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The Analytic Continuation of the Resolvent Kernel and Scattering Operator Associated with the Schroedinger Operator[1,2]

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1. INTRODUCTION

In this paper we shall apply some fundamental facts from the general theory of functional analysis in order to study the analytic properties of the resolvent kernel and the scattering operator associated with the Schroedinger operator

$$A = -\Delta + M(q).$$

Here Δ denotes the Laplacian in 3-dimensional Euclidean space, while $M(q)$ denotes the operation of multiplication by the real-valued potential function $q(x)$, i.e.

$$M(q)f(x) = q(x)f(x), \qquad x \in E_3 .$$

The conditions we shall impose upon $q(x)$ are the following:

(i) $q(x)$ is locally Hoelder continuous except at a finite number of singularities;

(ii) $q(x)$ is locally square-integrable;

(iii) $|q(x)| \leqslant Ce^{-\alpha|x|}$ for $|x|$ sufficiently large, C and α being positive constants.

It is important to note that we do not require that q possess any symmetry properties such as spherical symmetry.

With these conditions on $q(x)$, it is known [1] that the operator A, operating in the Hilbert space $L^2(E_3)$, has a unique self-adjoint extension, whose domain is in fact the same as that of the self-adjoint extension of $-\Delta$. From now on, A will be used to denote the self-adjoint extension. Furthermore, it is proved in [2] that the resolvent operator $R(\lambda, A) = (A - \lambda I)^{-1}$ is an integral

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operator having a kernel $G(x, y, \kappa)$ ($\kappa = \sqrt{\lambda}$, im $\kappa > 0$) of Carleman type. (Further details are given in Section 2.) The kernel is a meromorphic function of κ provided that im $\kappa > 0$, and its poles occur at a finite point set $\{\kappa_n\}$ on the positive imaginary axis (17, 18). The values $\kappa_n{}^2$ constitute the point spectrum of the operator A.

The first goal of this paper is to obtain an analytic continuation of $G(x, y, \kappa)$ into the half-plane im $\kappa < 0$. As will be seen, the Hilbert space $L^2(E_3)$ is too "small" to accommodate such a continuation, and an appropriate larger Hilbert space has to be introduced. The continuation is then accomplished in the larger space by reducing the problem to that of solving an infinite set of linear equations and employing the theory of infinite determinants.

The second goal in the paper is to obtain an analytic continuation of the scattering operator. The scattering operator $S(\kappa)$ is generally believed (by causality requirements) to depend analytically upon the parameter κ in the half-plane im $\kappa > 0$, the so-called "physical sheet" of the energy plane associated with the Schroedinger operator A. However, it is the singularities of $S(\kappa)$ on the "unphysical sheet" im $\kappa < 0$ which enter into the impedance theory [3] and which are associated with the physical theories of resonance scattering and unstable particles, for example; and it is these singularities which we shall obtain. Our results agree with those known for several special radially-symmetric cases.

Several authors have done work in this same direction. Ladyjzenskaya, in a paper [4] dealing with the principle of limiting amplitude, obtained a continuation of the resolvent kernel, but the problem discussed by her was so designed as to exclude explicitly any singularities in the continuation. Grossmann [5] has studied the possibility of obtaining analytic continuations of operator-valued functions in a Hilbert space by suitable imbeddings of the space. The analytic properties of the scattering operator have also been discussed by Hunziker [6] using the idea of functions of a complex variable with values in a Banach space, but his work is confined to spherically symmetric cases. For such cases, there is also the survey article by Newton [7]. Another approach to the analytic character of the scattering operator associated with the classical wave equation has been developed by Lax and Phillips [8]. While the scattering operator for the classical wave equation can in fact be related to the scattering operator $S(\kappa)$ for the Schroedinger equation, we shall not enter into the discussion of this more abstract approach.

Finally, it should be noted that, in theory at least, our method is constructive in nature and provides a basis for developing computational procedures for scattering problems.

Sections 2-4 discuss the continuation of the resolvent kernel, Section 5 the continuation of the scattering operator. Finally three illustrative examples are discussed in Sections 6-8.

2. The Continuation of the Resolvent Kernel

In this section and the two succeeding ones we show that the resolvent kernel $G(x, y, \kappa)$ can be continued analytically at least into part of the half plane $\operatorname{im} \kappa < 0$. This corresponds to a continuation into the second Riemann sheet in the λ-plane ($\kappa^2 = \lambda$). We recall that the assumptions we shall make concerning the potential $q(x)$ are the following:

(i) $q(x)$ is locally Holder continuous except at a finite number of singularities;

(ii) $q(x)$ is locally square-integrable;

(iii) $| q(x) | \leqslant Ce^{-\alpha|x|}$ for $| x |$ sufficiently large.

Our result involves the Hilbert space $\mathscr{H}$ of complex-valued Lebesgue-measurable functions $f(x)$ such that

$$\int_{E_3} | f(x) |^2 e^{-\alpha|x|} \, dx < \infty.$$

It is convenient to view the Hilbert space $\mathscr{H}$ as the space $L^2(E_3, d\mu)$ where the measure $d\mu$ is given by $d\mu = e^{-\alpha|x|} \, dx$. With this notation we have

Theorem 1. *Under the conditions* (i), (ii), (iii) *above on* $q(x)$, *the resolvent kernel* $G(x, y, \kappa)$ *associated with* $(A - \lambda I)^{-1}$ *defines for each fixed* $y \in E_3$ *a function* $\kappa \to G(\cdot, y, \kappa) \in \mathscr{H}$. *As an* $\mathscr{H}$*-valued function of* κ, $G(\cdot, y, \kappa)$ *is analytic in* $\operatorname{im} \kappa > 0$ *except for a finite number of poles on the positive imaginary axis. Moreover, it has an analytic continuation which is meromorphic in* $\operatorname{im} \kappa > -\frac{1}{2}\alpha$.

Proof. Since our conditions on $q(x)$ are more stringent than those of Ikebe [2], in that our condition for large $| x |$ is more restrictive, we can repeat Ikebe's analysis. His Theorem 1 tells us that if $\operatorname{im} \lambda \neq 0$, $\operatorname{im} \kappa > 0$, $G(x, y, \kappa)$ satisfies

$$G(x, y, \kappa) = \frac{e^{i\kappa|x-y|}}{4\pi \, | x - y |} - \int_{E_3} \frac{e^{i\kappa|x-z|}}{4\pi \, | x - z |} q(z) \, G(z, y, \kappa) \, dz \tag{2.1}$$

as a function of $\mathbf{x}$ almost everywhere in E_3 for almost every fixed $y \in E_3$. Further, G is a kernel of Carleman type, i.e.,

$$\int_{E_3} | G(x, y, \kappa) |^2 \, dx < \infty, \qquad \text{for a.e.} \quad y \in E_3,$$

$$\int_{E_3} | G(x, y, \kappa) |^2 \, dy < \infty, \qquad \text{for a.e.} \quad x \in E_3;$$

and the only solution of (2.1) which is of Carleman type is $G(x, y, \kappa)$.

Ikebe now turns to Fourier transforms, and defines $g(x, k, \kappa)$ to be the conjugate Fourier transform of $G(x, y, \kappa)$ (which exists since G is of Carleman type if im $\kappa > 0$, im $\kappa^2 \neq 0$), i.e.,

$$g(x, k, \kappa) = (2\pi)^{-3/2} \int_{E_3} G(x, y, \kappa)\, e^{ik\cdot y}\, dy.$$

g is a bounded function of **x**, **k** for all **x**, $\mathbf{k} \in E_3$, and **g** satisfies the equations

$$g(x, k, \kappa) = (2\pi)^{-3/2} \frac{e^{ik\cdot x}}{|\, k\,|^2 - \kappa^2} - \frac{1}{4\pi} \int_{E_3} \frac{e^{i\kappa|x-y|}}{|\, x - y\,|}\, q(y)\, g(y, k, \kappa)\, dy.$$

If we finally put

$$h(x, k, \kappa) = (2\pi)^{3/2}\, (|\, k\,|^2 - \kappa^2)\, g(x, k, \kappa),$$

then **h** satisfies

$$h(x, k, \kappa) = e^{ik\cdot x} - \frac{1}{4\pi} \int_{E_3} \frac{e^{i\kappa|x-y|}}{|\, x - y\,|}\, q(y)\, h(y, k, \kappa)\, dy. \tag{2.2}$$

What we shall in fact do is analytically continue both G and **h** (or **g**) from the half-plane im $\kappa > 0$, in which they are so far defined, to the strip $-\frac{1}{2}\alpha <$ im $\kappa \leqslant 0$, and we do this through the integral equations which we know them to satisfy. We must however note that although G and **g** are Fourier transforms of one another for im $\kappa > 0$, this is no longer necessarily true of their continuations, a fact which is easily seen by considering the simplest possible case, that in which $q(x) \equiv 0$. For then, if im $\kappa > 0$,

$$G(x, y, \kappa) = \frac{e^{i\kappa|x-y|}}{4\pi\, |\, x - y\,|},$$

and

$$g(x, k, \kappa) = (2\pi)^{-3/2} \frac{e^{ik\cdot x}}{|\, k\,|^2 - \kappa^2}.$$

The continuations of these are clearly $e^{i\kappa|x-y|}/(4\pi\, |\, x - y\,|)$, $(2\pi)\,(-\frac{3}{2}\, e^{ik\cdot x}/|\, k\,|^2 - \kappa^2)$; but these are not Fourier transforms of one another if im $\kappa < 0$. For then $e^{i\kappa|x-y|}/(4\pi\, |\, x - y\,|)$, being exponentially large, has no Fourier transform in $L^2(E_3)$ at all, while $(2\pi)\,(-\frac{3}{2}\, e^{ik\cdot x}/|\, k\,|^2 - \kappa^2)$ still is square-summable with respect to **k** and does have a Fourier transform. (Indeed, its transform is $e^{-i\kappa|x-y|}/(4\pi\, |\, x - y\,|)$.)

This means the two continuations do not have the simple connection between them that we might have hoped for. At the same time, we shall be able to prove that both are meromorphic and that their poles coincide.

3. To carry out the continuation we shall use an idea due to Tamarkin [9]. What Tamarkin discusses is an integral equation of the form

$$f(s) = g(s) + \int_a^b K(s, t, \lambda) f(t)\, dt,$$

where $a \leqslant s,\ t \leqslant b$ and g and K are known functions. Tamarkin treats the problem by setting it in a Hilbert space—in this case $L^2(a, b)$—and reducing the integral equations for the reciprocal kernel $H(s, t, \lambda)$, namely,

$$H(s, t, \lambda) + K(s, t, \lambda) = \int_a^b H(s, r, \lambda)\, K(r, t, \lambda)\, dr$$

$$= \int_a^b K(s, r, \lambda)\, H(r, t, \lambda)\, dr,$$

to an infinite set of linear equations, which are then solved with the aid of the theory of infinite determinants. Not only does the existence of H follow, but also the fact that, if K is analytic in λ, then H is meromorphic in λ. What follows is an adaptation of Tamarkin's argument.

Looking at the integral equations (2.1) and (2.2), we see at once that the Hilbert space $L^2(E_3)$ will not suit our purposes, since $e^{i\kappa|x-y|}/(4\pi\,|\,x - y\,|)$ does not belong to $L^2(E_3)$ for im $\kappa < 0$. Instead we choose the larger Hilbert space $\mathscr{H}$. We set $p(x) = e^{\alpha|x|}\, q(x)$, and define the operator T_κ (im $\kappa > -\frac{1}{2}\alpha$) for functions $f \in \mathscr{H}$ by

$$(T_\kappa f)(x) = -\frac{1}{4\pi} \int_{E_3} \frac{e^{i\kappa|x-y|}}{|\,x - y\,|}\, p(y) f(y)\, d\mu(y). \tag{3.1}$$

With the above notation, the integral equations (2.1) and (2.2) take the form

$$(I - T_\kappa) f = g(\kappa), \tag{3.2}$$

where $g(\kappa)$ is an element of $\mathscr{H}$ which depends analytically on κ. (Since our object is only to perform the continuations to the strips im $\kappa > -\frac{1}{2}\alpha$, we will assume until further notice that im $\kappa > -\frac{1}{2}\alpha$.)

We now take up the study of the integral operator T_κ. It is a simple matter to verify that T_κ is an operator belonging to the Hilbert-Schmidt class. Indeed, suppose first that $\kappa = \sigma - i\tau$ where $0 \leqslant \tau < \frac{1}{2}\alpha$. To prove that T_κ

is a Hilbert-Schmidt operator, it suffices to show that $N(T_\kappa) < \infty$, where

$$N(T_\kappa) = \int_{E_3 \times E_3} \left| \frac{e^{i\kappa|x-y|}}{|x-y|} p(y) \right|^2 d\mu(x)\, d\mu(y)$$

$$= \int_{E_3} \int_{E_3} \frac{e^{2\tau|x-y|}\, |p(y)|^2}{|x-y|^2} d\mu(x)\, d\mu(y).$$

Set

$$h(y) = \int_{E_3} \frac{e^{2\tau|x-y|}}{|x-y|^2} d\mu(x)$$

$$= \left\{ \int_{|x-y| \leqslant 1} + \int_{|x-y| \geqslant 1} \right\} \frac{e^{2\tau|x-y|}}{|x-y|^2} d\mu(x)$$

$$= h_1(y) + h_2(y),$$

say. The estimate

$$h_1(y) \leqslant \int_{|x-y| \leqslant 1} \frac{e^{2\tau|x-y|}}{|x-y|^2} dx = C \int_0^1 e^{2\tau r}\, dr \tag{3.3}$$

shows that $h_1(y)$ is bounded, while the estimate

$$h_2(y) \leqslant \int_{E_3} e^{2\tau|x-y|}\, e^{-\alpha|x|}\, dx \leqslant e^{2\tau|y|} \int_{E_3} e^{-(\alpha - 2\tau)|x|}\, dx$$

shows that $e^{-2\tau|y|}\, h_2(y)$ is bounded. It follows, then, that $h(y) = h_1(y) + h_2(y)$ is bounded on compact subsets of E_3 and behaves not worse than $e^{2\tau|y|}$ as $|y| \to \infty$. Thus $h \in L^1(d\mu)$, for $2\tau < \alpha$, and since $p(y)$ is locally square-summable and is bounded for large $|y|$, it follows that $h\,|p|^2$ belongs to $L^1(d\mu)$. Hence $N(T_\kappa) < \infty$, as required.

We have been tacitly supposing in the above calculation that $-\frac{1}{2}\alpha < \operatorname{im} \kappa \leqslant 0$, but for $\operatorname{im} \kappa > 0$, the proof that $N(T_\kappa) < \infty$ is even simpler. (We shall not go through the details.) In fact, for future purposes, we note that we can make $N(T_\kappa)$ as small as we please by choosing $\operatorname{im} \kappa$ sufficiently large and positive, i.e., τ sufficiently large and negative. For integrating (3.3), we obtain at once that $h_1(y) = O(\tau^{-1})$ as $\tau \to -\infty$, while

$$h_2(y) \leqslant \int_{|x-y| \geqslant 1} e^{2\tau|x-y|}\, dx$$

$$= \int_{|t| \geqslant 1} e^{2\tau|t|}\, dt \qquad (t = x - y)$$

$$= O(\tau^{-1}).$$

Carrying these estimates in place of the earlier ones for h_1, h_2, we readily see that $N(T_\kappa) = O(\tau^{-1})$.

In view of the form of (3.2), it is evidently relevant to establish that $(I - T_\kappa)^{-1}$ is meromorphic (throughout $-\frac{1}{2}\alpha < \operatorname{im}\kappa$), and this is done by appeal to the theory of infinite determinants. We first note the relation

$$(I - T_\kappa)^{-1} = (I - T_\kappa^2)^{-1}(I + T_\kappa),$$

valid certainly if $(I - T_\kappa^2)^{-1}$ exists, and this assures us that $(I - T_\kappa)^{-1}$ will be meromorphic if $(I - T_\kappa^2)^{-1}$ is. It is in fact $(I - T_\kappa^2)^{-1}$ that we will investigate.

Let $\{\phi_i\}$, $(i = 1, 2, ...)$ be a complete orthonormal set of functions for the Hilbert space $\mathscr{H}$ and denote by $f^+ \equiv \{f_i^+\}$ the element of l_2 (the Hilbert space of square-summable sequences of complex numbers) having components $f_i^+ = (f, \phi_i)$. Further, let $k(x, y, \kappa)$ denote the kernel of the Hilbert-Schmidt operator T_κ^2. Then the functions

$$k_i(x, \kappa) = (T_\kappa^2\phi_i)(x) \qquad (i = 1, 2, \cdots)$$

belong to $\mathscr{H}$, and the doubly infinite sequence

$$k_{ij}(\kappa) = (T_\kappa^2\phi_i, \phi_j)$$

satisfies

$$\sum_{i,j} | k_{ij}(\kappa) |^2 = N(T_\kappa^2) < \infty$$

in the half-plane $\operatorname{im}\kappa > -\frac{1}{2}\alpha$. The equation

$$(I - T_\kappa^2)f = g(\kappa)$$

is represented in l_2 by the infinite system of linear equations

$$f_i^+ - \sum_{j=1}^{\infty} k_{ji}(\kappa) f_j^+ = g_i^+(\kappa) \qquad (i = 1, 2, ...). \tag{3.4}$$

Now it is a basic result from the theory of infinite determinants ([10], Ch. II), that the conditions

(i) $\sum\sum_{i,j=1}^{\infty} | k_{ij}(\kappa) |^2 < \infty$,

(ii) $\sum_{i=1}^{\infty} | k_{ii}(\kappa) | < \infty$

are sufficient for the simultaneous absolute convergence of the infinite determinant

$$\Delta(\kappa) \equiv | \delta_{ij} - k_{ji}(\kappa) |$$

and all its minors, δ_{ij} being the Kronecker delta. As we have already seen, condition (i) holds, while (ii) also holds since

$$\sum_{i=1}^{\infty} | k_{ii}(\kappa) | = \sum_{i=1}^{\infty} | (T_\kappa^2 \phi_i , \phi_i) |$$

$$= \sum_{i=1}^{\infty} | (T_\kappa \phi_i , T_\kappa^* \phi_i) |$$

$$\leqslant \left\{ \sum_{i=1}^{\infty} \| T_\kappa \phi_i \|^2 \right\}^{1/2} \left\{ \sum_{i=1}^{\infty} \| T_\kappa^* \phi_i \|^2 \right\}^{1/2}$$

$$< \infty,$$

since T_κ is Hilbert-Schmidt. Further, since the bounds implied in conditions (i) and (ii) hold uniformly in κ for im $\kappa \geqslant - \frac{1}{2} \alpha + \epsilon$, for any fixed $\epsilon > 0$, the infinite determinant $\Delta(\kappa)$ and its minors will converge uniformly, and so will be analytic functions of κ for im $\kappa > - \frac{1}{2} \alpha$. Finally, Tamarkin shows that, for those values of κ for which $\Delta(\kappa) \neq 0$, the infinite set of linear equations (3.4) has, if $g^+(\kappa) \in l_2$ (i.e., $g(\kappa) \in \mathscr{H}$), one and only one solution $f^+ \in l_2$ (i.e., $f \in \mathscr{H}$), and this solution is given by the usual formula for solving a finite set of linear equations with $\Delta(\kappa)$ and its minors replacing the finite determinant. Since the solution $f_i^+(\kappa)_i^+$ is thus given as a quotient of two analytic functions, it must be meromorphic, unless $\Delta(\kappa) \equiv 0$. Hence $f^+(\kappa)$, thought of as a function of κ with values in $\mathscr{H}$ is (weakly) analytic in im $\kappa > - \frac{1}{2} \alpha$.

We have thus established that the Green's function $G(\cdot, y, \kappa)$ and its Fourier transform $g(\cdot, k, \kappa)$, thought of as functions of κ with values in $\mathscr{H}$, can be continued meromorphically throughout im $\kappa > - \frac{1}{2} \alpha$, provided that $\Delta(\kappa) \not\equiv 0$, and we exclude this possibility in Section 4 below. This implies that $(I - T_\kappa^2)^{-1}$ is meromorphic, and as we saw before, so also is $(I - T_\kappa)^{-1}$. Further, the poles of G and $\mathbf{g}$ will be found among those of $(I - T_\kappa)^{-1}$, and so will be effectively the same except that at certain of the poles the singularity of G or $\mathbf{g}$ may be removable.

One further remark may usefully be made in this section. It is that we can establish very nearly as much under wider circumstances by using a theorem in [11], p. 592, rather than the work of Tamarkin. In our notation, the theorem of [11] states that, if T_κ is compact, which it certainly is, and is analytic in κ in a connected domain D, then either $I - T_\kappa$ has a bounded inverse for no point in D, or else this inverse exists except at a countable number of isolated points. This is virtually what we want except for the meromorphic character, but it seems difficult to extend the argument in [11] so as to obtain this.

4. We now have to establish that $\Delta(\kappa) \not\equiv 0$. But if we choose $\operatorname{im} \kappa$ sufficiently large and positive, we have already seen that $N(T_\kappa)$ can be made as small as we please. That is to say that the Hilbert-Schmidt norm of T_κ can be made as small as we please and since the Hilbert-Schmidt norm is never less than the Hilbert-space norm, the Hilbert-space norm can also be made as small as we please. Hence $(I - T_\kappa)^{-1}$ and $(I + T_\kappa)^{-1}$, and so $(I - T_\kappa^2)^{-1}$, all exist, which would be contradicted by the assumption $\Delta(\kappa) = 0$.

This completes the proof of Theorem 1, but for the application to the scattering operator, we require a little more. We now know that $G(x, y, \kappa)$ can be continued and that the continuation is meromorphic. Furthermore, in the region $\operatorname{im} \kappa > 0$, $G(x, y, \kappa)$ is analytic except for possible poles on the imaginary axis. This result about the position of the poles is not a consequence of anything we have done in this paper, but an appeal to the well-known fact that $G(x, y, \kappa)$, as a function of $\lambda = \kappa^2$, is analytic except on the real axis, where the spectrum lies. We would like to say the same about the poles of $(I - T_\kappa)^{-1}$, that they also are, in the region $\operatorname{im} \kappa > 0$, confined to the imaginary axis, but this requires proof, since a pole of $(I - T_\kappa)^{-1}$ does not necessarily imply a pole of

$$G(x, y, \kappa) = (I - T_\kappa)^{-1} \frac{e^{i\kappa|x-y|}}{4\pi \mid x - y \mid},$$

the singular part conceivably vanishing in the course of operating on $e^{i\kappa|x-y|}/(4\pi \mid x - y \mid)$.

We proceed now, therefore, to prove

THEOREM 2. *Under the conditions of Theorem 1, in the region* $\operatorname{im} \kappa \geqslant 0$, *the poles of* $(I - T_\kappa)^{-1}$ *are confined to the imaginary axis.*

PROOF. There are perhaps two points worth noting before giving the proof. The first is that we are considering the region $\operatorname{im} \kappa \geqslant 0$ rather than just the region $\operatorname{im} \kappa > 0$. This means in effect that we are excluding the possibility of positive λ-eigenvalues, and is in line with similar results (for more general $q(x)$) given by Ikebe [2] and Kato [12]). The second point is that $\kappa = 0$ lies on the imaginary axis, so that we are not excluding the possibility of a pole at $\kappa = 0$, and in fact examples can be given in which this possibility becomes an actuality. In the appendix there is an example of a potential $q(x)$ with compact support for which zero is an eigenvalue with an

L^2-eigenfunction. This would imply that $\kappa = 0$ is a pole of $(I - T_\kappa)^{-1}$. Another example shows that a pole of $(I - T_\kappa)^{-1}$ at $\kappa = 0$ can arise with an "eigenfunction" which is not L^2.

The proof of Theorem 2 is an almost immediate consequence of the following

LEMMA. *If $f(x) \in \mathscr{H}$ and satisfies $f = T_\kappa f$ for some κ in* $\operatorname{im} \kappa \geqslant 0$, *then $f(x)$ is bounded.*

For once the lemma is proved, so that any eigenfunction **f** of $I - T_\kappa$ belonging to $\mathscr{H}$ is necessarily bounded, it follows by reference to [2], in the remark preceding Lemma 4.2, that $f \in B$, where B is the Banach space introduced by Ikebe, and then by reference to Lemmas 4.4 and 4.5 of [2] that $(I - T_\kappa)^{-1}$ exists except for possible poles on the imaginary axis.

It only remains to prove the lemma. The statement $f = T_\kappa f$ written out in full is

$$f(x) = -\frac{1}{4\pi} \int_{E_3} \frac{e^{i\kappa|x-y|}}{|x-y|} q(y) f(y)\, dy.$$

Since $\operatorname{im} \kappa \geqslant 0$, we have that

$$\begin{aligned} |f(x)| &\leqslant \frac{1}{4\pi} \int_{E_3} \frac{|q(y)f(y)|}{|x-y|}\, dy \\ &= \frac{1}{4\pi} \left\{ \int_{|y|\leqslant R} + \int_{|y|\geqslant R} \right\} \frac{|q(y)f(y)|}{|x-y|}\, dy \\ &= I_1 + I_2, \end{aligned} \tag{4.1}$$

say, where R is chosen sufficiently large that $|q(y)| \leqslant Ce^{-\alpha|y|}$ for $|y| \geqslant R$. (C will denote various positive constants, not necessarily the same at each appearance.) Then

$$\begin{aligned} I_2 &\leqslant C \int_{|y|\geqslant R} \frac{|e^{-\alpha|y|} f(y)|}{|x-y|}\, dy \\ &\leqslant C \left\{ \int_{E_3} |f(y)|^2 e^{-\alpha|y|}\, dy \right\}^{1/2} \left\{ \int_{|y|\geqslant R} \frac{e^{-\alpha|y|}}{|x-y|^2}\, dy \right\}^{1/2} \\ &\leqslant C, \end{aligned} \tag{4.2}$$

since $f \in \mathscr{H}$ and the second integral in (4.2) is seen to be bounded by splitting it over the ranges $|x-y| \leqslant 1$, $|x-y| > 1$. Hence, returning to (4.1),

$$|f(x)| \leqslant C + \frac{1}{4\pi} \int_{|y|\leqslant R} \frac{|q(y)f(y)|}{|x-y|}\, dy.$$

Iterating this, we obtain

$$|f(x)| \leqslant C + C \int_{|y| \leqslant R} \frac{|q(y)|}{|x-y|}\, dy + C \int_{|y| \leqslant R} \frac{|q(y)|}{|x-y|}\, dy \int_{|z| \geqslant R} \frac{|q(z) f(z)|}{|y-z|}\, dz. \tag{4.3}$$

The second term on the right of (4.3) is seen to be bounded by a straight application of the Cauchy-Schwarz inequality. In the third term we change the order of integration in the integral, and the inner integral is then

$$\int_{|y| \leqslant R} \frac{|q(y)|}{|x-y|\,|y-z|}\, dy \leqslant \left\{ \int_{|y| \leqslant R} |q(y)|^2\, dy \right\}^{1/2} \times \left\{ \int_{|y| \leqslant R} \frac{dy}{|x-y|^2\, |y-z|^2} \right\}^{1/2} \leqslant \frac{C}{|x-z|^{1/2}}.$$

Hence (4.3) reduces to

$$|f(x)| \leqslant C + C \int_{|y| \leqslant R} \frac{|q(y) f(y)|}{|x-y|^{1/2}}\, dy.$$

If we iterate this, and go through the same manoeuvres, we obtain

$$|f(x)| \leqslant C + C \int_{|z| \leqslant R} |q(z) f(z)|\, dz \left\{ \int_{|y| \leqslant R} \frac{dy}{|x-y|\,|y-z|} \right\}^{1/2}.$$

But

$$\int_{|y| \leqslant R} \frac{dy}{|x-y|\,|y-z|} \leqslant \frac{1}{2} \int_{|y| \leqslant R} \frac{dy}{|x-y|^2} + \frac{1}{2} \int_{|y| \leqslant R} \frac{dy}{|y-z|^2} \leqslant C,$$

and so

$$|f(x)| \leqslant C + C \int_{|z| \leqslant R} |q(z) f(z)|\, dz \leqslant C$$

since both $\mathbf{q}$, $\mathbf{f}$ are locally L^2. This concludes the proof of the lemma and so of Theorem 2.

5. Analytic Continuation of the Scattering Operator

We shall now emply the results of the preceding sections to obtain an analytic continuation of the scattering operator. The method of continuation will be based upon a representation of the scattering operator which has been obtained by Ikebe [13] for potentials of the sort we are considering, but satisfying the weaker asymptotic condition $q(x) = O(|x|^{-3-\delta})$ as $|x| \to \infty$ for some $\delta > 0$.

Let $\hat{u}$ denote the Fourier transform of the function $u \in L^2(E_3)$. The scattering operator S takes the following form in the Fourier transform space:

$$(Su)\hat{}(k) = \hat{u}(k) - i\,|k| \int_{\Omega} F(|k|, \omega, \omega')\, \hat{u}(|k|, \omega')\, d\omega'. \tag{5.1}$$

Here Ω denotes the totality of unit vectors $\omega' \in E_3$, and $\hat{u}(|k|, \omega') \equiv \hat{u}(k')$, where $k' = |k|\,\omega'$. The kernel $F(|k|, \omega, \omega')$ appearing in (5.1) is given in terms of generalized eigenfunctions $\phi(x, k)$ $(x,\ k \in E_3)$ associated with the generalized eigenvalues $|k|^2 > 0$ of the continuous spectrum of the Schroedinger operator $-\Delta + M(q)$. The defining equation for F is

$$F(|k|, \omega, \omega') = \frac{1}{8\pi^2} \int_{E_3} e^{i|k|\omega'\cdot x}\, q(x)\, \phi(x, |k|, -\omega)\, dx, \tag{5.2}$$

where $\phi(x, |k|, -\omega) \equiv \phi(x, k)$, $k = -|k|\,\omega$. The kernel $F(|k|, \omega, \omega')$ defines an integral operator of Hilbert-Schmidt type on the space $L^2(\Omega)$ for $|k| > 0$, as has been demonstrated by Ikebe in [13]. The generalized eigenfunctions $\phi(x, k)$ are solutions of the Schroedinger equation

$$-\Delta_x \phi + q\phi = |k|^2 \phi,$$

and are obtained as bounded solutions of the corresponding integral equation

$$\phi(x, k) = e^{ik\cdot x} - \frac{1}{4\pi} \int_{E_3} \frac{e^{i|k||x-y|}}{|x-y|}\, q(y)\, \phi(y, k)\, dy. \tag{5.3}$$

A closer examination of the above integral equation satisfied by ϕ shows that ϕ has the asymptotic expansion

$$\phi(x, k) = e^{ik\cdot x} - \frac{2\pi}{|x|} e^{i|k||x|} F(|k|, -\omega_k, -\omega_x) + 0(|x|^{-1}) \quad \text{as} \quad |x| \to \infty,$$

where $\omega_k = k/|k|$, $\omega_x = x/|x|$.

As in the case of the resolvent kernel, it is the integral equation (5.3) satisfied by ϕ which allows us to obtain an analytic continuation. We begin by writing (5.3) in the form

$$\phi(x, \kappa, \omega) = e^{i\kappa x\cdot\omega} - \frac{1}{4\pi} \int_{E_3} \frac{e^{i\kappa|x-y|}}{|x-y|}\, q(y)\, \phi(y, \kappa, \omega)\, dy, \tag{5.4}$$

where $\kappa = |k|$, $\omega = k/|k|$. Equation (5.4) has an obvious extension to complex values of κ, and when extended in this manner, takes the form

$$\phi(x, \kappa, \omega) = \psi_\kappa(x, \omega) + (T_\kappa \phi)(x, \kappa, \omega), \tag{5.5}$$

with $\psi_\kappa(x, \omega) = e^{i\kappa x\cdot\omega}$. Since ψ_κ is an analytic $\mathscr{H}$-valued function of κ for κ in the strip $|\operatorname{im} \kappa| < \frac{1}{2}\alpha$, the Equation (5.5) has the meromorphic $\mathscr{H}$-valued solution

$$\phi = (I - T_\kappa)^{-1} \psi_\kappa$$

for κ belonging to the strip $|\operatorname{im} \kappa| < \frac{1}{2}\alpha$.

Let $Q : \mathscr{H} \to L^1(E_3)$ denote the transformation of multiplication by **q**:

$$(Qf)(x) = q(x) f(x).$$

It is readily observed that Q is a bounded transformation. As a matter of fact, since $|q(x)| \leqslant Ce^{-\alpha|x|}$ for $|x|$ sufficiently large, say $|x| \geqslant R$, we have (by Schwarz's inequality)

$$\int_{|x|\geqslant R} |q(x) f(x)|\, dx \leqslant C \int_{|x|\geqslant R} |f(x)|\, e^{-\alpha|x|}\, dx \leqslant C \|f\|_{\mathscr{H}},$$

while

$$\left(\int_{|x|\leqslant R} |q(x) f(x)|\, dx\right)^2 \leqslant \int_{|x|\leqslant R} |q(x)|^2 e^{\alpha|x|}\, dx \int_{|x|\leqslant R} |f|^2 e^{-\alpha|x|}\, dx$$
$$\leqslant C \|f\|_{\mathscr{H}}^2,$$

which shows that $\|Qf\|_{L^1} \leqslant C \|f\|_{\mathscr{H}}$. As a consequence,

$$(Q\phi)(x, \kappa, \omega) \equiv q(x)\phi(x, \kappa, \omega)$$

is a meromorphic $L^1(E_3)$-valued function.

By considering ϕ in the form given by (5.5), it can be established without difficulty (by estimates similar to those of the previous sections) that

$$\phi(x, \kappa, \omega) = O(e^{|\operatorname{Im}\kappa||x|})$$

as $|x| \to \infty$, uniformly in $\omega \in \Omega$. It follows directly that the product function $g(x) = q(x)\phi(x, \kappa, \omega)$ behaves no worse than $O(e^{-\frac{1}{2}\alpha|x|})$ as $|x| \to \infty$ so long as κ is restricted to lie in the strip $|\operatorname{im} \kappa| < \frac{1}{2}\alpha$. Because of this exponential decay of the function **g**, the Fourier transform

$$\hat{g}(k) = \frac{1}{(2\pi)^{3/2}} \int_{E_3} e^{ik\cdot x} g(x)\, dx$$
$$= \frac{1}{(2\pi)^{3/2}} \int_{E_3} e^{i|k|\omega\cdot x} g(x)\, dx$$

possesses an analytic continuation to the strip $|\operatorname{im} \xi| < \frac{1}{2}\alpha$, the continuation being given by

$$\hat{g}(\xi, \omega) = \frac{1}{(2\pi)^{3/2}} \int_{E_3} e^{i\xi\omega \cdot x} g(x)\, dx.$$

Consequently the function

$$H(\xi, \kappa, \omega, \omega') = \frac{1}{(2\pi)^{3/2}} \int_{E_3} e^{i\xi\omega \cdot x} q(x)\, \phi(x, \kappa, -\omega')\, dx \tag{5.6}$$

is regular analytic in the variable ξ for $|\operatorname{im} \xi| < \frac{1}{2}\alpha$, and is a meromorphic function of κ for $|\operatorname{im} \kappa| < \frac{1}{2}\alpha$.

If we now compare (5.6) with the expression (5.2) for $F(|k|, \omega, \omega')$, it becomes immediately apparent that $F(|k|, \omega, \omega')$ has an analytic continuation to the strip $|\operatorname{im} \kappa| < \frac{1}{2}\alpha$ given by

$$F(\kappa, \omega, \omega') = (8\pi)^{-1/2} \cdot H(\kappa, \kappa, \omega, \omega'), \tag{5.7}$$

and is a meromorphic function of κ having poles only in the half-strip $-\frac{1}{2}\alpha < \operatorname{im} \kappa < 0$ and on the nonnegative imaginary axis. For each fixed κ other than a pole, $F(\kappa, \omega, \omega')$ is a kernel of Hilbert-Schmidt class on the Hilbert space $L^2(\Omega)$. This follows from the uniform continuity of $F(\kappa, \omega, \omega')$ in $\omega, \omega' \in \Omega$, a fact which scarcely requires proof.

Let $S(\kappa) : L^2(\Omega) \to L^2(\Omega)$ be the operator defined by

$$[S(\kappa)\, u]\,(\omega) = u(\omega) - i\kappa \int_{\Omega} F(\kappa, \omega, \omega')\, u(\omega')\, d\omega'.$$

For $\kappa > 0$, $S(\kappa)$ is the scattering operator associated with the fixed value $\kappa^2 > 0$ of the kinetic energy. The above discussion shows that, aside from the poles on the non-negative imaginary axis (which, with the possible exception of $\kappa = 0$, correspond to bound states of the Schroedinger operator), the scattering operator $S(\kappa)$ admits no other singularities in that part of its domain of analyticity which lies in the closed upper half-plane $\operatorname{im} \kappa \geqslant 0$. This analytic property of $S(\kappa)$ on the so-called "physical sheet" $\operatorname{im} \kappa \geqslant 0$ is generally ascribed to the scattering operator by physicists. A second property, the symmetry relation

$$F(\kappa, \omega, \omega') = \overline{F(-\bar{\kappa}, \omega, \omega')}, \tag{5.8}$$

is an easy consequence of the formula (5.7), (5.6), and (5.4). From (5.8) we deduce that the poles of $S(\kappa)$ on the "unphysical sheet" $\operatorname{im} \kappa < 0$ are symmetrically placed with respect to the negative imaginary axis.

Finally, we note that in the case the potential $q(x)$ has compact support, or else vanishes faster than every exponential $e^{-\beta|x|}$ as $|x| \to \infty$, the constant α appearing in the foregoing analysis may be chosen arbitrarily large, with the result that $S(\kappa)$ is a meromorphic operation defined in the entire κ-plane.

To sum up this section, we have

THEOREM 3. *The scattering operator, under the conditions of Theorem 1, can be continued analytically through* $|\operatorname{im} \kappa| < \frac{1}{2}\alpha$, *and it is in this region meromorphic with poles confined to the subregion* $-\frac{1}{2}\alpha < \operatorname{im} \kappa < 0$ *and to the nonnegative imaginary axis. The poles in* $\operatorname{im} \kappa < 0$ *are symmetrically placed with respect to the negative imaginary axis.*

6. EXAMPLES

In the next three sections we look at three examples, and examine them in the light of the general theory developed in the earlier parts of the paper. The first example is that of the potential arising from a partially transparent sphere of radius **a**, and this gives us a delta-function potential:

$$q(x) = V\delta(\rho - a) \qquad (\rho = |x|),$$

where V is some constant, of either sign.
(This has been previously discussed by Nussenzveig ([14], p. 121) and Petzold [15].) Strictly speaking, it does not come within the conditions of our theorems since the potential is not locally square-integrable, but the formal analysis is very straightforward, and it is interesting to see how it reflects the earlier results.)

The second example is that of a box potential:

$$q(x) = \begin{cases} V & (\rho \leqslant a), \\ 0 & (\rho > a). \end{cases}$$

The third is that of an exponential potential:

$$q(x) = -Ve^{-\rho/a}.$$

Again in both cases V is a constant, of either sign.

In every case, what we do is to look for nontrivial spherically symmetric solutions of the equation

$$(I - T_\kappa)f = 0.$$

(Clearly, spherically symmetric solutions are not the only possible ones, but they are the only ones for which the analysis is reasonable.) In every case, we find that the condition for such nontrivial solutions can be put in the form

$F(\kappa) = 0$, where F is an integral function of κ (different, of course for the three cases.) This suggests, although it certainly does not prove, that $(I - T_\kappa)^{-1}$ is a meromorphic function of κ, with poles at the zeros of $F(\kappa)$.

There is an essential difference between the first two cases and the third, in that the derivation in the case of the first two suggests no other restrictions on κ; and this is in accordance with the general theory, for the potentials are $O(e^{-\alpha|x|})$ for any α, and so we would expect $(I - T_\kappa)^{-1}$ to be meromorphic throughout the entire κ-plane. But convergence difficulties in the exponential case force us to restrict ourselves to im $\kappa > (2a)^{-1}$, and this is just the limitation imposed by the general theory. In some sense therefore the general theory is best-possible.

Since the examples are at best no more than suggestive, the discussion of the analytical details has been kept fairly formal. We discuss the delta-function potential in the remainder of this section, the box potential in Section 7, and the exponential case in Section 8.

The equation

$$(I - T_\kappa)f = 0$$

is in full

$$f(x) = -\frac{1}{4\pi}\int_{E_3} \frac{e^{i\kappa|x-y|}}{|x-y|} q(y) f(y)\, dy. \tag{6.1}$$

If we suppose that we look only for solutions which are spherically symmetric, and let ρ, θ, ϕ; ρ', θ', ϕ' be the spherical polar coordinates of x, y, respectively, then, since our potentials are all spherically symmetric, we obtain

$$f(\rho) = -\frac{1}{4\pi}\int_0^\infty q(\rho') f(\rho')\, \rho'^2\, d\rho' \int_{\theta',\phi'} \frac{e^{i\kappa|x-y|}}{|x-y|} \sin\theta'\, d\theta'\, d\phi'.$$

If, fixing $\mathbf{x}$ temporarily, we choose the direction of the axes so that θ' is the angle between $\mathbf{x}, \mathbf{y}$, the inner integrand in the last formula line is a function of θ' but not of ϕ', and the integral evaluates easily by use of the transformation $\cos\theta' = t$. The final result is

$$f(\rho) = -\frac{1}{2i\kappa\rho}\int_0^\infty [e^{i\kappa(\rho+\rho')} - e^{i\kappa|\rho-\rho'|}]\, q(\rho') f(\rho')\, \rho'\, d\rho',$$

or, with $g(\rho) = \rho f(\rho)$,

$$g(\rho) = -\frac{1}{2i\kappa}\int_0^\infty [e^{i\kappa(\rho+\rho')} - e^{i\kappa|\rho-\rho'|}]\, q(\rho') g(\rho')\, d\rho'. \tag{6.2}$$

Now substitute the delta-function for $\mathbf{q}$ and carry out the integration to give

$$g(\rho) = -\frac{V}{2i\kappa}\{e^{i\kappa(\rho+a)} - e^{i\kappa|\rho-a|}\}\, g(a), \tag{6.3}$$

which implies that, for a nontrivial solution, we must have $g(a) \neq 0$. But if we insert $\rho = a$ in (6.3), we have

$$g(a) \left\{ 1 + \frac{V}{2i\kappa} (e^{2i\kappa a} - 1) \right\} = 0,$$

so that $g(a) \neq 0$ implies that

$$e^{2i\kappa a} - 1 = - \frac{2i\kappa}{V}. \tag{6.4}$$

If we take $V > 0$, it is easy to verify, by trying a solution for κ of the form $\kappa = s + it$ $(t > 0)$, that such a solution is impossible, and that even if $t = 0$, then the only possibility is $s = 0$. Hence the solutions of (6.4), except for $\kappa = 0$, are restricted to im $\kappa < 0$, and this is in accordance with the general theory. Even if $V < 0$, the solutions $\kappa = s + it$ $(t > 0)$ must have $s = 0$ according to the general theory, but it is not so easy to see this in the particular example.

7. We now turn to the case of the box potential, also discussed by Nussenzveig ([14], p. 97). Inserting the new value for $\mathbf{q}$ in (6.2), we obtain

$$-\frac{2i\kappa g(\rho)}{V} = e^{i\kappa\rho} \int_0^a e^{i\kappa\rho'} g(\rho')\, d\rho' - e^{i\kappa\rho} \int_0^\rho e^{-i\kappa\rho'} g(\rho')\, d\rho' - e^{-i\kappa\rho} \int_\rho^a e^{i\kappa\rho'} g(\rho')\, d\rho'. \tag{7.1}$$

(It will be sufficient to consider $\rho \leqslant a$.)

Now we know that a solution of (6.1) is also, formally at least, a solution of Schroedinger's equation, and this could alternatively be verified by differentiating (7.1). We thus have

$$g'' + (\kappa^2 - V) g = 0,$$

whence

$$g = Ae^{i\tau\rho} + Be^{-i\tau\rho},$$

A, B arbitrary constants, $\tau^2 = \kappa^2 - V$. Since, by definition, we must have $g(0) = 0$, we obtain that $B = -A$,

$$g = A(e^{i\tau\rho} - e^{-i\tau\rho}).$$

Substituting this back into (7.1), we see that a nontrivial solution implies that

$$-\frac{2i\kappa}{V}(e^{i\tau\rho}-e^{-i\tau\rho}) = e^{i\kappa\rho}\int_0^a \{e^{i(\kappa+\tau)\rho'}-e^{i(\kappa-\tau)\rho'}\}\,d\rho'$$
$$-e^{i\kappa\rho}\int_0^\rho \{e^{i(-\kappa+\tau)\rho'}-e^{i(-\kappa-\tau)\rho'}\}\,d\rho'$$
$$-e^{-i\kappa\rho}\int_\rho^a \{e^{i(\kappa+\tau)\rho'}-e^{i(\kappa-\tau)\rho'}\}\,d\rho'.$$

Evaluating the various integrals, we readily verify that this reduces to

$$-\frac{2i\kappa}{V}(e^{i\tau\rho}-e^{-i\tau\rho}) = e^{i\tau\rho}\left\{\frac{1}{i(\kappa+\tau)}+\frac{1}{i(\kappa-\tau)}\right\}$$
$$+e^{-i\tau\rho}\left\{\frac{1}{-i(k+\tau)}-\frac{1}{i(\kappa-\tau)}\right\}$$
$$+2i\sin\kappa\rho\, e^{i\kappa a}\left\{\frac{e^{i\tau a}}{i(\kappa+\tau)}-\frac{e^{-i\tau a}}{i(\kappa-\tau)}\right\}$$
$$=\frac{2\kappa e^{i\tau\rho}}{iV}-\frac{2\kappa e^{-i\tau\rho}}{iV}$$
$$+\frac{2\sin\kappa\rho\, e^{i\kappa a}}{V}\{(\kappa-\tau)\,e^{i\tau a}-(\kappa+\tau)\,e^{-i\tau a}\},$$

since

$$\kappa^2-\tau^2=V.$$

We therefore conclude that for a nontrivial solution the last term in the last formula-line vanishes for all ρ, i.e.,

$$(\kappa-\tau)\,e^{i\tau a}-(\kappa+\tau)\,e^{-i\tau a}=0,$$

or

$$\tau\cot\tau a = i\kappa.$$

We can cast this into the form $F(\kappa)=0$, where $F(\kappa)$ is a integral function of κ. The zeros of $F(\kappa)$ presumably provide the poles of the meromorphic function $(I-T_\kappa)^{-1}$.

8. We now turn to the last example, in which

$$q(x) = -\,Ve^{-\rho/a} \qquad (\rho = |\,x\,|).$$

(Cf. Ma [16].)

Once again, we look for solutions of $(I - T_\kappa) f = 0$ which are spherically symmetric. Since any solution must be also a solution of Schroedinger's equation, we have as before that, if $g(\rho) = \rho f(\rho)$, then

$$g'' + \{\kappa^2 + Ve^{-\rho/a}\} g = 0,$$

whence it is well-known that

$$g = A J_{2i\kappa a}(2aV^{1/2} e^{-\rho/2a}) + B J_{-2i\kappa a}(2aV^{1/2} e^{-\rho/2a}),$$

or, since $g(0) = 0$,

$$g = C\{J_{-2i\kappa a}(2aV^{1/2}) J_{2i\kappa a}(2aV^{1/2} e^{-\rho/2a}) - J_{2i\kappa a}(2aV^{1/2}) J_{-2i\kappa a}(2aV^{1/2} e^{-\rho/2a})\}. \tag{8.1}$$

(A, B, C are arbitrary constants.)

Since, as before, the integral equation satisfied by **g** is

$$g(\rho) = \frac{V}{2i\kappa} \int_0^\infty [e^{i\kappa(\rho+\rho')} - e^{i\kappa|\rho-\rho'|}] e^{-\rho'/a} g(\rho') \, d\rho',$$

we can substitute from (8.1), and if we write

$$\nu = 2i\kappa a, \qquad \alpha = 2aV^{1/2}, \qquad t = 2aV^{1/2} e^{-\rho/2a},$$

we obtain (for a nontrivial solution) that

$$\begin{aligned} &\frac{2i\kappa\{J_{-\nu}(\alpha) J_\nu(t) - J_\nu(\alpha) J_{-\nu}(t)\}}{V} \\ &= \left(\frac{\alpha}{t}\right)^\nu \int_0^\alpha \left(\frac{2a}{\alpha}\right) \left(\frac{\alpha}{t'}\right)^{\nu-1} \{J_{-\nu}(\alpha) J_\nu(t') - J_\nu(\alpha) J_{-\nu}(t')\} \, dt' \\ &\quad - \left(\frac{\alpha}{t}\right)^\nu \int_t^\alpha \left(\frac{2a}{\alpha}\right) \left(\frac{\alpha}{t'}\right)^{-\nu-1} \{J_{-\nu}(\alpha) J_\nu(t') - J_\nu(\alpha) J_{-\nu}(t')\} \, dt' \\ &\quad - \left(\frac{\alpha}{t}\right)^{-\nu} \int_0^t \left(\frac{2a}{\alpha}\right) \left(\frac{\alpha}{t'}\right)^{\nu-1} \{J_{-\nu}(\alpha) J_\nu(t') - J_\nu(\alpha) J_{-\nu}(t')\} \, dt'. \end{aligned} \tag{8.2}$$

(Since $J_\nu(t)$ behaves for small **t** like t^ν, we note that the convergence of the various integrals at the origin demands that $\operatorname{re} \nu < 1$.)

We now use the relations

$$\int t^{1-\nu} J_\nu(t) \, dt = - t^{1-\nu} J_{\nu-1}(t),$$

$$\int t^{1-\nu} J_{-\nu}(t) \, dt = t^{1-\nu} J_{-\nu+1}(t),$$

$$\int t^{\nu+1} J_\nu(t) \, dt = t^{\nu+1} J_{\nu+1}(t),$$

$$\int t^{\nu+1} J_{-\nu}(t) \, dt = - t^{\nu+1} J_{-\nu-1}(t),$$

whence (8.2) simplifies to

$$\frac{i\kappa\{J_{-\nu}(\alpha)\, J_\nu(t) - J_\nu(\alpha)\, J_{-\nu}(t)\}}{V}$$

$$= \left(\frac{\alpha}{t}\right)^\nu \left(\frac{a}{\alpha}\right) [\alpha^{\nu-1}\, J_{-\nu}(\alpha)\, \{-\, \alpha^{1-\nu}\, J_{\nu-1}(\alpha) + c_\nu\} - \alpha^{\nu-1}\, J_\nu(\alpha)\, \{\alpha^{1-\nu}\, J_{-\nu+1}(\alpha)\}]$$

$$- \left(\frac{\alpha}{t}\right)^\nu \left(\frac{a}{\alpha}\right) [\alpha^{-\nu-1}\, J_{-\nu}(\alpha)\, \{\alpha^{\nu+1}\, J_{\nu+1}(\alpha) - t^{\nu+1}\, J_{\nu+1}(t)\}$$

$$+ \alpha^{-\nu-1}\, J_\nu(\alpha)\, \{\alpha^{\nu+1}\, J_{-\nu-1}(\alpha) - t^{\nu+1}\, J_{-\nu-1}(t)\}]$$

$$- \left(\frac{\alpha}{t}\right)^{-\nu} \left(\frac{a}{\alpha}\right) [\alpha^{\nu-1}\, J_{-\nu}(\alpha)\, \{-\, t^{1-\nu}\, J_{\nu-1}(t) + c_\nu\} - \alpha^{\nu-1}\, J_\nu(\alpha)\, \{t^{1-\nu}\, J_{-\nu+1}(t)\}], \tag{8.3}$$

where

$$c_\nu = \lim_{t\to 0} t^{1-\nu}\, J_{\nu-1}(t)$$

and where the term

$$\lim_{t\to 0} t^{1-\nu}\, J_{-\nu+1}(t)$$

vanishes because re $\nu < 1$. By using the identity

$$2\nu\, J_\nu(t) = t\{J_{\nu-1}(t) + J_{\nu+1}(t)\},$$

we find that the terms involving Bessel functions in **t** cancel on the two sides of (8.3); while the terms involving $J_{\nu-1}(\alpha)$, $J_{-\nu+1}(\alpha)$, $J_{\nu+1}(\alpha)$, $J_{-\nu-1}(\alpha)$ similarly disappear in the right-hand side. In order therefore that (8.3) should hold for all **t**, we are forced to conclude that $J_{-\nu}(\alpha) = 0$, i.e.,

$$J_{-2i\kappa a}(2aV^{1/2}) = 0.$$

This is the required analytic equation in κ. There is nothing in this equation which implies a restriction of the form im $\kappa > -1/2a$, which our general theory leads us to expect. On the other hand, reasons of convergence have led us throughout to demand re $\nu < 1$, and this is precisely im $\kappa > -1/2a$.

Appendix

The object of this appendix is to provide two examples which were mentioned in Section 4. The first is of a potential with compact support which has zero as an eigenvalue with an L^2-eigenfunction. The second is of a potential for which $(I - T_\kappa)^{-1}$ has a pole at $\kappa = 0$ with an ,'eigenfunction'' which is not L^2.

Both examples are constructed similarly. To take the first, consider the function $u(x)$ given by

$$u(x) = \begin{cases} \dfrac{1}{r^2}\cos\theta & (r \geqslant a), \\ f(r)\cos\theta & (r < a), \end{cases} \tag{1}$$

where (r, θ, ϕ) are the usual three-dimensional spherical polars, **a** is some positive number and $f(r)$ is a function, three times continuously differentiable, which is to be further determined later. In order that $u(x)$ be continuously differentiable, we must have

$$f(a) = \frac{1}{a^2}, \qquad f'(a) = -\frac{2}{a^3}. \tag{2}$$

For $r > a$, it is clear that $u(x)$ satisfies $\Delta u = 0$. For $r < a$, we readily verify that it satisfies the equation

$$\Delta u = qu,$$

where

$$q = \frac{\left\{\dfrac{1}{r^2}\dfrac{d}{dr}(r^2 f') - \dfrac{2f}{r^2}\right\}}{f}.$$

Now we can define the required potential by

$$q(x) = \begin{cases} 0 & (r > a), \\ \dfrac{\left\{\dfrac{1}{r^2}\dfrac{d}{dr}(r^2 f') - \dfrac{2f}{r^2}\right\}}{f} & (r < a), \end{cases}$$

the required eigenfunction being given by (1). The only restrictions necessary on **f** are that (2) hold; that

$$\frac{\left\{\dfrac{1}{r^2}\dfrac{d}{dr}(r^2 f') - \dfrac{2f}{r^2}\right\}}{f} \tag{3}$$

and its derivative vanish at $r = a$, if we want to insist that **q** be continuously differentiable and so Hoelder continuous; that **f** not vanish in $r < a$ in order that **q** have no singularities there; and finally, in order to cope with the exceptional point $r = 0$, that (3) tend to a limit as $r \to 0$. This last condition can be met if we insist that **f** behave sufficiently like **r** as $r \to 0$, and there is no difficulty in choosing an **f** which satisfies this and the other conditions as well.

To get the second example, we consider $v(x)$ given by

$$v(x) = \begin{cases} \dfrac{1}{r} & (r \geqslant a), \\ f(r) & (r < a), \end{cases}$$

and as before construct a $\mathbf{q}$ with compact support for which $\Delta v = qv$. Certainly $(I - T_\kappa)^{-1}$ now has a pole at $\kappa = 0$, for $\mathbf{v}$ is a solution in $\mathscr{H}$ of $v = T_0 v$. But $\mathbf{v}$ is not L^2.

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