Spectra of Products and Numerical Ranges

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1. INTRODUCTION

If \( A \) is bounded linear transformation from a complex Hilbert space \( H \) into itself, then the numerical range of \( A \) is by definition the set

\[
W(A) = \{ \langle Ax, x \rangle : \| x \| = 1 \}.
\]

It is well-known and easy to prove that if \( \sigma(A) \) denotes the spectrum of \( A \), then

\[
\sigma(A) \subseteq \overline{W(A)},
\]

where the bar indicates closure.

The purpose of this paper is two-fold. We first present an extension of the foregoing relation and the proceed to indicate how the extension may be used in two other situations, namely bounded linear operators on a Banach space, and certain nonlinear transformations on a real or complex Hilbert space. The extension is mild, Specifically, we will show that if \( 0 \notin \overline{W(A)} \), then

\[
\sigma(A^{-1}B) \subseteq \overline{W(B)}/\overline{W(A)}
\]

for any operator \( B \) on \( H \). Here the set on the right is by definition the set of quotients \( b/a \) with \( b \in \overline{W(B)} \) and \( a \in \overline{W(A)} \).

The extension has interesting consequences. For example it implies that if \( A \) is strictly positive and \( B \geq 0 \), then the product \( AB \) has a nonnegative spectrum. Also, if \( A \) is positive and \( B \) is self-adjoint then the product \( AB \) has real spectrum.

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2. LINEAR OPERATORS ON A HILBERT SPACE

We begin with the proof of the extension.

**Theorem 1.** Let $A$ and $B$ operators on the complex Hilbert space $H$. If $0 \notin \overline{W(A)}$ then
\[ \sigma(A^{-1}B) \subset \frac{\overline{W(B)}}{\overline{W(A)}}. \]

**Proof.** Observe first of all that since $\sigma(A) \subset \overline{W(A)}$, the hypothesis guarantees that $A^{-1}$ exists (as a bounded linear operator on $H$). Secondly, the identity
\[ A^{-1}B - \lambda = A^{-1}(B - \lambda A) \]
shows that if $\lambda \in \sigma(A^{-1}B)$, then $0 \in \sigma(B - \lambda A)$. This in turn implies that
\[ 0 \in \overline{W(B - \lambda A)} \subset \overline{W(B)} - \lambda \overline{W(A)}, \]
and this means that
\[ \lambda \in \frac{\overline{W(A)}}{\overline{W(A)}}. \]

We indicated two corollaries above. To get another we recall that any operator $A$ on $H$ has a “polar decomposition”
\[ A = UP, \]
and that if $A$ is invertible, then $U$ is unitary and $P$ is strictly positive. Following Berberian [1] we call the unitary operator $U$ cramped if its spectrum is contained in an arc of the unit circle with central angle $< \pi$.

**Corollary (Berberian).** If $0 \notin \overline{W(A)}$, then the unitary part of $A$ is cramped.

**Proof.** Use the fact that $\overline{W(A)}$ is convex to see that if $0 \notin \overline{W(A)}$, then $\overline{W(A)}$ is contained in a sector
\[ S = \{re^{i\theta} : r > 0 ; \theta_1 \leq \theta \leq \theta_2 \} \]
with $\theta_2 - \theta_1 < \pi$. Then write $U = A \cdot P^{-1}$ and apply the theorem to see that $\sigma(U)$ is a subset of the arc
\[ \{e^{i\theta} : \theta_1 \leq \theta \leq \theta_2 \}. \]

**Remark.** (i) The inclusion $\sigma(A^{-1}B) \subset \overline{W(B)}/\overline{W(A)}$ is not valid with the
weaker assumption that \( A \) is merely invertible. Indeed if \( A \) and \( B \) are self-adjoint \( \sigma(AB) \) need not even be real. This follows from the computation

\[
\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]

in two-dimensional Hilbert space.

(ii). The more symmetric statement

\[ \sigma(AB) \subseteq W(A) \cdot W(B) \quad \text{if} \quad 0 \notin W(A) \cup W(B) \]

is also not valid. To see this let \( A \) be the operator

\[ A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \]

Then \( W(A) = W(A^*) \) is the disk of radius 1/2 about 1 and so the set \( W(A) \cdot W(A^*) \) lies to the left of \( \text{Re} z = 9/4 \). On the other hand \( 9/4 < 1/2(3 + \sqrt{5}) \in \sigma(AA^*) \).

Returning to the theorem, the reader will note that the proof really does not concern operators on a Hilbert space at all. Indeed, the essential ingredients are these: An algebra \( \mathcal{A} \) with unit, and two mappings \( A \rightarrow \sigma(A), \ A \rightarrow W(A) \) from \( \mathcal{A} \) to subsets of the complex plane which have the following properties:

1. \( W(A + B) \subseteq W(A) + W(B) \)
2. \( W(\lambda A) \subseteq \lambda W(A) \)
3. \( \sigma(A) \subseteq W(A) \)
4. \( \lambda \notin \sigma(A) \) if and only if \( (A - \lambda)^{-1} \notin \mathcal{A} \).

(We write \( B^{-1} \notin \mathcal{A} \) to mean that the element \( B \) of \( \mathcal{A} \) has an inverse and that this inverse in fact belongs to \( \mathcal{A} \).) In what follows we will indicate how this observation extends the theorem to two other situations.

3. Linear Operators on a Banach Space

For our first application we need a few facts about Banach spaces. First, if \( X \) is a Banach space then the Hahn-Banach theorem guarantees that for each \( x \in X \) there is an \( x^* \in X^* \) of norm 1 such that \( \langle x, x^* \rangle = \| x \| \). The space \( X \) (or more properly, the unit ball of \( X \)) is called smooth [2] if there is exactly one such \( x^* \) for each \( x \in X \). Thus in a smooth space there is a unique map \( \varphi \) form \( X \) to \( X^* \) such that

\[
\| \varphi(x) \| = \| x \|, \quad \langle x, \varphi(x) \rangle = \| x \|^2 \quad (x \in X).
\]
As an example the reader can easily verify that $L^p$ is smooth for $1 < p < \infty$. The isometry $\varphi$ sends $f \in L^p$ to
\[ f \frac{|f|^{p-2}}{\|f\|^{p-2}}. \]

If $X$ is smooth and $\varphi$ is the indicated mapping, then it is easy to see that $\varphi$ is conjugate homogeneous:
\[ \varphi(\alpha x) = \bar{\alpha} \varphi(x), \quad \alpha \text{ complex}. \]
(However, if $\varphi$ is additive, then the norm in $X$ satisfies the parallelogram law and hence $X$ is a Hilbert space.) Again, if $X$ is smooth and $f \in X^*$ attains its supremum on the unit ball of $X$, then $f$ belongs to the range of $\varphi$. Now a result of Bishop and Phelps [3] states that for any Banach space $X$ the collection of bounded linear functionals on $X$ which attain their suprema on the unit ball of $X$ is always (norm) dense in $X^*$. By using this fact and the preceding remark it follows that if $X$ is smooth, then the range of $\varphi$ is dense in $X^*$.

Now using the function $\varphi$ we can define a "semi-inner-product" on $X$ by
\[ [x, y] = \langle x, \varphi(y) \rangle \quad (x, y \in X). \]
It is readily verified that the following hold:
\[
\begin{align*}
[x, x] &= \|x\|^2 \\
[x_1 + x_2, y] &= [x_1, y] + [x_2, y] \\
[\lambda x, y] &= \lambda [x, y], \quad [x, \lambda y] = \bar{\lambda} [x, y] \\
\| [x, y] \| &\leq \| x \| \| y \|. 
\end{align*}
\]

If now $A$ is a bounded linear operator on $X$ we can define the numerical range of $A$ by setting
\[ W(A) = \{ [Ax, x] : \| x \| = 1 \}. \]
Clearly we will have
\[ W(A + B) \subseteq W(A) + W(B), \]
\[ W(\lambda A) \subseteq \lambda W(A). \]
Lumer [4] also shows that the boundary of $\sigma(A)$ is a subset of $\overline{W(A)}$. We need the following stronger result:

**Proposition.** $\sigma(A) \subseteq \overline{W(A)}$.

**Proof.** The argument parallels the linear case: If $\lambda$ is at a positive distance $\delta$ from $\overline{W(A)}$, then for unit vectors $x$
\[ \| (A - \lambda)x \| \geq \| [(A - \lambda)x, x] \| - \| [Ax, x] - \lambda x \| \geq \delta - \delta \| x \|. \]
and

\[ \| (A - \lambda)^* \varphi(x) \| \geq | \langle x, (A - \lambda)^* \varphi(x) \rangle | = | (A - \lambda)x, x \| \geq \delta = \delta \| \varphi(x) \| . \]

The first of these implies that \( A - \lambda \) is one-to-one with a closed range. The second implies that \( (A - \lambda)^* \) is bounded below on the range of \( \varphi \) and since this is dense in \( X^* \), \( (A - \lambda)^* \) is bounded below, hence one-to-one, and this means that \( A - \lambda \) has a dense range. It now follows from the Open Mapping Theorem that \( A - \lambda \) has a bounded inverse. Hence \( \lambda \notin W(A) \) implies \( \lambda \notin \sigma(A) \) as asserted.

We may summarize the preceding discussion as follows:

\textbf{Theorem 2.} Let \( X \) be a smooth Banach space and define \( W(A) \) as above. Then if \( 0 \notin W(A) \) we have

\[ \sigma(A^{-1}B) \subset \overline{W(B)W(A)} \]

for any operator \( B \) on \( X \).

If the Banach space \( X \) is not smooth then there will be many isometries \( \varphi_\alpha \) from \( X \) to \( x^* \) satisfying

\[ \langle x, \varphi_\alpha(x) \rangle = \| x \|^2 \quad (x \in X). \]

Each of these maps defines a semi-inner product \([ , ]_\alpha\) on \( X \) and a bounded linear operator \( T \) on \( X \) has corresponding numerical ranges \( W_\alpha(T) \). It is natural to define the \textit{numerical range} of \( T \) on \( X \) by

\[ W(T) = \bigcup \alpha W_\alpha(T). \]

The argument used for the smooth case is easily adapted to prove that \( \sigma(T) \subset \overline{W(T)} \) is still valid and so we can conclude that Theorem 2 holds without the hypothesis that \( X \) is smooth.

In this connection Lumer has shown [4] that \( W(T) \) is real (or positive) if and only if some \( W_\alpha(T) \) is real (or positive). Thus \( T = T^* \) (or \( T \geq 0 \)) has intrinsic meaning and with these conventions we can state the following corollary:

\textbf{Corollary.} If \( A > 0 \), \( B \geq 0 \) and \( C = C^* \), then \( \sigma(AB) \) is positive and \( \sigma(AC) \) is real.

4. \textbf{Nonlinear Operators on a Hilbert Space}

Our final application is more delicate. Here we let \( H \) be a real or complex Hilbert space and let \( \mathcal{A} \) be the collection of maps from \( H \) to itself which are
continuous and which send bounded sets into bounded sets. Clearly $\mathcal{A}$ is an algebra with unit. We take the numerical range of $A \in \mathcal{A}$ to be

$$W(A) = \left\{ \frac{\langle Ax_1 - Ax_2, x_1 - x_2 \rangle}{\| x_1 - x_2 \|^2} : x_1 \neq x_2 \right\}.$$ 

There are two possible definitions of the spectrum of $A \in \mathcal{A}$, namely, $\sigma(A)$, and $\sigma_1(A)$ defined respectively as the complements of the sets

$$\rho(A) = \{ \lambda : (A - \lambda)^{-1} \notin \mathcal{A} \}$$

$$\rho_1(A) = \{ \lambda : (A - \lambda)^{-1} \text{ exists and is Lipschitzian} \}.$$ 

(By definition, $B$ is Lipschitzian if

$$\| Bx_1 - Bx_2 \| \leq M \cdot \| x_1 - x_2 \|$$

for some constant $M > 0$ and all $x_1, x_2$.)

It is easy to see that $\sigma(A) \subset \sigma_1(A)$. Moreover, a theorem of Zarantonello [5] asserts that, with $W(A)$ as defined above, we have the inclusion

$$\sigma_1(A) \subset \overline{W(A)}.$$ 

Taking $\sigma(A)$ as the definition of the spectrum of $A$ and applying Theorem 1, we get the following result:

**Theorem 3.** Let $A$ and $B$ be bounded and continuous on $H$. If $0 \notin \overline{W(A)}$, then for each $\lambda \notin \overline{W(B)} \setminus \overline{W(A)}$ the mapping $A^{-1}B - \lambda$ has a bounded, continuous inverse defined on $H$.

Taking $\sigma_1(A)$ as the definition of the spectrum of $A$ we get:

**Theorem 4.** Let $B$ be bounded and continuous, let $A$ be Lipschitzian and suppose $0 \notin \overline{W(A)}$. Then for each $\lambda$ outside the set $\overline{W(B)} \setminus \overline{W(A)}$ the transformation $A^{-1}B - \lambda$ has a Lipschitzian inverse defined on $H$.

**Proof.** If $0 \notin \sigma_1(B - \lambda A)$, then $(B - \lambda A)^{-1}$ exists and is Lipschitzian. Hence the product $(B - \lambda A)^{-1}A$ is also Lipschitzian. Since however

$$(A^{-1}B - \lambda)(B - \lambda A)^{-1}A = A^{-1}(B - \lambda A)(B - \lambda A)^{-1}A = I,$$

this implies that $A^{-1}B - \lambda$ has a Lipschitzian inverse and so $\lambda \notin \sigma_1(A^{-1}B)$. In other words,

$$\lambda \in \sigma_1(A^{-1}B) \Rightarrow 0 \in \sigma_1(B - \lambda A)$$

and the remainder of the proof is as before.
I am indebted to Professors William A. Porter and Richard A. Volz for stimulating discussions which led to the results of this paper. I am further indebted to Professor Lumer for his observations (a) that my proof of the inclusion \( \sigma(T) \subseteq \overline{W(T)} \) is valid in an arbitrary Banach space so that (b) the corollary of Theorem 2 does not require smoothness of \( X \). He has also informed me of the following elegant proof of that corollary for Hilbert spaces: \( \sigma(AB) = \sigma(BA) \pm (0) \) so that if \( A \succ 0 \), then \( \sigma(AB) = \sigma(A^{1/2}BA^{1/2}) \pm (0) \), and the operator \( A^{1/2}BA^{1/2} \) is self-adjoint (or positive) if \( R = R^* \) (or \( B \succ 0 \)).

REFERENCES