

Some Theorems on Čebyšev Approximation II

HAROLD S. SHAPIRO

The University of Michigan, Ann Arbor, Michigan

Submitted by R. P. Boas

This note is intended as a sequel to [1], and consists of two parts. In Section 1 we consider a problem of approximation in a Cartesian product space, obtaining a result analogous to Theorem 1 of [1]. In Section 2 we consider certain configurations which play an important role in problems of best Čebyšev approximation (called in [2] "extremal signatures"), in the case that the approximating functions are ordinary polynomials of fixed degree in several variables. Our results here extend somewhat those of Collatz [3]. The present note may be read independently of [1], but some acquaintance with the general theory of best Čebyšev approximation, as expounded in [2] or [4], will be helpful.

1. APPROXIMATION IN A CARTESIAN PRODUCT SPACE

In this section X denotes a compact Hausdorff and $C(X)$ the Banach space of real-valued continuous functions on X normed by $\|f\| = \max |f(x)|$. Let F be an m -dimensional subspace of $C(X)$ and $f_i(x)$, $i = 1, \dots, m$ a basis for F ; similarly, Y denotes a compact Hausdorff space, G an n -dimensional subspace, and $g_j(y)$, $j = 1, \dots, n$ a basis for G . $X \times Y$ denotes the Cartesian product of X and Y with the usual product topology.

THEOREM 1. *Let P denote the set of functions*

$$p(x, y) = \sum_{i=1}^m f_i(x)b_i(y) + \sum_{j=1}^n a_j(x)g_j(y), \tag{1}$$

where $a_j(x)$, $b_i(y)$ denote arbitrary functions of $C(X)$, $C(Y)$ respectively. Let $f(x) \in C(X)$, $g(y) \in C(Y)$. Then, the expression $\|f(x)g(y) - p(x, y)\|$, the norm being in the space $C(X \times Y)$, attains a minimum as p ranges over P . A minimizing element p^* is given by

$$p^*(x, y) = f^*(x)g(y) + f(x)g^*(y) - f^*(x)g^*(y), \tag{2}$$

and

$$\|f(x)g(y) - p^*(x, y)\| = \|f - f^*\| \cdot \|g - g^*\|. \tag{3}$$

Here f^*, g^* denote closest elements to f, g in F, G respectively; the norms are with respect to the spaces $C(X \times Y), C(X), C(Y)$, respectively.

PROOF. It is known from the general theory of Čebyšev approximation (see [2] or [4]) that there is a measure $d\sigma$ concentrated on a set of at most $m + 1$ points of X such that (i) $\int |d\sigma| = 1$, (ii) $\int f_i(x) d\sigma = 0, i = 1, \dots, m$, and (iii) $\int f d\sigma = \|f - f^*\|$. Similarly, a measure λ exists, concentrated on a set of at most $n + 1$ points of Y , such that (i') $\int |d\lambda| = 1$, (ii') $\int g_j(y) d\lambda = 0, j = 1, \dots, n$, and (iii') $\int g d\lambda = \|g - g^*\|$. If then β is the product measure $\sigma \times \lambda$ on $X \times Y$ we have $\int |d\beta| = 1$, and so for any $p \in P$,

$$\|f(x)g(y) - p(x, y)\| \geq \left| \int [f(x)g(y) - p(x, y)] d\beta \right| \tag{4}$$

Now, $\int p(x, y) d\beta = 0$ since

$$\int f_i(x)b_i(y) d\beta = \left(\int f_i d\sigma\right)\left(\int b_i d\lambda\right) = 0$$

by (ii) and

$$\int a_j(x)g_j(y) d\beta = \left(\int a_j d\sigma\right)\left(\int g_j d\lambda\right) = 0$$

by (ii'). Hence the right side of (4) equals

$$\left| \int f(x)g(y) d\beta \right| = \left| \left(\int f d\sigma\right)\left(\int g d\lambda\right) \right| = \|f - f^*\| \cdot \|g - g^*\|.$$

Since for the choice of p^* given by (2), we have

$$f(x)g(y) - p^*(x, y) = [f(x) - f^*(x)][g(y) - g^*(y)]$$

this completes the proof of the theorem.

REMARKS. For the proof of Theorem 1, the discreteness of the measures σ, λ really is not needed, hence the proof could be based simply on the Hahn-Banach theorem. On the other hand, the construction of *discrete* measures (in the present instance β) which annihilate all the admissible approximating functions is of importance for the solution of concrete problems of approximation.

Theorem 1 extends to complex-valued functions. Also, we remark that theorem 1 possesses an obvious extension to Cartesian products of three or more spaces; we leave to the reader the task of formulating this in detail.

EXAMPLES. We give two simple examples to illustrate Theorem 1.

Example 1. $X = Y = [-1, 1]$; F : polynomials in x of degree $m - 1$, G : polynomials in y of degree $n - 1$. Choosing $f(x) = x^m$, $g(y) = y^n$ we get

$$\left\| x^m y^n - \sum_0^{m-1} b_i(y)x^i - \sum_0^{n-1} a_j(x)y^j \right\| \geq \| T_m(x)T_n(y) \| = 2^{-(m+n-2)}, \quad (5)$$

where the norm on the left denotes the maximum over the unit square. T_n denotes the n th Čebyšev polynomial, normalized so that the leading coefficient is one. Specializing b_i and a_j in (5) we see in particular that $x^m y^n$ cannot be approximated closer than $2^{-(m+n-2)}$ on the unit square by polynomials of degree less than $m + n$ in x and y .

Example 2. We shall prove: $f(x) = x_1 x_2 \dots x_k$ admits, on the sphere $S: \sum_1^k x_i^2 = 1$, 0 as a best approximation among all polynomials $h(x_1, \dots, x_k)$ of degree not greater than $k - 1$. This fact was mentioned, without proof, in [5], p. 217.

Indeed, by the arithmetic-geometric mean inequality we have, for $x \in S$,

$$|x_1, \dots, x_k|^2 \leq \left(\frac{\sum_1^k x_i^2}{k} \right)^k = k^{-k},$$

hence $|f(x)|$ attains its maximum $k^{-k/2}$ at 2^k distinct points of S , namely the Cartesian product S_0 of the sets $x_i = \pm k^{-1/2}$ ($i = 1, \dots, k$). By using Theorem 1 (extended to a product of k spaces) the result follows at once; indeed, we obtain a sharper result in which S is replaced by S_0 , and the class of admissible approximating functions is extended to sums $\sum_{i=1}^k h_i$, where h_i is any function not containing the variable x_i .

2. POLYNOMIAL APPROXIMATION IN SEVERAL VARIABLES

In attempting to extend the classical alternation criterion of Čebyšev to polynomial approximation in more than one variable, one encounters the complication that there is no one simple class of sign patterns (signatures) which plays the role of the alternating sign patterns in the one variable case. For approximation by *linear* polynomials in a plane region, Collatz [3] found that there are three basic types of patterns. These are illustrated in Fig. 1.

The significance of these patterns is as follows. Suppose $p^*(x, y)$ is a best linear approximation to $f(x, y)$ on some compact set K in the plane. To avoid degenerate cases we suppose $m = \|f - p^*\| > 0$. Then it is possible to find a subset of K of one of the three types illustrated in figure 1 such that $f - p^*$

takes the value $+m$ on the “black” points, and $-m$ on the “white” points (or vice versa). Conversely, if for any linear polynomial $p, f - p = \pm \|f - p\|$ on such a configuration, p is a best linear approximation to f on K . The distinguishing characteristic of the patterns illustrated in figure 1 is that the convex hull of the black points intersects the convex hull of the white points; and the patterns are minimal with respect to this property. The analogous property also characterizes the relevant patterns (let us for simplicity call them Čebyšev patterns) for linear approximation in more variables. For approximation by higher degree polynomials to characterize the Čebyšev patterns seems very difficult—it is easy to do so in terms of a notion of polynomial convexity, but this is little more than a restatement of the problem.

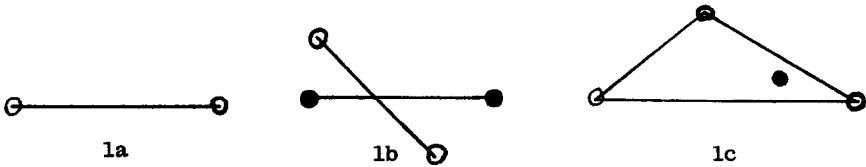


FIG. 1

To our knowledge, the only study of Čebyšev patterns for polynomials of degree higher than one is that of Collatz [3]; a method for constructing special patterns was obtained incidentally in [1]; for another, related, study see also [6]. All the results are quite fragmentary.

We propose in this section to give a new method for constructing Čebyšev patterns. Actually our “method” is little more than the observation that a certain classical formula of Euler and Jacobi has relevance to the problem, but it does furnish a large variety of new configurations.

THEOREM 2. *Let $P_1(\xi), P_2(\xi), \dots, P_k(\xi)$ denote polynomials of degrees m_1, m_2, \dots, m_k respectively in $\xi = (x_1, x_2, \dots, x_k)$ such that*

- (i) *the system of equations*

$$P_1(\xi) = P_2(\xi) = \dots = P_k(\xi) = 0$$

has precisely $M = m_1 m_2 \dots m_k$ distinct roots ξ^r in real Euclidean k -space

- (ii) *the numbers $J_r (r = 1, 2, \dots, M)$ are all different from zero, where J_r denotes the value of the Jacobian $(\partial P_i / \partial x_j)$ at $\xi = \xi^r$.*

Then, no polynomial $f(\xi)$ of degree not exceeding $m_1 + m_2 + \dots + m_k - k - 1$ is positive at those ξ^r where $J_r > 0$ and negative at those ξ^r where $J_r < 0$.

PROOF. According to a formula of Euler and Jacobi (for a proof see [7] p. 135), under the hypotheses of Theorem 2 we have

$$\sum_{r=1}^M \frac{f(\xi_r)}{J_r} = 0 \tag{1}$$

for every f of degree $\leq m_1 + \dots + m_k - k - 1$, and this implies the stated result (indeed, in the stronger form that if f is non-negative where $J_r > 0$ and non-positive where $J_r < 0$ then f vanishes at all the ξ^r).

EXAMPLES. We confine ourselves to the case $k = 2$, and write $x_1 = x$, $x_2 = y$.

(a) We can construct a large class of examples by choosing for $P_i(x, y)$ a product of m_i linear polynomials ($i = 1, 2$). We thus obtain a figure where m_1 "red" lines intersect m_2 "black" lines in $m_1 m_2$ points. To determine the sign of the Jacobian at each of these points no calculation is required. For, thinking of the pair of functions P_1, P_2 as defining a transformation we have only to determine whether at each point where a black line meets a red line the sense of rotation is preserved or reversed by this transformation. For this purpose, consider the regions into which the $m_1 + m_2$ lines dissect the plane. Choose one of these at random, and suppose the P_i so normalized that each is positive in this region. Write in this region $++$ and now continue to mark the remaining regions with the four signatures $++$, $+-$, $-+$, $--$ according to the scheme: if two regions meet along a red line, their signatures differ in the first position; if they meet along a black line, their signatures differ in the second position. When all the regions have been so marked, consider any point where a black line meets a red line. Draw a circle

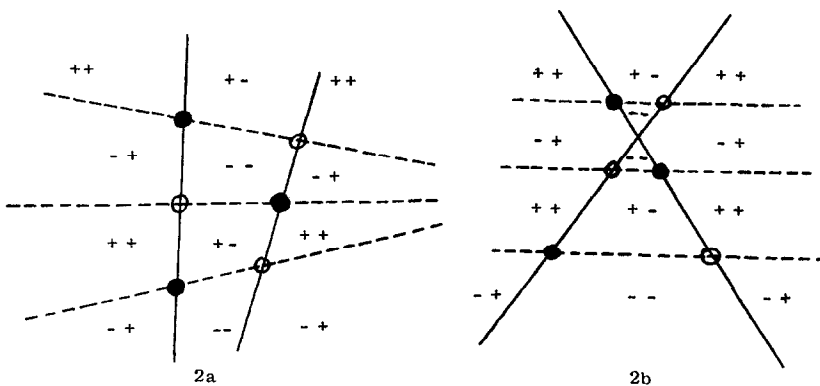


FIG. 2

around this point which passes from the $++$ region to the $+ -$ region. According as the sense of rotation of this circle is clockwise or counterclockwise, we divide the points into two classes. The points of these two classes may be then taken as the "black" and "white" points of a configuration such that no polynomial of degree not greater than $m_1 + m_2 - 3$ is positive on the black points and negative on the white points. For two illustrations this procedure in the case $m_1 = 3, m_2 = 2$, see Fig. 2 (these simple cases could of course be readily verified without Theorem 2).

The construction just outlined can obviously be carried out for curves of any degree, providing they intersect in the prescribed number of points; and with suitable modifications to surfaces in 3 space, etc.

(b) As an application of the preceding ideas we show: Let K be an ellipse; the configuration formed by $2n$ points of K , marked alternately "black" and "white," is a Čebyšev pattern for polynomials of degree $n - 1$.

Indeed, divide the $2n$ points into pairs in any fashion whatever, and pass a line through each of these point pairs. The system of lines, of degree n , has $2n$ distinct intersections with the ellipse, hence Theorem 2 is applicable. Simple considerations as in the preceding paragraph show that the sign of the Jacobian alternates as we travel around the ellipse. The situation is illustrated in Fig. 3, for $n = 2$. That the configuration is minimal is readily proved by induction.

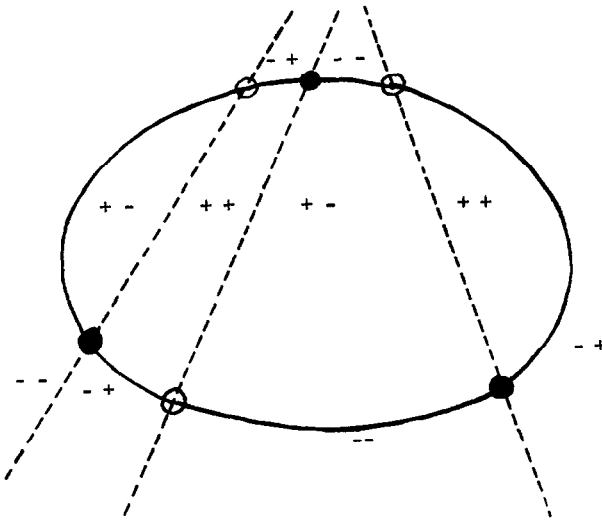


FIG. 3

In conclusion, we remark some points we have not been able to settle:

(i) are the configurations obtained from Theorem 2 always minimal? (ii) are there Čebyšev patterns not accessible to the method of Theorem 2? Concerning this point, we note that for $k = 1$, as well as for degree one when $k = 2$, the method does give all Čebyšev patterns. That it could give all in the general case seems unlikely since, for example, in the case of degree 2, $k = 2$ we must have $m_1 + m_2 - 2 = 2$ hence $m_1 = 4$, $m_2 = 1$ or $m = 3$, $m_2 = 2$ are essentially the only possibilities, i.e. only configurations with four or six points are obtainable.

Note added in proof. See also the recent paper of Bruno Brosowski, Über Extremalsignaturen linearen Polynome in n Veränderlichen. *Numer. Math.* 7 (1965), 396-405.

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