

The Structure of the Level Surfaces of a Lyapunov Function

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Let f be a real-valued C^1 function which is defined on Euclidian space R^n . We are interested in characterizing the noncritical level surfaces of f near its isolated relative maxima and minima. The technique which is used for this investigation is to study the relationship between the trajectories of a differential equation and its Lyapunov function. As an application of interest, we obtain characterizations of the level surfaces of a Lyapunov function and of the domain of asymptotic stability of an asymptotically stable critical point. The domain of asymptotic stability is diffeomorphic to R^n , and the level surfaces are manifolds (as smooth as the defining function) which are homotopically equivalent to the $(n - 1)$ -sphere S^{n-1} . It follows from the generalized Poincaré conjecture that the level surfaces are spheres if $n \neq 4, 5$. When $n = 5$, the problem of whether or not the level surface is homeomorphic to the sphere is equivalent to the Poincaré conjecture. The paper concludes with a discussion of similar statements for asymptotically stable sets and nonautonomous systems.

1. ASYMPTOTIC STABILITY IN THE LARGE

We shall say that a function is differentiable of class C^r ($0 \leq r \leq \infty$) if its first r derivatives exist and are continuous. Consider the differential equation

$$dx/dt = F(x), \quad F(0) = 0, \quad (1)$$

defined on R^n , where F is C^1 on $R^n - 0$ and satisfies a Lipschitz condition at 0. The trajectories of this equation satisfy the uniqueness condition and vary differentially with respect to their initial points on $R^n - 0$. Con-

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sequently, if a subset A of $R^n - 0$ is translated for finite time along the trajectories of (1), then the image set is diffeomorphic to A . Denote the trajectory through the point x by $f(x, t)$.

We shall say that 0 is an *asymptotically stable* critical point of (1) if for every neighborhood U of 0 , there is a neighborhood U' of 0 such that every trajectory which begins at a point x of U' satisfies

1. $f(x, t)$ is in U for all positive t .
2. $\lim_{t \rightarrow +\infty} f(x, t) = 0$.

If moreover, for any compact set K in R^n and for any neighborhood U of 0 there is a positive constant T such that $f(K, T)$ is contained in U , then we say that 0 is asymptotically stable *in the large*. By a *Lyapunov function* V for (1) on a domain D which contains 0 , we mean a C^∞ function $V : D \rightarrow R$ that satisfies

1. $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$.
2. $\dot{V}(x) = (d/dt) Vf(x, t)|_{t=0} = \mathbf{grad} V_x \cdot F(x) < 0$ on $D - 0$.
3. V tends to a constant value at the boundary of D (possibly infinite).

The relationship between a Lyapunov function and stability is described in the following theorem, whose proof is due to several authors. A proof for V continuous and differentiable along trajectories is given in [7]. The full differentiability of V is established in [4] and [6].

THEOREM 1.1. *A necessary and sufficient condition for 0 to be asymptotically stable in the domain D is that there exist a Lyapunov function for (1) on D .*

For any constant c [$0 < c < \sup_D V(x)$], we define the level surface $V_c = V^{-1}(c)$ of V . The Condition 2 in the definition of a Lyapunov function and the Implicit-Function Theorem together imply that V_c is a C^∞ manifold. Any two manifolds which are embedded in D transverse to the trajectories of (1) have a diffeomorphism defined between them by the flow. Hence it makes sense to speak of the diffeomorphism class of the level surfaces of the Lyapunov functions of (1) on D without specifying a particular function.

We shall now discuss the topological notion of homotopy. Let f and g be continuous mappings of the space X into the space Y . Then we say that f and g are *homotopic* if there exists a continuous mapping $h : X \times I \rightarrow Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all x in X . We say that X and Y are *homotopically equivalent* if there are mappings $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that fg and gf are homotopic to the respective identity mappings. The subset A of X is called a *deformation retract* of X if there is a mapping $f : X \times I \rightarrow X$ such that $f|X \times 0$ is the identity, $f|X \times 1$ has its image in A , and $f|A \times t$ is the identity on A for all t . In this case, it is clear that A and X are homotopically equivalent. Since homotopy equivalence is an

equivalence relation, it follows that if A and B are deformation retracts of X , then A and B are homotopically equivalent (see [2], for instance). A homotopy sphere is a manifold which is homotopically equivalent to a sphere of the same dimension. The Poincaré conjecture asserts that a homotopy sphere is homeomorphic to a real sphere. It is true for differentiable manifolds of dimensions 1, 2, and ≥ 5 , and unsolved for dimensions 3 and 4[5]. Our spheres will always have the standard differentiable structure, since it is induced by the embedding in R^n .

THEOREM 1.2. *Let $V : R^n \rightarrow R$ be a Lyapunov function for (1). Then $V_c \times R$ is diffeomorphic to $S^{n-1} \times R$, and consequently V_c is a homotopy sphere.*

Proof. Each trajectory of (1) crosses V_c exactly once. Using the parameterization one obtains a diffeomorphism of $V_c \times R$ onto $R^n - 0$. But $R^n - 0$ is diffeomorphic to $S^{n-1} \times R$. The last assertion follows from the above discussion.

COROLLARY 1.3. *If $n \neq 4, 5$, then V_c is diffeomorphic to S^{n-1} .*

By [1; Theorem 2], every four-dimensional homotopy sphere can occur as the level surface of a Lyapunov function. Thus the question of whether or not V_c is homeomorphic to S^4 is equivalent to the Poincaré conjecture when $n = 5$.

Now let M be a compact subset of R^n which is invariant under the system

$$dx/dt = F(x) \tag{2}$$

where F is a C^1 function on R^n . The analogous definition of asymptotic stability, asymptotic stability in the large, and Lyapunov function can be made for M . If M is asymptotically stable in the large under (2), then we can define a new system on S^n by taking ∞ as a critical point, and smoothing $F(x)$ to 0 as $x \rightarrow \infty$ [this will not change the trajectories of (2) as pointsets; it will just change the velocity of the motion along the trajectories]. The phrase "in the large" means that ∞ is asymptotically stable in the negative sense. By Theorem 1.1, we can find a C^∞ Lyapunov function W for ∞ defined on $S^n - M$, and having supremum $+1$. $V(x) = 1 - W(x)$ is a Lyapunov function for M . Thus we have

THEOREM 1.4. *If M is asymptotically stable in the large for (2), and V is a Lyapunov function, then V_c is a homotopy sphere. If $n \neq 4, 5$, then V_c is diffeomorphic to a sphere.*

THEOREM 1.5. *A necessary and sufficient condition for M to be an invariant set which is asymptotically stable in the large for some differential equation, is that $R^n - M$ be diffeomorphic to $S^{n-1} \times R$.*

Proof. Necessity will follow from the local version of Theorem 1.2 (see Theorem 2.2). For sufficiency, let $h : S^{n-1} \times R \rightarrow R^n - M$ be a diffeomorphism. Define the desired differential equation by $F(0) = 0$ for x in M , and $F(x) = d(x, M) \cdot v_x$ for x not in M [$d(x, M)$ denotes the distance from x to M and v_x is the unit tangent vector to the curve $h(p, t)$ which passes through x].

COROLLARY 1.6. *A periodic solution of (2) cannot be asymptotically stable in the large.*

2. LOCAL ASYMPTOTIC STABILITY

Suppose that 0 is an asymptotically stable solution of (1) on a domain D which is possibly smaller than R^n . Our previous analysis will apply if we can show that the domain of stability D is diffeomorphic to R^n . We shall use

LEMMA 2.1 (Brown–Stallings) [5]. *Let M be a paracompact manifold such that every compact subset is contained in an open set which is diffeomorphic to R^n . Then M is diffeomorphic to R^n .*

THEOREM 2.2. *The domain of asymptotic stability of a critical point of (1) is diffeomorphic to R^n .*

Proof. Let K denote an arbitrary compact subset of D , and let U denote an open ball at 0 which is contained in D . Then there is a $T > 0$ such that $f(U, -T)$ is a neighborhood of K . Therefore the conditions of the lemma are satisfied, and D is diffeomorphic to R^n .

COROLLARY 2.3. *The level surfaces of a Lyapunov function for (1) on D are homotopy spheres and hence spheres if $n \neq 4, 5$.*

THEOREM 2.4. *If f is a C^1 function on R^n which has an isolated critical point p which is a relative maximum or minimum, then the level surfaces of f near p are homotopy spheres, and hence spheres if $n \neq 4, 5$.*

Proof. If f is C^2 , then $\mathbf{grad} f$ is a C^1 vector field on R^n which is asymptotically stable (unstable) at p . Therefore the above results apply. If f is only C^1 , then $\mathbf{grad} f$ is only C^0 , and may not have unique trajectories. However, we could draw the same conclusion if we could find a system like (1) which has f for a Lyapunov function. Certainly any system which approximates $\mathbf{grad} f$ to within $\frac{1}{2} |\mathbf{grad} f_x|$ would do. But by an adaptation of the Stone–Weierstrass theorem, we can always find such a system.

3. ASYMPTOTICALLY STABLE SETS

In this section, we shall be interested in the case of a closed subset A which is asymptotically stable under (2). In the case where A is not compact, we can add a compactness condition to the convergence as follows: A is *uniformly asymptotically stable* if there is a constant $r > 0$ such that for each $\epsilon > 0$, there is a $T(\epsilon)$ such that $d(f(x, t), A) < \epsilon$ when $d(x, A) < r$ and $t > T(\epsilon)$. It is easy to verify that compact asymptotically stable sets always have this property.

It will be necessary to use more powerful topological machinery in the discussions which follow, and no attempt will be made to keep this section self-contained from the topological viewpoint. We begin by restating the problem in a more general context. Let M^n be a paracompact manifold on which a vector field F is given. Let A be a closed subset of M^n which is uniformly asymptotically stable on a domain D under the flow $f(x, t)$ which is induced by F . The following generalization of Theorem 1.1 is proved in [6].

THEOREM 3.1. *A necessary and sufficient condition for a closed subset A to be uniformly asymptotically stable on the domain D under F , is that there exist a C^∞ Lyapunov function for F and A defined on D .*

THEOREM 3.2. *A necessary and sufficient condition for a closed subset A to be a uniformly asymptotically stable subset under some flow, is that there exist a neighborhood D of A and a diffeomorphism $h : N \times R \rightarrow D - A$ such that N is a C^∞ manifold, and $\lim_{t \rightarrow \infty} d(h(p, t), A) = 0$ for all $p \in N$.*

Proof. Necessity of the condition is clear by our previous arguments, Suppose that $h : N \times R \rightarrow D - A$ is given as described. Then there is a differentiable flow $f(x, t)$ whose trajectories are the pointsets $h(p \times R)$, defined by

$$f(x, t) = h[h^{-1}(x)_1, h^{-1}(x)_2 + t],$$

where $h^{-1}(x) = [h^{-1}(x)_1, h^{-1}(x)_2] \in N \times R$. Note that if $j : N \times R \rightarrow N \times R$ is any diffeomorphism which carries $p \times R$ onto $p \times R$ in an increasing manner for all $p \in N$, then there is a flow $g(x, t)$ which is induced by jh . The trajectories of $f(x, t)$ and $g(x, t)$ coincide as directed pointsets, and hence have the same limit sets, i.e., j gives another parameterization of the same trajectories. Now for each $p \in N$, there is a function $j_p(t, \epsilon)$ such that $d[f(h(p, 0), t), A] < \epsilon$ when $t > j_p(t, \epsilon)$. It is possible to define a diffeomorphism $j : N \times R \rightarrow N \times R$ such that $j(p, t) \geq j_p(t, 1/t)$ for all $p \in N$ and $t \geq 1$. Let $g(x, t)$ be the flow which is induced by jh . Then A is uniformly asymptotically stable under $g(x, t)$.

Note that any differentially embedded submanifold N of M^n can be the

uniformly asymptotically stable subset of some vector field, since the tubular neighborhood satisfies the above conditions. We will now show that the domain of stability of a submanifold N must always have this form.

LEMMA 3.3. *If W is a paracompact manifold containing N , and if every compact subset of W is contained in some tubular neighborhood of N , then W is diffeomorphic to an open tubular neighborhood of N .*

Proof. The proof is analogous to the proof of Lemma 2.1 in [5], using the uniqueness of tubular neighborhoods up to isotopy [3] in place of the Cerf–Palais lemma.

THEOREM 3.4. *Let F be a vector field on M^n , and let N be a uniformly asymptotically stable submanifold. Then the domain D of stability of N is diffeomorphic to an open tubular neighborhood of N .*

Proof. Let U denote a tubular neighborhood of N which is contained in D . Given any compact subset K of D , we can define an isotopy of D by pushing backwards along trajectories, so that U is carried onto a neighborhood of K and so that a smaller neighborhood of N is held fixed. The theorem now follows from the previous lemma.

COROLLARY 3.5. *The level surfaces of a Lyapunov function for F and N are homotopically equivalent to the boundary of a closed tubular neighborhood of N .*

COROLLARY 3.6. *Let $f: M^n \rightarrow R$ be a C^1 function which has an isolated critical submanifold N which is a relative maximum or minimum. Then the level surfaces of f near N are homotopically equivalent to the boundary of a closed tubular neighborhood of N .*

Observe that Theorem 3.4 and Corollary 3.5 apply in particular to nonautonomous systems. The level surfaces and the domain of stability will always have the form of the product of a manifold with R in this case.

THEOREM 3.7. *Let V be a Lyapunov function for a nonautonomous n -dimensional differential equation which is uniformly asymptotically stable at the origin. Then the level surfaces of V are diffeomorphic to $S^n \times R$.*

Proof. By Corollary 3.5, the level surfaces are homotopically equivalent to $S^n \times R$, and so they are diffeomorphic to $S^n \times R$ if $n \neq 3, 4$. But if M is a four-dimensional homotopy sphere, or a three-dimensional homotopy sphere which bounds a homotopy disk, then Hirsch has shown that $M \times R = S^n \times R [J]$. Thus the theorem has been established for $n \neq 3$.

The condition $dt/dt = 1$ says that the level surface intersects the hyperplane $t = 0$ transversally, and so M bounds a homotopy disk, which completes the proof of the theorem.

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