ \ T\ ;$	$0\ 0$	$\frac{3H_1}{[2(\omega_1-1)(\omega_1+1)(\omega_1+2)(\omega_1+4)]^{\frac{1}{2}}}$
$0\ H_1+1\ T+1;$	$-1\ 1$	$-\frac{3}{2}\left[\frac{T(T+2)(\omega_1-H_1-T)(\omega_1+3+H_1+T)}{(\omega_1-1)(\omega_1+1)(\omega_1+2)(\omega_1+4)(T+1)(2T+1)}\right]^{\frac{1}{2}}$
$0\ H_1+1\ T-1;$	$-1\ 1$	$\frac{3}{2}\left[\frac{(T-1)(T+1)(\omega_1+1-H_1+T)(\omega_1+2+H_1-T)}{(\omega_1-1)(\omega_1+1)(\omega_1+2)(\omega_1+4)T(2T+1)}\right]^{\frac{1}{2}}$

$\beta'\ H'_1T'$	$;H''_1T''$	$(\bar{\omega}_1 1) = (\omega-1\ 1)$	$(\bar{\omega}_1 1) = (\omega_1+1,\ 1)$
$0\ H_1-1\ T$	$;\ 1\ 1$	$-\frac{1}{2}\left[\frac{(\omega_1+2)(\omega_1+1-H_1-T)(\omega_1+2-H_1+T)}{T(T+1)(\omega_1+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\frac{1}{2}\left[\frac{(\omega_1+1)(\omega_1+1+H_1-T)(\omega_1+2+H_1+T)}{T(T+1)\omega_1(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$
$0\ H_1+1\ T$	$;-1\ 1$	$\frac{1}{2}\left[\frac{(\omega_1+2)(\omega_1+1+H_1-T)(\omega_1+2+H_1+T)}{T(T+1)(\omega_1+1)(\omega_1+3)(2\omega_1+3)}\right]$	$-\frac{1}{2}\left[\frac{(\omega_1+1)(\omega_1+1-H_1-T)(\omega_1+2-H_1+T)}{T(T+1)\omega_1(\omega_1+2)(2\omega_1+3)}\right]^{\frac{1}{2}}$
$0\ H_1\ T\pm1$	$;\ \ 0\ 1$	0	0

TABLE 11e

$\langle (tt)H'_1T'; (11)H''_1T''\|(\bar{t}\bar{t})H_1T\rangle$

H'_1T';	H''_1T''	$(\bar{t}\bar{t}) = (tt)$	$(\bar{t}\bar{t}) = (t+1, t+1)$	$(\bar{t}\bar{t}) = (t-1, t-1)$
$H_1-1\ T+1$;	1 1	$-\left[\frac{(t-T)(t+T+2)(T-H_1+2)(T-H_1+1)}{2t(t+2)(2T+1)(2T+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(t-T)(t-T+1)(T-H_1+1)(T-H_1+2)}{2(t+1)(2t+1)2(T+1)(2T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(t+T+2)(t+T+1)(T-H_1+1)(T-H_1+2)}{2(t+1)(2t+3)2(T+1)(2T+1)}\right]^{\frac{1}{2}}$
$H_1-1\ T$;	1 1	$(t+1)\left[\frac{(T+H_1)(T-H_1+1)}{2t(t+2)2T(T+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(t+T+2)(t-T+1)(T+H_1)(T-H_1+1)}{2(t+1)(2t+1)2T(T+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(t-T)(t+T+1)(T+H_1)(T-H_1+1)}{2(t+1)(2t+3)2T(T+1)}\right]^{\frac{1}{2}}$
$H_1-1\ T-1$;	1 1	$-\left[\frac{(t+T+1)(t-T+1)(T+H_1)(T+H_1-1)}{2t(t+2)(2T+1)2T}\right]^{\frac{1}{2}}$	$\left[\frac{(t+T+1)(t+T+2)(T+H_1)(T+H_1-1)}{2(t+1)(2t+1)2T(2T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(t-T)(t-T+1)(T+H_1-1)(T+H_1)}{2(t+1)(2t+3)2T(2T+1)}\right]^{\frac{1}{2}}$
$H_1\ T+1$;	0 1	0	$-\left[\frac{(t-T)(t-T+1)(T+H_1+1)(T-H_1+1)}{2(t+1)(2t+1)(2T+1)(T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(t+T+2)(t+T+1)(T+H_1+1)(T-H_1+1)}{2(t+1)(2t+3)(2T+1)(T+1)}\right]^{\frac{1}{2}}$
$H_1\ T$;	0 1	$\left[\frac{T(T+1)}{2t(t+2)}\right]^{\frac{1}{2}}$	$H_1\left[\frac{(t-T+1)(t+T+2)}{2(t+1)(2t+1)T(T+1)}\right]^{\frac{1}{2}}$	$-H_1\left[\frac{(t-T)(t+T+1)}{2(t+1)(2t+3)T(T+1)}\right]^{\frac{1}{2}}$
$H_1\ T-1$;	0 1	0	$\left[\frac{(t+T+1)(t+T+2)(T+H_1)(T-H_1)}{2(t+1)(2t+1)T(2T+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(t-T+1)(t-T)(T+H_1)(T-H_1)}{2(t+1)(2t+3)T(2T+1)}\right]^{\frac{1}{2}}$
$H_1\ T$;	0 0	$\frac{H_1}{[2t(t+2)]^{\frac{1}{2}}}$	$\left[\frac{(t-T+1)(t+T+2)}{2(t+1)(2t+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(t-T)(t+T+1)}{2(t+1)(2t+3)}\right]^{\frac{1}{2}}$
$H_1+1\ T+1$;	-1 1	$\left[\frac{(t-T)(t+T+2)(T+H_1+2)(T+H_1+1)}{2t(t+2)((2T+1)(2T+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(t-T)(t-T+1)(T+H_1+1)(T+H_1+2)}{2(t+1)(2t+1)2(T+1)(2T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(t+T+2)(t+T+1)(T+H_1+1)(T+H_1+2)}{2(t+1)(2t+3)2(T+1)(2T+1)}\right]^{\frac{1}{2}}$
$H_1+1\ T$;	-1 1	$(t+1)\left[\frac{(T-H_1)(T+H_1+1)}{2t(t+2)2T(T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(t-T+1)(t+T+2)(T-H_1)(T+H_1+1)}{2(t+1)(2t+1)2T(T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(t-T)(t+T+1)(T-H_1)(T+H_1+1)}{2(t+1)(2t+3)2T(T+1)}\right]^{\frac{1}{2}}$
$H_1+1\ T-1$,	-1 1	$\left[\frac{(t+T+1)(t-T+1)(T-H_1)(T-H_1-1)}{2t(t+2)(2T+1)2T}\right]^{\frac{1}{2}}$	$\left[\frac{(t+T+1)(t+T+2)(T-H_1)(T-H_1-1)}{2(t+1)(2t+1)2T(2T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(t-T)(t-T+1)(T-H_1-1)(T-H_1)}{2(t+1)(2t+3)2T(2T+1)}\right]^{\frac{1}{2}}$

TABLE 12

$\langle(\omega_1 0)H_1 T;(20)0T''\|(\omega_1 0)H_1 T\rangle$

H''_1	T''	Coefficient
0	0	$\dfrac{\{\omega_1(\omega_1+3)+5H_1^2-5T(T+1)\}}{3[\omega_1(\omega_1+3)(2\omega_1+1)(2\omega_1+5)]^{\frac{1}{2}}}$
0	2	$-\left[\dfrac{5T(T+1)(2T-1)(2T+3)}{\omega_1(\omega_1+3)(2\omega_1+1)(2\omega_1+5)}\right]\left\{\dfrac{2H_1^2-2T(T+1)-(2\omega^2_1+6\omega_1+3)}{3(2T-1)(2T+3)}\right\}$

$\langle(\omega_1 0)H_1 T;(22)0T''\|(\omega_1 0)H_1 T\rangle$

H''_1	T''	Coefficient
0	0	$\dfrac{\{4H_1^2+2T(T+1)-\omega_1(\omega_1+3)\}}{3[\omega_1(\omega_1-1)(\omega_1+3)(\omega_1+4)]^{\frac{1}{2}}}$
0	1	$-H_1\left[\dfrac{6T(T+1)}{\omega_1(\omega_1-1)(\omega_1+3)(\omega_1+4)}\right]^{\frac{1}{2}}$
0	2	$\left[\dfrac{2T(T+1)(2T-1)(2T+3)}{\omega_1(\omega_1-1)(\omega_1+3)(\omega_1+4)}\right]^{\frac{1}{2}}\left\{\dfrac{H_1^2+5T(T+1)-(\omega^2_1+3\omega_1+6)}{3(2T-1)(2T+3)}\right\}$

TABLE 13

$\langle (\omega_1 \tfrac{1}{2})^{\mathrm{e}}_{\mathrm{o}} H_1 T;\, (20)\, 0\, T'' \| (\omega_1 \tfrac{1}{2})^{\mathrm{e}}_{\mathrm{o}} H_1 T \rangle$

H''_1	T''	Coefficient
0	0	$\dfrac{\{5H_1^2-5T(T+1)+(\omega^2_1+3\omega_1+\frac{3}{4}\}}{3[(\omega_1+1)(\omega_1+2)(2\omega_1-1)(2\omega_1+7)]^{\frac{1}{2}}}$
0	2	$\left[\dfrac{5T(T+1)(2T-1)(2T+3)}{(\omega_1+1)(\omega_1+2)(2\omega_1-1)(2\omega_1+7)}\right]^{\frac{1}{2}} \left\{\dfrac{T(T+1)-H_1^2+(\omega_1+\frac{3}{2})^2}{6T(T+1)}\right\}$

$\langle (\omega_1 \tfrac{1}{2})^{\mathrm{e}}_{\mathrm{o}} H_1 T;\, (22)\, 0\, T'' \| (\omega_1 \tfrac{1}{2})^{\mathrm{e}}_{\mathrm{o}} H_1 T \rangle_\rho$

H''_1	T''	ρ	Coefficient
0	0	1	$\dfrac{2\{4H_1^2+2T(T+1)-(\omega^2_1+3\omega_1+\frac{3}{4})\}}{3[(2\omega_1-1)(2\omega_1+7)(\omega_1^2+3\omega_1+\frac{1}{4})]^{\frac{1}{2}}}$
0	0	2	$-\dfrac{\{7(2\omega_1+3)[4H_1^2+2T(T+1)]\mp 3H_1(2T+1)(4\omega^2_1+12\omega_1+1)-(2\omega_1+3)(4\omega^2_1+12\omega_1+\frac{9}{2})\}}{3[2(\omega_1+1)(\omega_1+2)(2\omega_1-3)(2\omega_1-1)(2\omega_1+7)(2\omega_1+9)(\omega^2_1+3\omega_1+\frac{1}{4})]^{\frac{1}{2}}}$
0	1	1	$\dfrac{\{4H_1[3T(T+1)-\frac{1}{4}]\mp(\omega_1+\frac{3}{2})(2T+1)\}}{[6T(T+1)(2\omega_1-1)(2\omega_1+7)(\omega^2_1+3\omega_1+\frac{1}{4})]^{\frac{1}{2}}}$
0	1	2	$\dfrac{\{2H_1(2\omega_1+3)[(\omega_1+1)(\omega_1+2)-21T(T+1)]\pm 7(\omega_1+\frac{3}{2})^2(2T+1)\pm(2T+1)(2\omega^2_1+6\omega_1+\frac{1}{2})[3H_1^2+3T(T+1)-(\omega_1+\frac{3}{2})^2]\}}{2[3T(T+1)(\omega_1+1)(\omega_1+2)(2\omega_1-3)(2\omega_1-1)(2\omega_1+7)(2\omega_1+9)(\omega^2_1+3\omega_1+\frac{1}{4})]^{\frac{1}{2}}}$
0	2	1	$\left[\dfrac{(2T-1)(2T+3)T(T+1)}{2(2\omega_1-1)(2\omega_1+7)(\omega^2_1+3\omega_1+\frac{1}{4})}\right]^{\frac{1}{2}} \left\{\dfrac{H_1^2+5T(T+1)-(\omega_1+\frac{3}{2})^2}{3T(T+1)}\right\}$
0	2	2	$-\dfrac{[(2T-1)(2T+3)]^{\frac{1}{2}}\{7(2\omega_1+3)[H_1^2+5T(T+1)-(\omega_1+\frac{3}{2})^2]-(4\omega^2_1+12\omega_1+1)[\frac{3}{2}(2\omega_1+3)\pm 3H_1(2T+1)]\}}{12[T(T+1)(\omega_1+1)(\omega_1+2)(2\omega_1-3)(2\omega_1-1)(2\omega_1+7)(2\omega_1+9)(\omega^2_1+3\omega_1+\frac{1}{4})]^{\frac{1}{2}}}$

The quantum number ρ has been chosen so that the coefficients with $\rho = 1$, T'' even, are even functions of H_1.

TABLE 14a

$$\langle(\omega_1 1)\beta = 0 H_1 T; (20) 0 T'' \| (\omega_1 1)\beta = 0 H_1 T\rangle_\rho$$

H''_1	T''	ρ	Coefficient
0	0	1	$-\frac{2}{3}\left[\frac{(\omega_1+1)(\omega_1+2)}{3(2\omega^2_1+6\omega_1-1)}\right]^{\frac{1}{2}}$
0	0	2	$\frac{[3(2\omega^2_1+6\omega_1-1)]^{\frac{1}{2}}}{[2(\omega_1-1)(\omega_1+1)(\omega_1+2)(\omega_1+4)(2\omega_1+1)(2\omega_1+5)]^{\frac{1}{2}}}\left\{H_1{}^2-T(T+1)+\frac{(\omega_1+1)(\omega_1+2)(2\omega_1{}^2+6\omega_1+13)}{9(2\omega^2_1+6\omega_1-1)}\right\}$
0	2	1	$-\frac{1}{3}\left[\frac{(\omega_1+1)(\omega_1+2)5(2T-1)(2T+3)}{3T(T+1)(2\omega^2_1+6\omega_1-1)}\right]^{\frac{1}{2}}$
0	2	2	$\frac{[3(2\omega^2_1+6\omega_1-1)]^{\frac{1}{2}}\left\{9+(2T-1)(2T+3)\left[(2\omega^2_1+6\omega_1+7)-2H_1{}^2+2T(T+1)-\frac{(\omega_1+1)(\omega_1+2)(4\omega^2_1+12\omega_1-37)}{9(2\omega^2_1+6\omega_1-1)}\right] -3T(T+1)[(2\omega^2_1+6\omega_1+7)-2H_1{}^2+2T(T+1)]\right\}}{[10T(T+1)(2T-1)(2T+3)(\omega_1-1)(\omega_1+1)(\omega_1+2)(\omega_1+4)(2\omega_1+1)(2\omega_1+5)]^{\frac{1}{2}}}$

TABLE 14b

The ρ sums

$$\sum_{\rho} \langle(\omega_1 1)\beta = 0H_1 T;\ (22)0T''\|(\omega_1 1)\beta = 0H_1 T\rangle_\rho\ U\begin{pmatrix}(\omega_1 1)(11);\ (\omega'_1\omega'_2)_\alpha - \\ (11)(\omega_1 1);\ (22) - \rho\end{pmatrix}$$

H''_1	T''	$(\omega'_1\omega'_2)_\alpha$	Value of ρ sum
0	0	$(\omega_1 0)$	$-\dfrac{\{2H_1^2+4T(T+1)-(\omega_1+1)(\omega_1+2)\}}{3[6\omega_1(\omega_1+1)(\omega_1+2)(\omega_1+3)]^{\frac{1}{2}}}$
0	0	$(\omega_1-1, 0)$	$\dfrac{\{2H_1^2-2T(T+1)-(\omega_1+1)\}}{3[6(\omega_1+1)(\omega_1+3)(2\omega_1+1)(2\omega_1+3)]^{\frac{1}{2}}}$
0	0	$(\omega_1+1, 0)$	$\dfrac{\{2H_1^2-2T(T+1)+(\omega_1+2)\}}{3[6\omega_1(\omega_1+2)(2\omega_1+3)(2\omega_1+5)]^{\frac{1}{2}}}$
0	1	$(\omega_1 0)$	$-\dfrac{H_1\{6T(T+1)-1\}}{6[T(T+1)\omega_1(\omega_1+1)(\omega_1+2)(\omega_1+3)]^{\frac{1}{2}}}$
0	1	$(\omega_1-1, 0)$	$\dfrac{H_1[(2\omega_1+1)]^{\frac{1}{2}}}{6[T(T+1)(\omega_1+1)(\omega_1+3)(2\omega_1+3)]^{\frac{1}{2}}}$
0	1	$(\omega_1+1, 0)$	$-\dfrac{H_1[(2\omega_1+5)]^{\frac{1}{2}}}{6[T(T+1)\omega_1(\omega_1+2)(2\omega_1+3)]^{\frac{1}{2}}}$
0	2	$(\omega_1 0)$	$-\dfrac{\{H_1^2[10T(T+1)-3]-\omega_1(\omega_1+3)[2T(T+1)+3]+[2T^2(T+1)^2-7T(T+1)-6]\}}{6[T(T+1)(2T-1)(2T+3)3\omega_1(\omega_1+1)(\omega_1+2)(\omega_1+3)]^{\frac{1}{2}}}$
0	2	$(\omega_1-1, 0)$	$\dfrac{\{9\omega^2_1-H_1^2[8T(T+1)+3]+\omega_1[16T(T+1)+15]+[8T^2(T+1)^2+13T(T+1)+6]\}}{6[T(T+1)(2T-1)(2T+3)3(\omega_1+1)(\omega_1+3)(2\omega_1+1)(2\omega_1+3)]^{\frac{1}{2}}}$
0	2	$(\omega_1+1, 0)$	$\dfrac{\{9\omega^2_1-H_1^2[8T(T+1)+3]-\omega_1[16T(T+1)-39]+[8T^2(T+1)^2-35T(T+1)+42]\}}{6[T(T+1)(2T-1)(2T+3)3\omega_1(\omega_1+2)(2\omega_1+3)(2\omega_1+5)]^{\frac{1}{2}}}$

Wigner coefficients for specific values of ρ can be obtained with the aid of the U-coefficients of table 16. A very arbitrary choice of ρ is implied by this choice of U-coefficients.

TABLE 15

$\langle (tt)H_1T;\ (20)0T''\|(tt)H_1T\rangle$

H''_1	T''	Coefficient
0	0	$-\dfrac{\{5T(T+1)-5H_1^2-2t(t+2)\}}{[6t(t+2)(2t-1)(2t+5)]^{\frac{1}{2}}}$
0	2	$-\dfrac{\sqrt{10}\{H_1^2[T(T+1)-3(t+1)^2]+T(T+1)[(t+1)^2-T(T+1)+\frac{1}{2}]\}}{[3t(t+2)(2t-1)(2t+5)T(T+1)(2T-1)(2T+3)]^{\frac{1}{2}}}$

$\langle (tt)H_1T;\ (22)0T''\|(tt)H_1T\rangle$

H''_1	T''	Coefficient
0	0	$\dfrac{\{2H_1^2+T(T+1)-t(t+2)\}}{[3t(t+2)(2t-1)(2t+5)]^{\frac{1}{2}}}$
0	1	$\dfrac{H_1\{3T(T+1)-(t+1)^2\}}{[2T(T+1)t(t+2)(2t-1)(2t+5)]^{\frac{1}{2}}}$
0	2	$\dfrac{\{H_1^2[T(T+1)-3(t+1)^2]+T(T+1)[(t+1)^2+5T(T+1)-4]\}}{[6T(T+1)(2T-1)(2T+3)t(t+2)(2t-1)(2t+5)]^{\frac{1}{2}}}$

TABLE 16
Some U-coefficients

$U\begin{pmatrix}(\omega_1 0)(\frac12\frac12);\ (\omega'_1\omega'_2)\\ (\frac12\frac12)(\omega_1 0);\ (\omega''_1\omega''_2)\end{pmatrix}$

$(\omega'_1\omega'_2)$ \ $(\omega''_1\omega''_2)$	(00)	(11)
$(\omega_1+\frac12, \frac12)$	$\left[\frac{(\omega_1+3)}{(2\omega_1+3)}\right]^{\frac12}$	$\left[\frac{\omega_1}{(2\omega_1+3)}\right]^{\frac12}$
$(\omega_1-\frac12, \frac12)$	$\left[\frac{\omega_1}{(2\omega_1+3)}\right]^{\frac12}$	$-\left[\frac{(\omega_1+3)}{(2\omega_1+3)}\right]^{\frac12}$

$U\begin{pmatrix}(\omega_1 0)(\frac12\frac12);\ (\omega'_1\omega'_2)\\ (\frac12\frac12)(\omega_1 1);\ (\omega''_1\omega''_2)\end{pmatrix}$

$(\omega'_1\omega'_2)$ \ $(\omega''_1\omega''_2)$	(10)	(11)
$(\omega_1+\frac12, \frac12)$	$-\left[\frac{(\omega_1+2)}{(2\omega_1+3)}\right]^{\frac12}$	$\left[\frac{(\omega_1+1)}{(2\omega_1+3)}\right]^{\frac12}$
$(\omega_1-\frac12, \frac12)$	$\left[\frac{(\omega_1+1)}{(2\omega_1+3)}\right]^{\frac12}$	$\left[\frac{(\omega_1+2)}{(2\omega_1+3)}\right]^{\frac12}$

$U\begin{pmatrix}(\omega_1 \frac12)(\frac12\frac12);\ (\omega'_1\omega'_2)-\\ (\frac12\frac12)(\omega_1 \frac12);\ (\omega''_1\omega''_2)\rho_{1,23}\end{pmatrix}$

$(\omega'_1\omega'_2)$ \ $(\omega''_1\omega''_2)$	(00)	(10)	$(11)\rho_{1,23}=1$	$(11)\rho_{1,23}=2$
$(\omega_1+\frac12, 0)$	$-\frac12\left[\frac{(\omega_1+2)}{2\omega_1+1}\right]^{\frac12}$	$-\frac12\left[\frac{3(\omega_1+1)}{2\omega_1+1}\right]^{\frac12}$	$\frac{-(2\omega_1-3)[(\omega_1+2)]^{\frac12}}{4[(2\omega_1+1)(\omega^2_1+3\omega_1+\frac34)]^{\frac12}}$	$\frac14\left[\frac{3(\omega_1+1)(2\omega_1-1)(2\omega_1+7)}{(2\omega_1+1)(\omega^2_1+3\omega_1+\frac34)}\right]^{\frac12}$
$(\omega_1-\frac12, 0)$	$-\frac12\left[\frac{(\omega_1+1)}{(2\omega_1+5)}\right]^{\frac12}$	$\frac12\left[\frac{3(\omega_1+2)}{(2\omega_1+5)}\right]^{\frac12}$	$\frac{(2\omega_1+9)[(\omega_1+1)]^{\frac12}}{4[(2\omega_1+5)(\omega^2_1+3\omega_1+\frac34)]^{\frac12}}$	$\frac14\left[\frac{3(\omega_1+2)(2\omega_1-1)(2\omega_1+7)}{(2\omega_1+5)(\omega^2_1+3\omega_1+\frac34)}\right]^{\frac12}$

$$U\begin{pmatrix}(\omega_1\frac12)(\frac12\frac12);\ (\omega_1+\frac12 0)\\ (\frac12\frac12)(\omega_1+1, \frac12);\ (10)\end{pmatrix} = -\frac12 \qquad U\begin{pmatrix}(\omega_1\frac12)(\frac12\frac12);\ (\omega_1+\frac12 0)\\ (\frac12\frac12)(\omega_1+1, \frac12);(11)\end{pmatrix} = \frac{\sqrt3}{2}$$

$$U\begin{pmatrix}(\omega_1\frac12)(\frac12\frac12);\ (\omega_1-\frac12 0)\\ (\frac12\frac12)(\omega_1-1, \frac12);\ (10)\end{pmatrix} = \frac12 \qquad U\begin{pmatrix}(\omega_1\frac12)(\frac12\frac12);\ (\omega_1-\frac12 0)\\ (\frac12\frac12)(\omega_1-1, \frac12);\ (11)\end{pmatrix} = \frac{\sqrt3}{2}$$

$$U\begin{pmatrix}(\omega_1 1)(\frac12\frac12);\ (\omega_1+\frac12\frac12)\\ (\frac12\frac12)(\omega_1+1, 1);\ (10)\end{pmatrix} = -\frac{1}{\sqrt3} \qquad U\begin{pmatrix}(\omega_1 1)(\frac12\frac12);\ (\omega_1+\frac12, \frac12)\\ (\frac12\frac12)(\omega_1+1, 1);\ (11)\end{pmatrix} = \sqrt{\frac23}$$

$$U\begin{pmatrix}(\omega_1 1)(\frac12\frac12);\ (\omega_1-\frac12, \frac12)\\ (\frac12\frac12)(\omega_1-1, 1);\ (10)\end{pmatrix} = \frac{1}{\sqrt3} \qquad U\begin{pmatrix}(\omega_1 1)(\frac12\frac12);\ (\omega_1-\frac12\frac12)\\ (\frac12\frac12)(\omega_1-1, 1);\ (11)\end{pmatrix} = \sqrt{\frac23}$$

TABLE 16 (continued)

$U\begin{pmatrix}(\omega_1 1)(\frac{1}{2}\frac{1}{2});\ (\omega'_1\omega'_2)\\ (\frac{1}{2}\frac{1}{2})(\omega_1 0);\ (\omega''_1\omega''_2)\end{pmatrix}$

$(\omega'_1\omega'_2)$ \ $(\omega''_1\omega''_2)$	(10)	(11)
$(\omega_1+\frac{1}{2}, \frac{1}{2})$	$-\left[\frac{(\omega_1+2)}{(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\omega_1+1)}{(2\omega_1+3)}\right]^{\frac{1}{2}}$
$(\omega_1-\frac{1}{2}, \frac{1}{2})$	$\left[\frac{(\omega_1+1)}{(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\omega_1+2)}{(2\omega_1+3)}\right]^{\frac{1}{2}}$

$U\begin{pmatrix}(tt)(\frac{1}{2}\frac{1}{2});\ (\omega'_1\omega'_2)\\ (\frac{1}{2}\frac{1}{2})(tt);\ (\omega''_1\omega''_2)\end{pmatrix}$

$(\omega'_1\omega''_2)$ \ $(\omega''_1\omega''_2)$	(00)	(10)	(11)
$(t+\frac{1}{2}, t+\frac{1}{2})$	$\left[\frac{(t+2)}{2(2t+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{t}{2(2t+1)}\right]^{\frac{1}{2}}$	$\left[\frac{t}{(2t+1)}\right]^{\frac{1}{2}}$
$(t-\frac{1}{2}, t-\frac{1}{2})$	$-\left[\frac{t}{2(2t+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(t+2)}{2(2t+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(t+2)}{(2t+3)}\right]^{\frac{1}{2}}$

$U\begin{pmatrix}(\omega_1 1)(\frac{1}{2}\frac{1}{2});\ (\omega'_1\omega'_2)-\\ (\frac{1}{2}\frac{1}{2})(\omega_1 1);\ (\omega''_1\omega''_2)\ \rho_{1,23}\end{pmatrix}$

$(\omega'_1\omega'_2)$ \ $(\omega''_1\omega''_2)$	(00)	(10)	$(11)\rho_{1,23}=1$	$(11)\rho_{1,23}=2$
$(\omega_1-\frac{1}{2}, \frac{1}{2})$	$-\left[\frac{(\omega_1+1)(\omega_1+2)}{3(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\frac{(\omega_1+2)\sqrt{2}}{[3(\omega_1+3)(2\omega_1+3)]^{\frac{1}{2}}}$	$\frac{(\omega_1+5)}{[3(\omega_1+3)(2\omega_1+3)]^{\frac{1}{2}}}$	$-\left[\frac{2(\omega_1-1)(\omega_1+4)}{3(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$
$(\omega_1+\frac{1}{2}, \frac{1}{2})$	$-\left[\frac{(\omega_1+1)(\omega_1+2)}{3\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\frac{-(\omega_1+1)\sqrt{2}}{[3\omega_1(2\omega_1+3)]^{\frac{1}{2}}}$	$\frac{-(\omega_1-2)}{[3\omega_1(2\omega_1+3)]^{\frac{1}{2}}}$	$-\left[\frac{2(\omega_1-1)(\omega_1+4)}{3\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$

$U\begin{pmatrix}(\omega_1\frac{1}{2})(11);\ (\omega'_1\omega'_2)-\\ (11)(\omega_1\frac{1}{2});\ (22)\rho_{1,23}\end{pmatrix}$

$(\omega'_1\omega'_2)$ \ $(\omega''_1\omega''_2)$	$(22)\rho_{1,23}=1$	$(22)\rho_{1,23}=2$
$(\omega_1+1, \frac{1}{2})$	$\frac{-(2\omega_1-5)[2\omega_1-1]^{\frac{1}{2}}}{3[2(2\omega_1+1)(4\omega_1^2+12\omega_1+1)]^{\frac{1}{2}}}$	$\frac{1}{3}\left[\frac{(\omega_1+1)(2\omega_1-1)(2\omega_1-3)(2\omega_1+9)}{(\omega_1+2)(2\omega_1+1)(4\omega_1^2+12\omega_1+1)}\right]^{\frac{1}{2}}$
$(\omega_1-1, \frac{1}{2})$	$\frac{-(2\omega_1+11)[2\omega_1+7]^{\frac{1}{2}}}{3[2(2\omega_1+5)(4\omega_1^2+12\omega_1+1)]^{\frac{1}{2}}}$	$-\frac{1}{3}\left[\frac{(\omega_1+2)(2\omega_1-3)(2\omega_1+7)(2\omega_1+9)}{(\omega_1+1)(2\omega_1+5)(4\omega_1^2+12\omega_1+1)}\right]^{\frac{1}{2}}$

$$U\begin{pmatrix}(\omega_1 1)(11);\ (\omega'_1\omega'_2)\rho_{12}\rho_{12,3} \\ (11)(\omega_1 1);\ (\omega''_1\omega''_2)\rho_{1,23}\end{pmatrix}^{a)}$$

$(\omega'_1\omega'_2)$ \ $(\omega''_1\omega''_2)$	(00)	(10)	$(11)\rho_{1,23}=1$
$(\omega_1 0)$	$\left[\frac{(\omega_1+1)(\omega_1+2)}{30\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$	$\frac{-1}{3[\omega_1(\omega_1+3)]^{\frac{1}{2}}}$	$\frac{-4}{3[2\omega_1(\omega_1+3)]^{\frac{1}{2}}}$
$(\omega_1+1, 0)$	$\left[\frac{(\omega_1+2)(2\omega_1+5)}{30\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\frac{1}{3}\left[\frac{(\omega_1+1)(2\omega_1+5)}{\omega_1(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\frac{(\omega_1-2)[2\omega_1+5]^{\frac{1}{2}}}{3[2\omega_1(\omega_1+1)(2\omega_1+3)]^{\frac{1}{2}}}$
$(\omega_1-1, 0)$	$\left[\frac{(\omega_1+1)(2\omega_1+1)}{30(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$-\frac{1}{3}\left[\frac{(\omega_1+2)(2\omega_1+1)}{(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\frac{-(\omega_1+5)[2\omega_1+1]^{\frac{1}{2}}}{3[2(\omega_1+2)(\omega_1+3)(2\omega_1+3)]^{\frac{1}{2}}}$

$(\omega'_1\omega'_2)$ \ $(\omega''_1\omega''_2)$	$(11)\rho_{1,23}=2$	$(20)\rho_{1,23}=1$	$(20)\rho_{1,23}=2$
$(\omega_1 0)$	$\frac{1}{3}\left[\frac{(\omega_1-1)(\omega_1+4)}{\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$	$\frac{-14(\omega_1+1)(\omega_1+2)}{9[10\omega_1(\omega_1+3)(2\omega^2_1+6\omega_1-1)]^{\frac{1}{2}}}$	$-\frac{1}{9}\left[\frac{5(\omega_1-1)(\omega_1+4)(2\omega_1+1)(2\omega_1+5)}{\omega_1(\omega_1+3)(2\omega^2_1+6\omega_1-1)}\right]^{\frac{1}{2}}$
$(\omega_1+1, 0)$	$\frac{1}{3}\left[\frac{(\omega_1-1)(\omega_1+4)(2\omega_1+5)}{\omega_1(\omega_1+1)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\frac{(\omega_1+2)(16\omega_1+1)[2\omega_1+5]^{\frac{1}{2}}}{9[10\omega_1(\omega_1+1)(2\omega_1+3)(2\omega^2_1+6\omega_1-1)]^{\frac{1}{2}}}$	$\frac{1}{9}\left[\frac{5(\omega_1-1)(\omega_1+1)(\omega_1+4)(2\omega_1+1)}{\omega_1(2\omega_1+3)(2\omega^2_1+6\omega_1-1)}\right]^{\frac{1}{2}}$
$(\omega_1-1, 0)$	$\frac{1}{3}\left[\frac{(\omega_1-1)(\omega_1+4)(2\omega_1+1)}{(\omega_1+2)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\frac{(\omega_1+1)(16\omega_1+47)[2\omega_1+1]^{\frac{1}{2}}}{9[10(\omega_1+2)(\omega_1+3)(2\omega_1+3)(2\omega^2_1+6\omega_1-1)]^{\frac{1}{2}}}$	$\frac{1}{9}\left[\frac{5(\omega_1-1)(\omega_1+2)(\omega_1+4)(2\omega_1+5)}{(\omega_1+3)(2\omega_1+3)(2\omega^2_1+6\omega_1-1)}\right]^{\frac{1}{2}}$

a) Only some of the U-coefficients are given. The full matrix for this case, for example, is 12×12. The three missing columns correspond to $(\omega''_1\omega''_2) = (21)\ \rho_{1,23} = 1, 2, 3$. The nine missing rows correspond to $(\omega'_1\omega'_2) = (\omega_1+1, 2)$, $(\omega_1-1, 2)$, $(\omega_1 2)$, $(\omega_1+1, 1)$, $(\omega_1-1, 1)$; and $(\omega'_1\omega'_2) = (\omega_1\omega_2)$ with $\rho_{12} = 1$, $\rho_{12,3} = 1$ and 2, and with $\rho_{12} = 2$, $\rho_{12,3} = 1$ and 2.

TABLE 16 (continued)

$(\omega'_1\omega'_2) \backslash (\omega''_1\omega''_2)$	$(22)\rho_{1,23} = 1$	$(22)\rho_{1,23} = 2$
$(\omega_1 0)$	$-\frac{1}{3}\left[\frac{7(\omega^2_1+3\omega_1-2)}{2\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$	0 [a]
$(\omega_1+1, 0)$	$\frac{(3\omega_1-2)[(\omega_1+1)(2\omega_1+5)]^{\frac{1}{2}}}{3[14\omega_1(2\omega_1+3)(\omega^2_1+3\omega_1-2)]^{\frac{1}{2}}}$	$\frac{-(\omega_1+2)[(\omega_1+1)(\omega_1-1)(\omega_1+4)(2\omega_1+1)(2\omega_1+5)]^{\frac{1}{2}}}{6[7\omega_1(2\omega_1+3)(\omega^2_1+3\omega_1-2)(3\omega^3_1+30\omega^2_1+91\omega_1+79)]^{\frac{1}{2}}}$
$(\omega_1-1, 0)$	$\frac{(3\omega_1+11)[(\omega_1+2)(2\omega_1+1)]^{\frac{1}{2}}}{3[14(\omega_1+3)(2\omega_1+3)(\omega^2_1+3\omega_1-2)]^{\frac{1}{2}}}$	$\frac{2}{3}\left[\frac{(\omega_1-1)(\omega_1+4)(3\omega^3_1+30\omega^2_1+91\omega_1+79)}{7(\omega_1+2)(\omega_1+3)(2\omega_1+3)(\omega^2_1+3\omega_1-2)}\right]^{\frac{1}{2}}$

$(\omega'_1\omega'_2) \backslash (\omega''_1\omega''_2)$	$(22)\rho_{1,23} = 3$
$(\omega_1 0)$	0 [a]
$(\omega_1+1, 0)$	$\frac{1}{6}\left[\frac{5(\omega_1-1)(\omega_1-2)(\omega_1+4)(\omega_1+5)(2\omega_1+3)}{\omega_1(\omega_1+1)(3\omega^3_1+30\omega^2_1+91\omega_1+79)}\right]^{\frac{1}{2}}$
$(\omega_1-1, 0)$	0 [a]

[a]) The quantum numbers ρ for the coupling $(\omega_1 1)\times(22) \rightarrow (\omega_1 1)$ have been chosen such that the U-coefficients with $\rho = 3$ and $(\omega'_1\omega'_2) = (\omega_1 0)$, $(\omega_1-1, 0)$ and with $\rho = 2$, $(\omega'_1\omega'_2) = (\omega_1 0)$, are equal to zero.

TABLE 16 (continued)

$$U\begin{pmatrix}(\omega_1 1)(10); (\omega'_1\omega'_2)- \\ (10)(\omega_1 1); (\omega''_1\omega''_2)\rho_{1,23}\end{pmatrix}$$

$(\omega'_1\omega'_2)$ \ $(\omega''_1\omega''_2)$	(00)	$(11)\rho_{12,3}=1$	$(11)\rho_{1,23}=2$
$(\omega_1 0)$	$\left[\frac{(\omega_1+1)(\omega_1+2)}{15\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{3}{2\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\omega_1-1)(\omega_1+4)}{3\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$
$(\omega_1-1, 1)$	$\left[\frac{(2\omega_1+1)(\omega_1-1)(\omega_1+2)}{5\omega_1(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\omega_1-1)(\omega_1+3)(2\omega_1+1)}{2\omega_1(\omega_1+1)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(2\omega_1+1)(\omega_1+4)}{\omega_1(\omega_1+1)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$
$(\omega_1+1, 1)$	$\left[\frac{(\omega_1+1)(\omega_1+4)(2\omega_1+5)}{5\omega_1(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{\omega_1(\omega_1+4)(2\omega_1+5)}{2(\omega_1+2)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\omega_1-1)(2\omega_1+5)}{\omega_1(\omega_1+2)(\omega_1+3)(2\omega_1+3)}\right]^{\frac{1}{2}}$

$(\omega'_1\omega'_2)$ \ $(\omega''_1\omega''_2)$	$(20)\rho_{1,23}=1$	$(20)\rho_{1,23}=2$
$(\omega_1 0)$	$\left[\frac{3(2\omega^2_1+6\omega_1-1)}{10\omega_1(\omega_1+3)}\right]^{\frac{1}{2}}$	0 [a]
$(\omega_1-1, 1)$	$\frac{-(2\omega_1+9)[(\omega_1-1)(\omega_1+1)(2\omega_1+1)]^{\frac{1}{2}}}{3[10\omega_1(\omega_1+3)(2\omega_1+3)(2\omega^2_1+6\omega_1-1)]^{\frac{1}{2}}}$	$\frac{(\omega_1+2)[5\omega_1(\omega_1+4)(2\omega_1+5)]^{\frac{1}{2}}}{3[(\omega_1+1)(\omega_1+3)(2\omega_1+3)(2\omega^2_1+6\omega_1-1)]^{\frac{1}{2}}}$
$(\omega_1+1, 1)$	$\frac{-(2\omega_1-3)[(\omega_1+2)(\omega_1+4)(2\omega_1+5)]^{\frac{1}{2}}}{3[10\omega_1(\omega_1+3)(2\omega_1+3)(2\omega^2_1+6\omega_1-1)]^{\frac{1}{2}}}$	$\frac{(\omega_1+1)[5(\omega_1-1)(\omega_1+3)(2\omega_1+1)]^{\frac{1}{2}}}{3[\omega_1(\omega_1+2)(2\omega_1+3)(2\omega^2_1+6\omega_1-1)]^{\frac{1}{2}}}$

[a]) The quantum numbers ρ for the coupling $(\omega_1 1)\times(20)\rightarrow(\omega_1 1)$ have been chosen such that this U-coefficient is equal to zero.

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