JOURNAL OF ALGEBRA 4, 442-470 (1966)

Pseudocompact Algebras, Profinite Groups and Class Formations

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Communicated by D. A. Buchsbaum

Received December 23, 1965

INTRODUCTION

A *pseudocompact ring* Λ is a complete Hausdorff topological ring which admits a system of open neighborhoods of 0 consisting of two sided ideals I for which Λ/I is an Artin ring. A complete Hausdorff topological Λ-module M is said to be *pseudocompact* if it has a system of open neighborhoods of 0 consisting of submodules N for which M/N has finite length. The category $\mathscr{C}_\Lambda$ of pseudocompact Λ-modules is an Abelian category with exact inverse limits and enough projectives. Such generalities, which are more or less well known, are gathered in the first section for the convenience of the reader.

To get more interesting results, we must introduce some commutativity by assuming that, in addition, Λ is a pseudocompact algebra over a commutative pseudocompact ring Ω (see definition in Section 2). We may then define a tensor product on $\mathscr{C}_\Lambda$ and introduce its derived functor $\mathscr{T}\!or$. The category $\mathscr{D}_\Lambda$ of discrete Λ-modules, which is dual to $\mathscr{C}_{\Lambda^0}$ by Proposition 2.3, also plays an important role through the bifunctor Hom : $\mathscr{C}_\Lambda \times \mathscr{D}_\Lambda \to \mathscr{D}_\Omega$ and its derived functor $\mathscr{E}\!xt$. We then have the proper setting for doing homological algebra. This is done in Section 3 which generalizes the elementary results on homological dimension in complete Noetherian semilocal rings. As an immediate application we find that if $\Omega\{\{x_i\}\}$ is the algebra of noncommutative formal power series in $\{x_i\}$ over Ω, then $\text{gl dim}\,\Omega\{\{x_i\}\} = \text{gl dim}\,\Omega + 1$ (Theorem 3.9).

We recall that a profinite group is a compact totally disconnected topological group, i.e., an inverse limit of finite groups. We define the complete group

* This paper was written while the author was T. H. Hildebrandt Instructor at The University of Michigan. The work was completed with the support of NSF grant No. GP4124. The results were announced in *Bull. Amer. Math. Soc.* **72** (1966), 321-324.

algebra $\Omega[\![G]\!]$ as the inverse limit of the ordinary group algebras $\Omega[G/U]$ as U runs through the open normal subgroups of G. Then $\Omega[\![G]\!]$ is a pseudocompact Ω-algebra which can be expected to play an important role in the study of profinite groups. For instance $\mathcal{E}xt^n_{\Omega[\![G]\!]}(\Omega, A)$ is simply the cohomology group $H^n(G, A)$ so that it is not too surprising that

$$\text{gl dim}\,\Omega[\![G]\!] = \text{gl dim}\,\Omega + cd_\Omega G,$$

where $cd_\Omega G$ is the sup of the cohomological p-dimension of G over all primes p which are not units in Ω (Theorem 4.9). In particular if G is a pro-p-group of finite cohomological dimension and Ω is a complete regular local ring in which p is not a unit, then $\Omega[\![G]\!]$ is a complete noncommutative local ring of finite global dimension. Does it behave like a regular local ring? In the special case where G is free and Ω is a field, the work of Cohn [*5*] implies that $\Omega[\![G]\!]$ can be embedded in a division algebra and that it is a unique factorization domain.

In Section 5, we obtain criteria for determining the cohomological dimension of a profinite group G represented as a quotient F/N of a free profinite group. The most striking is given by Proposition 5.7: $scdG = 2$ if and only if $N/[N, N]$ is a projective $\hat{\mathbf{Z}}[\![G]\!]$-module and $N \cap [V, V] = [N, V]$ for every open subgroup V of F containing N, where $\hat{\mathbf{Z}}$ is the total completion of the integers in the ideal topology.

As a consequence of much deeper considerations, Tate has shown that the Galois groups of the formations of local and global class field theory have strict cohomological dimension 2 (this is not quite true for number fields; cf. Section 6). It is possible to unify these results in the following general theorem on class formations: if the kernel and co-kernel of the reciprocity map are cohomologically trivial then the strict cohomological dimension of the Galois group of the formation is equal to 2 (Theorem 6.1).

In the appendix we have included a number of simple technical lemmas on limits which could not find their way into the body of the paper.

1. Generalities on Pseudocompact Rings

Let Λ be a pseudocompact ring, then a Λ-module is pseudocompact if and only if it is the inverse limit of Λ-modules of finite length. It follows that the category $\mathcal{C}$ of pseudocompact Λ-modules is an Abelian category with exact inverse limits (cf. [*7*], [*10*], [*11*], [*17*]).

Lemma 1.1. *Let $f : A \to B$ be an epimorphism in $\mathcal{C}$. Then there is a continuous section $s : B \to A$ such that $fs(b) = b$ for all b in B.*

The proof is left to the reader who may imitate that of Proposition 1 in Reference [*14*].

Let I be any index set and let $L(X)$ be the free Λ-module on a set X indexed by I. Let S be the collection of all submodules N of $L(X)$ which contain all but a finite number of elements of X and such that $L(X)/N$ has finite length. Then $F(X) = \varprojlim L(X)/N$ where N runs through S, is a pseudocompact Λ-module called the *free pseudocompact Λ-module on X* or simply the free Λ-module on X if no confusion ensues. We note that $F(X) = \Lambda^I$, where Λ^I is given the product topology.

LEMMA 1.2. *The morphisms of $F(X)$ into a pseudocompact Λ-module A are in one-to-one correspondence with sequences $\{a_i\}_{i\in I}$ in A which tend to zero on the filter of complements of finite subsets.*

In fact, we associate to the morphism f the sequence $\{f(x_i)\}_{i\in I}$.

COROLLARY 1.3. *$F(X)$ is projective in $\mathscr{C}$.*

Proof. Given a diagram in $\mathscr{C}$

$$\begin{array}{ccccc} & & F(X) & & \\ & \swarrow & & \searrow f & \\ A & & \xrightarrow{\pi} & B & \to 0 \end{array}$$

we can find a continuous section $s : B \to A$ passing through 0 by Lemma 1.1. The sequence $a_i = sf(x_i)$ satisfies the hypothesis of Lemma 1.2 so there is a morphism $g : F(X) \to A$ such that $g(x_i) = sf(x_i)$. The morphism g makes the diagram commutative and thus $F(X)$ is projective.

We denote by R the radical of Λ, i.e., R is the intersection of all open maximal two sided ideals of Λ. Thus R is a closed two sided ideal which is in fact the Jacobson radical of Λ. The following will play the role of the Nakayama lemma.

LEMMA 1.4. *Let A be an object of $\mathscr{C}$. Suppose that $MA = A$ for all open maximal ideals M of Λ. Then $A = 0$ and in particular $RA = A$ implies $A = 0$.*

Proof. Let B be an open submodule of A. Since A is a topological Λ-module, we may find an open ideal I_x and a neighborhood V_x of x in A such that $I_x V_x \subset B$ and we may choose V_0 to be an open submodule of B for which A/V_0 has finite length and *a fortiori* is finitely generated, say, by the images of $x_1, \cdots, x_n$. Let

$$I = I_0 \cap I_{x_1} \cap \cdots \cap I_{x_n};$$

then $IA \subset B$. Since Λ/I is an Artin ring with nilpotent radical R/I, it follows that for some finite set $M_1, \cdots, M_k$ of open maximal ideals and some integer m, we have

$$(M_1 M_2 \cdots M_k)^m \subset I.$$

But $A = (M_1 \cdots M_k)^m A \subset IA \subset B$ by hypothesis; hence A is contained in every open neighborhood B of 0 and thus $A = 0$.

COROLLARY 1.5. *Let A be in $\mathscr{C}$ and let $\{x_i\}$ be a family of elements of A. Then $\{x_i\}$ generate A as a topological Λ-module if and only if their images $\{\bar{x}_i\}$ generate A/RA as a topological Λ/R-module. If A/RA is finitely generated by $\bar{x}_1, \cdots, \bar{x}_n$, then $A = \Lambda x_1 + \cdots + \Lambda x_n$.*

Proof. Let B be the closed Λ-submodule generated by elements $\{x_i\}$ whose images generate A/RA. We note that if $\{x_i\}$ is a finite set, then we may take $B = \Lambda x_1 + \cdots + \Lambda x_n$ which is closed since it is the continuous image of Λ^n. Our hypothesis shows that $B + RA = A$, i.e., $R(A/B) = A/B$, hence $A = B$ by Lemma 1.4.

LEMMA 1.6. *Every object A in $\mathscr{C}$ is the quotient of a free object, i.e., there are enough projectives. In particular, the projectives are the direct summands of the free objects.*

Proof. The result follows from Lemma 1.2 once we find a generating set $\{x_i\}$ for A with the following property: each open submodule U of A contains all but a finite number of $\{x_i\}$. This is done as in Theorem 1.3 of [6].

Remark 1.7. There is a more categorical proof in Chap. IV of Reference [7] which is a convenient source for other facts mentioned in this section.

Dually, we have the following well-known result.

LEMMA 1.8. *Let Λ be a pseudocompact ring. Then the category $\mathscr{D}$ of discrete Λ-modules is an Abelian category with exact direct limits and enough injectives.*

Proof. We note that $\mathscr{D}$ is a full subcategory of the category of Λ-modules. For any Λ-module A, let A^0 be the set of elements of A annihilated by some open ideal of Λ. Then $A \rightsquigarrow A^0$ is a covariant functor from Λ-modules to $\mathscr{D}$ which takes injectives into injectives. The claims can be deduced easily from this observation.

2. $\mathscr{E}xt$ AND $\mathscr{T}or$ OVER PSEUDOCOMPACT ALGEBRAS

Let Ω be a commutative pseudocompact ring. The complete Hausdorff topological ring Λ will be said to be a *pseudocompact algebra over Ω* if:

(i) Λ is an Ω-algebra in the usual sense,

(ii) Λ admits a system of open neighborhoods of 0 consisting of two sided ideals I such that Λ/I has finite length as Ω-module. Clearly such a ring Λ is pseudocompact and a Λ-module A has finite length if and only if it has finite length as Ω-module.

We define a "tensor product" for Λ-modules by its universal property. Explicitly, let A be a right and B a left pseudocompact Λ-module; then their *complete tensor product* is a pseudocompact Ω-module $A \hat{\otimes}_\Lambda B$ and a Λ-bihomomorphism[1] $\alpha : A \times B \to A \hat{\otimes}_\Lambda B$ with the following property : given *any* Λ-bihomorphism f of $A \times B$ into a pseudocompact Ω-module C, there is a *unique* morphism of Ω-modules $g : A \hat{\otimes}_\Lambda B \to C$ such that $g\alpha = f$. We construct the tensor product as follows: $A \hat{\otimes}_\Lambda B = \varprojlim A/U \otimes_\Lambda B/V$, where U (resp. V) runs through the open submodules of A (resp. B). Since A/U and B/V are Ω-modules of finite length so is $A/U \otimes_\Lambda B/V$, and thus $A \hat{\otimes}_\Lambda B$ is a pseudocompact Ω-module. The natural bihomomorphisms $A \times B \to A/U \otimes_\Lambda B/V$ induce the desired bihomomorphism $\alpha : A \times B \to A \hat{\otimes}_\Lambda B$ upon passage to the limit.

The exact sequence

$$0 \to \operatorname{Im}(A \otimes_\Lambda V + U \otimes_\Lambda B) \to A \otimes_\Lambda B \to \frac{A}{U} \otimes_\Lambda \frac{B}{V} \to 0$$

shows that $A \hat{\otimes}_\Lambda B$ is the completion of $A \otimes_\Lambda B$ in the topology induced by taking $\operatorname{Im}(A \otimes_\Lambda V + U \otimes_\Lambda B)$ as a fundamental system of open neighborhoods of 0 (cf. References [*8*], [*10*] for the commutative case).

LEMMA 2.1. (i) *The functor $T(A, B) = A \hat{\otimes}_\Lambda B$ is an additive covariant left exact bifunctor from the category of pseudocompact Λ-modules to the category of pseudocompact Ω-modules.*

(ii) *If A is a finitely generated right Λ-module, then $A \hat{\otimes}_\Lambda B = A \otimes_\Lambda B$.*

(iii) *If A is a projective right Λ-module, then $T(A, *)$ is exact.*

(iv) *Similar statements for B.*

Proof. (i) Since inverse limits in $\mathscr{C}$ preserve exactness, (i) follows from the known properties of the tensor product.

(ii) The natural isomorphism

$$\Lambda^n \otimes_\Lambda B = \overbrace{B \oplus \cdots \oplus B}^{n}$$

[1] That is, α is a continuous morphism such that $\alpha(a\lambda, b) = \alpha(a, \lambda b)$ for $a \in A$ $b \in B$ and $\lambda \in \Lambda$.

of pseudocompact Ω-modules shows that

$$\Lambda^n \hat{\otimes}_\Lambda B = \Lambda^n \otimes_\Lambda B.$$

If A is finitely generated, there is an epimorphism $\Lambda^n \otimes_\Lambda B \to A \otimes_\Lambda B$. Thus $A \otimes_\Lambda B$ is pseudocompact and (ii) follows since $A \otimes_\Lambda B$ is dense in $A \hat{\otimes}_\Lambda B$.

(iii) Since T is an additive functor, it follows from Lemma 1.6 that we may suppose A to be a free Λ-module. If A is finitely generated, (iii) is a consequence of (ii). In the general case, write $A = \varprojlim A_i$ where A_i runs through the finitely generated free quotients of A and apply Lemma A.4 of the Appendix.

It follows from the preceeding lemma that we may define the left-derived functors $\mathscr{T}or_n^\Lambda(A, B)$ of $T(A, B) = A \hat{\otimes}_\Lambda B$ by using a projective resolution of either A or B according to established principles of homological algebra (cf. Reference [4]). We note that $\mathscr{T}or_n^\Lambda(A, B)$ is a pseudocompact Ω-module, that $\mathscr{T}or_0^\Lambda(A, B) = A \hat{\otimes}_\Lambda B$ and that $\mathscr{T}or_n^\Lambda(A, B) = 0$ for all $n \geqslant 1$ if A or B is projective.

The category $\mathscr{C}$ of pseudocompact Λ-modules has enough projectives by Lemma 1.6 and the category $\mathscr{D}$ of discrete Λ-modules has enough injectives by Lemma 1.8. The correspondence $A \times B \rightsquigarrow \mathrm{Hom}_\Lambda(A, B)$ is a left exact bifunctor from $\mathscr{C} \times \mathscr{D}$ to the category of discrete Ω-modules which is contravariant in A and covariant in B. We may define the right-derived functors $\mathscr{E}xt_\Lambda^n(A, B)$ by using either a projective resolution of A or an injective resolution of B. We note that $\mathscr{E}xt_\Lambda^0(A, B) = \mathrm{Hom}_\Lambda(A, B)$, and that $\mathscr{E}xt_\Lambda^n(A, B)$ is a discrete Ω-module. The following lemma shows that $\mathscr{E}xt_\Lambda^n(A, B) = 0$ for $n \geqslant 1$ if A is projective or B is injective.

LEMMA 2.2. (i) *If B is an injective in $\mathscr{D}$, then the functor* $\mathrm{Hom}_\Lambda(*, B)$ *is exact on $\mathscr{C}$.*

(ii) *If A is a projective in $\mathscr{C}$, then the functor* $\mathrm{Hom}_\Lambda(A, *)$ *is exact on $\mathscr{D}$.*

Proof. (i) Let $0 \to A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0$ be an exact sequence in $\mathscr{C}$. For any open submodule U_2 of A_2, let $U_1 = f^{-1}(U_2)$ and $U_3 = g(U_2)$, then we have an exact sequence of discrete Λ-module:

$$0 \to \frac{A_1}{U_1} \to \frac{A_2}{U_2} \to \frac{A_3}{U_3} \to 0$$

which gives rise to the exact sequence

$$0 \to \mathrm{Hom}_\Lambda\left(\frac{A_3}{U_3}, B\right) \to \mathrm{Hom}_\Lambda\left(\frac{A_2}{U_2}, B\right) \to \mathrm{Hom}_\Lambda\left(\frac{A_1}{U_1}, B\right) \to 0$$

since B is injective. Taking direct limits over all open submodules U_2, we conclude from Lemma A.3 of the Appendix that

$$0 \to \mathrm{Hom}_\Lambda(A_3, B) \to \mathrm{Hom}_\Lambda(A_2, B) \to \mathrm{Hom}_\Lambda(A_1, B) \to 0$$

is exact since direct limits preserve exactness for discrete modules and

$$A_i = \varprojlim \frac{A_i}{U_i} \qquad \text{for} \qquad i = 1, 2, 3.$$

(ii) is proved similarly.

We next show that the category $\mathscr{C}$ of right pseudocompact Λ-modules is dual to the category $\mathscr{D}$ of discrete left Λ-modules extending a result of Matlis and Gabriel. Let E be the *dualiser of* Ω, i.e., the injective envelope, in the category of discrete Ω-modules, of the module $\bigoplus \Omega/\mathfrak{M}$ where $\mathfrak{M}$ runs through the open maximal ideals of Ω. We define contravariant functors $S : \mathscr{C} \to \mathscr{D}$ by $A \rightsquigarrow \mathrm{Hom}_\Omega(A, E)$ and $T : \mathscr{D} \to \mathscr{C}$ by $C \rightsquigarrow \mathrm{Hom}_\Omega(C, E)$, where $T(C)$ is given the topology of pointwise convergence. The functor T is exact since E is injective while S is exact by Lemma 2.2(i).

PROPOSITION 2.3. *The functors S and T define a duality between $\mathscr{C}$ and $\mathscr{D}$. In fact, their composition is naturally equivalent to the identity functor on the respective category.*

Proof. This has been shown in the commutative case by Gabriel ([7], p. 400) for the full subcategory $\mathscr{F}$ of modules of finite length. The general case follows by writing $A = \varprojlim A/B$, where B runs through the open submodules B of A, and $C = \varinjlim D$ where D runs through the submodules of C of finite length. We then apply Lemma A.3 of the Appendix and the well known rule

$$\varprojlim \mathrm{Hom}_\Omega(A_i, E) = \mathrm{Hom}_\Omega(\varinjlim A_i, E)$$

for discrete modules A_i and E. The operations of Λ are preserved by functoriality.

LEMMA 2.4. *Let Λ and Γ be pseudocompact Ω-algebras. Let A be a right pseudocompact Λ-module, let B be a left (Λ, Γ)-pseudocompact module, and let C be a discrete left Γ-module. Then there is a unique isomorphism:*

$$S : \mathrm{Hom}_\Lambda(A, \mathrm{Hom}_\Gamma(B, C)) \to \mathrm{Hom}_\Gamma(B \mathbin{\hat{\otimes}}_\Lambda A, C)$$

such that $(S\varphi)(b \otimes a) = \varphi(a)(b)$. This morphism induces a natural equivalence of functors.

This is the topological analog of Proposition 5.2 on page 28 of Reference [4].

COROLLARY 2.5. *Same notation as above, but we assume that C is an injective Γ-module. Then we have a natural isomorphism,*

$$\mathscr{E}xt_\Lambda^n (A, \mathrm{Hom}_\Gamma (B, C)) \cong \mathrm{Hom}_\Gamma (\mathscr{T}or_n^\Lambda (B, A), C).$$

This follows from Lemma 2.4 by general principles (cf. Reference [4], p. 120).

COROLLARY 2.6. *Let Λ be a pseudocompact Ω-algebra and let E be the dualizer of Ω. Let A (resp. B) be a right (resp. left) pseudocompact Λ-module. Then we have a natural isomorphism:*

$$\mathscr{T}or_n^\Lambda (A, B) \cong \mathrm{Hom}_\Omega (\mathscr{E}xt_\Lambda^n (B, \mathrm{Hom}_\Omega (A, E)), E).$$

3. HOMOLOGICAL DIMENSION OF PSEUDOCOMPACT ALGEBRAS

We develop here a theory of dimension for a pseudocompact Ω-algebra Λ, thereby generalizing the homological theory of complete semilocal Noetherian rings. Many of the proofs will be omitted since they are formally the same as the classical ones.

We saw in Section 1 that the category $\mathscr{C}$ of pseudocompact Λ-modules has enough projectives. Thus, for any Λ-module A, we may define the ***homological dimension*** of A, written $\mathrm{hd}_\Lambda A$, as the least integer n for which we may find a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to A \to 0$$

in $\mathscr{C}$. If no such resolution exists, we write $\mathrm{hd}_\Lambda A = \infty$ and by convention we set $\mathrm{hd}_\Lambda 0 = -1$. We define the *global dimension*[2] *of* Λ by $\mathrm{gl\,dim}\, \Lambda = \sup \mathrm{hd}_\Lambda A$ as A ranges through $\mathscr{C}$.

PROPOSITION 3.1. *The following are equivalent for a pseudocompact Λ-module A.*

(i) *A is projective.*

(ii) *The functor* $C \rightsquigarrow \mathrm{Hom}_\Lambda (A, C)$ *is exact on the category of Λ-modules C of finite length.*

(iii) $\mathscr{E}xt_\Lambda^1 (A, C) = 0$ *for all simple discrete C.*

(iv) $\mathscr{T}or_1^\Lambda (C, A) = 0$ *for all simple pseudocompact C.*

Proof. We note that simple Λ-modules are necessarily of finite length. (i) → (ii) is a special case of Lemma 2.2(ii). (iii) is equivalent to (iv) by the duality established in Corollary 2.6.

[2] We can also define injective, weak, right and left global dimension but these are easily seen to be equivalent.

(ii) $\rightarrow$ (iii). It follows from Lemma A.3 of the Appendix that the functor $C \rightsquigarrow \mathrm{Hom}\,(A, C)$ is exact on the category $\mathscr{D}$ of discrete Λ-modules. We may embed the module C of finite length into an injective I in $\mathscr{D}$. Since $\mathscr{E}xt_\Lambda^1\,(A, I) = 0$ we conclude that $\mathscr{E}xt_\Lambda^1\,(A, C) = 0$ for all Λ-modules C of finite length.

(iii) $\rightarrow$ (i). In the Abelian category $\mathscr{C}$, we may define the right-derived functors of Hom, which we denote by $\mathscr{E}xt_\Lambda$ either by the Yoneda process via long exact sequences or by projective resolutions of the first variable. The second approach show that $\mathscr{E}xt_\Lambda$ agrees with our old $\mathscr{E}xt$ when the second variable has finite length. The Yoneda approach shows that if A satisfies (iii) then every exact sequence $0 \rightarrow C \rightarrow B \rightarrow A \rightarrow 0$ in $\mathscr{C}$ splits *if C is a simple Λ-module.* This is equivalent to the following: if $\pi : B_1 \rightarrow B_2$ is an epimorphism in $\mathscr{C}$ with simple kernel C, then every morphism $f : A \rightarrow B_2$ can be lifted to a morphism $g : A \rightarrow B_1$ such that $\pi g = f$. To prove that A is projective, it suffices to check that *every* exact sequence $0 \rightarrow C \rightarrow B \rightarrow A \rightarrow 0$ in $\mathscr{C}$ splits.

Consider the collection S of pairs (N, s) consisting of a submodule N of C and a splitting morphism $s : A \rightarrow B/N$ such that $\bar{\pi}s(a) = a$, where $\bar{\pi} : B/N \rightarrow A$ is the morphism induced by π. Since $B/\cap N_i = \varprojlim B/N_i$, we see that S with the obvious partial order is an inductively ordered set with a maximal element, by Zorn's lemma, say (N, s). If $N \neq 0$, we may find an open submodule M of N such that N/M is simple. Hence, by what we saw earlier, there is a morphism $t : A \rightarrow B/M$ making the following diagram commutative:

$$\begin{array}{ccccccccc} & & & & & & A & & \\ & & & & & \overset{t}{\swarrow} & \downarrow s & & \\ 0 & \rightarrow & \dfrac{N}{M} & \rightarrow & \dfrac{B}{M} & \rightarrow & \dfrac{B}{N} & \rightarrow & 0. \end{array}$$

The element (M, t) of S is strictly larger than (N, s). This contradiction proves that A is projective.

Remark. Part of this proof was suggested by that of Proposition 16 of [*14*].

COROLLARY 3.2. *The following are equivalent for a pseudocompact Λ-module A:*

(i) $\mathrm{hd}_\Lambda(A) < n$*;*

(ii) $\mathscr{E}xt_\Lambda^n\,(A, C) = 0$ *for all simple Λ-modules C;*

(iii) $\mathscr{T}or_n^\Lambda\,(A, C) = 0$ *for all simple Λ-modules C.*

COROLLARY 3.3. *Let* $(\Lambda_i, \lambda_{ij})$ *be an inverse system of pseudocompact* Ω*-algebras and let* (A_i, α_{ij}) *be an inverse system of projective* Λ_i*-modules. Suppose* λ_{ij} *and* α_{ij} *are surjective. Then* $A = \varprojlim A_i$ *is a projective module over* $\varprojlim \Lambda_i = \Lambda$.

Proof. Let C be a Λ-module of finite length, then ker λ_i is contained in the annihilator of C for some i by Lemma A.1. Hence C is a Λ_i-module for a cofinal set of i's. Let

$$0 \to B \to C \to D \to 0$$

be an exact sequence of Λ-modules of finite length and let i be as above; then

$$0 \to \mathrm{Hom}_{\Lambda_i}(A_i, B) \to \mathrm{Hom}_{\Lambda_i}(A_i, C) \to \mathrm{Hom}_{\Lambda_i}(A_i, D) \to 0$$

is exact and thus by Lemma A.3, the functor $C \rightsquigarrow \mathrm{Hom}_\Lambda(A, C)$ is exact for modules of finite length. The corollary follows from Proposition 3.1.

LEMMA 3.4. *Let* (A_i, α_{ij}) *be an inverse system of pseudocompact* Λ*-modules and let C be a discrete* Λ*-module. If all* α_{ij} *are epimorphisms, then we have natural isomorphisms,*

$$\varinjlim \mathcal{E}xt_\Lambda^n (A_i, C) = \mathcal{E}xt_\Lambda^n (\varprojlim A_i, C)$$

In particular,

$$\mathrm{hd}_\Lambda (\varprojlim A_i) \leqslant \sup \mathrm{hd}_\Lambda (A_i).$$

Proof. Lemma A.3 of the Appendix gives the case $n = 0$. The general case is deduced by induction using an injective resolution of C. The second assertion follows from 3.2.

THEOREM 3.5. *The following are equivalent for a pseudocompact* Ω*-algebra* Λ*:*

(i) gl dim $\Lambda < n$*;*

(ii) $\mathcal{E}xt_\Lambda^n (C, D) = 0$ *for all simple* Λ*-modules C and D;*

(iii) $\mathcal{T}or_n^\Lambda (C, D) = 0$ *for all simple* Λ*-modules C and D.*

Proof. (ii) and (iii) are equivalent by Corollary 2.6. Since (i) clearly implies (ii), it suffices to show that (ii) $\to$ (i). Let A be a pseudocompact Λ-module, we write $A = \varprojlim A/B$ where B runs through the open submodules of A, so that A/B has finite length and thus, by induction on its length we have $\mathcal{E}xt_\Lambda^n (A/B, D) = 0$ for all simple modules D. Hence $\mathrm{hd}_\Lambda A/B < n$ by Corollary 3.2 and therefore $\mathrm{hd}_\Lambda A < n$ by Lemma 3.4.

Remarks. (1) If Λ is Noetherian and A is finitely generated, then $\mathcal{T}or_n^{\Lambda}(A, B) = \mathrm{Tor}_n^{\Lambda}(A, B)$ and $\mathcal{E}xt_{\Lambda}^n(A, C) = \mathrm{Ext}_{\Lambda}^n(A, C)$. This follows from Lemma 2.1(ii), the fact that homomorphisms of A into the discrete Λ-module C are automatically continuous and the existence of finitely generated free resolutions of A. Thus the global dimension we have defined is the same as the usual one in this case.

(2) If Λ is a *local algebra* over Ω, i.e., if the radical is the only maximal ideal of Λ, then the only simple Λ-module is the residue class field l of Λ. Thus gl dim Λ is the least integer n such that $\mathcal{E}xt_{\Lambda}^{n+1}(l, l) = 0$.

The following may be proved like Case 4 on p. 348 of Reference [4].

LEMMA 3.6. *Let $f : \Lambda \to \Gamma$ be a morphism of pseudocompact Ω-algebras. Let A be a pseudocompact Γ-module and let C be a discrete Λ-module. Then we have a spectral sequence*

$$\mathcal{E}xt_{\Gamma}^p(A, \mathcal{E}xt_{\Lambda}^q(\Gamma, C)) \Rightarrow \mathcal{E}xt_{\Lambda}^n(A, C).$$

COROLLARY 3.7. *Let Λ be a pseudocompact local algebra over the local ring Ω. Suppose we have an augmentation morphism $\epsilon : \Lambda \to \Omega$. Then*

$$\mathrm{gl}\dim \Lambda = \mathrm{gl}\dim \Omega + \mathrm{hd}_{\Lambda}\Omega.$$

Proof. The only simple Λ-module is the residue class field k of Ω. Let $n = \mathrm{gl}\dim \Omega$ and let $m = \mathrm{hd}_{\Lambda}\Omega$. It follows from Corollary 3.2 that $\mathcal{E}xt_{\Lambda}^m(\Omega, k)$ is a nonzero discrete vector space over k and thus that

$$\mathcal{E}xt_{\Omega}^n(k, \mathcal{E}xt_{\Lambda}^m(\Omega, k)) \neq 0.$$

The spectral sequence

$$\mathcal{E}xt_{\Omega}^p(k, \mathcal{E}xt_{\Lambda}^q(\Omega, k)) \Rightarrow \mathcal{E}xt_{\Lambda}^n(k, k)$$

and Theorem 3.5 complete the proof.

Remark. The reader who prefers to avoid spectral sequences can do so by supposing that Ω is a Noetherian local ring. We have $k = \Omega/(x_1, \cdots, x_m)$ where $(x_1, \cdots, x_m)$ is an Ω-sequence and thus $\mathrm{hd}_{\Lambda}k = \mathrm{hd}_{\Lambda}\Omega + m$ as in Reference [2] and the corollary follows from Theorem 3.5. This assumption on Ω may not be too restrictive, since the author has no example of a non-Noetherian commutative pseudocompact local ring of finite global dimension.

By Theorem 2 of Reference [17], the ring Ω is the direct product of pseudocompact local ring with the product topology, say $\Omega = \prod \Omega_i$. Let $1 = (e_i)$ be the corresponding decomposition of the identity into primitive idempotents. If Λ is a pseudocompact Ω-algebra, then $e_i\Lambda$ is a pseudocompact ($e_i\Omega = \Omega_i$)-algebra. Suppose A is a pseudocompact Λ-module and C is a discrete Λ-module, then

$$\mathrm{Hom}_\Lambda (A, C) = \bigoplus \mathrm{Hom}_{e_i\Lambda} (e_iA, e_iC)$$

and therefore

$$\mathscr{E}xt_\Lambda{}^n(A, C) = \bigoplus \mathscr{E}xt^n_{e_i\Lambda} (e_iA, e_iC)$$

because e_iP is a projective $e_i\Lambda$-module whenever P is a projective Λ-module. The following immediate consequence sometimes reduces the computation of global dimension to the local case where Corollary 3.7 may be applied.

LEMMA 3.8. *(Same notation as above.) Then*

$$\mathrm{gl\,dim}\, \Lambda = \sup \mathrm{gl\,dim}\, e_i\Lambda.$$

The following generalization of a result essentially contained in Theorem 3.2 of Reference [5] is too tempting to be left out.

THEOREM 3.9. *Let Ω be a commutative pseudocompact ring and let $\Omega\{\{X\}\}$ be the algebra of noncommutative formal power series over Ω in a nonempty set $X = \{x_i\}$ of variables. Then*

$$\mathrm{gl\,dim}\, \Omega\{\{X\}\} = \mathrm{gl\,dim}\, \Omega + 1.$$

Proof. The algebra $\Omega\{\{X\}\}$ with the topology of pointwise convergence of the coefficients is a pseudocompact algebra over Ω. We may suppose by Lemma 3.8 that Ω is a local ring in which case so is $\Omega\{\{X\}\}$. We have a natural augmentation $\epsilon : \Omega\{\{X\}\} \to \Omega$ whose kernel is the ideal $\mathfrak{M}$ generated freely by X. Hence $\mathrm{hd}_{\Omega\{\{X\}\}}\Omega = 1$ and Theorem 3.5 proves our assertion.

4. GLOBAL DIMENSION OF GROUP ALGEBRAS

Throughout this section let Ω be a fixed commutative pseudocompact ring. Let G be a profinite group, i.e., the inverse limit of finite groups. We define the *complete group algebra*[3] $\Omega[\![G]\!]$ of G over Ω to be the inverse limit of the ordinary group ring of the finite quotients G/U of G over Ω, i.e., $\Omega[\![G]\!] = \varprojlim \Omega[G/U]$. We may alternatively define $\Omega[\![G]\!]$ as the completion of $\Omega[G]$ in the topology induced by taking as a system of open neighborhoods of 0 the kernels of the natural epimorphisms $\Omega[G] \to \Omega/N[G/U]$, where N runs through the open ideals of Ω and U runs through the open normal

[3] This algebra was introduced under somewhat more restrictive hypotheses by Golod and Šafarevič in their paper on the class tower problem, and by Lazard [*10*].

subgroups of G. The map $G \rightsquigarrow \Omega[\![G]\!]$ defines a covariant functor from the category of profinite groups to that of pseudocompact algebras over Ω. In particular, let $f: G \to G'$ be an epimorphism with kernel N, then we have an epimorphism $f: \Omega[\![G]\!] \to \Omega[\![G']\!]$ whose kernel will be denoted by $I(N)$ where the dependence on Ω and G is understood. It is sometimes useful to note that $I(N)$ is the closed left (right) ideal of $\Omega[\![G]\!]$ generated by the elements $\{1 - n \mid n \in N\}$, since N is normal.

We shall use freely the results on the cohomology of profinite groups found in [*14*] and in [*6*]. We recall that the cohomological p-dimension of G, written $\mathrm{cd}_p G$ is the smallest integer n such that $H^{n+1}(G, C) = 0$ for all discrete p-primary G-modules C. We define the *residual set of* Ω, written $r(\Omega)$, to be the set of the nonzero characteristics of the residue class fields $\Omega/\mathfrak{M}$ where $\mathfrak{M}$ runs through the open maximal ideals of Ω. We write $\mathrm{cd}_\Omega G = \sup_{p \in r(\Omega)} \mathrm{cd}_p G$. The central goal of this section is the proof of the following result.

THEOREM 4.1. *Let Ω be a commutative pseudocompact ring and let G be a profinite group. Then*

$$\mathrm{gl\,dim}\, \Omega[\![G]\!] = \mathrm{gl\,dim}\, \Omega + \mathrm{cd}_\Omega G.$$

EXAMPLES: (1) If G is a finite group, we find the well known result that $\mathrm{gl\,dim}\, \Omega[G] = \infty$ unless the order of G is a unit in Ω in which case $\mathrm{gl\,dim}\, \Omega[\![G]\!] = \mathrm{gl\,dim}\, \Omega$.

(2) If Ω is a local ring whose residue class field has characteristic p and G is a pro-p-group, then $\Omega[\![G]\!]$ is a local ring whose maximal ideal is generated by $I(G)$ and the maximal ideal of Ω since the only irreducible representation of a p-group over a field of characteristic p is the trivial one (cf. Reference [*10*]). In particular, if G is the direct sum of n copies of the p-adic integers, then $\Omega[\![G]\!]$ is the algebra $\Omega[\![x_1, \cdots, x_n]\!]$ of formal power series in n variables over Ω since

$$\Omega[\![G \times H]\!] = \Omega[G] \mathbin{\hat{\otimes}}_\Omega \Omega[H].$$

We find again a known result:

$$\mathrm{gl\,dim}\, \Omega[\![x_1, \cdots, x_n]\!] = \mathrm{gl\,dim}\, \Omega + n.$$

(3) If G is a free pro-p-group on a set X, then $\Omega[\![G]\!]$ is the algebra of noncommuting formal power series over Ω on the set $\{1 + x \mid x \in X\}$ by a result of Lazard ([*10*]), and we have a special case of Theorem 3.9.

Whenever there is no danger of confusion, we write $\mathscr{E}xt_G$ for $\mathscr{E}xt_{\Omega[G]}$ and $\mathscr{T}or^G$ for $\mathscr{T}or^{\Omega[G]}$. The following result establishes the connection of these functors with the cohomology groups of G. We note that an Ω-module of finite length is a G-module if and only if it is an $\Omega[\![G]\!]$ module, hence this is true of pseudocompact and discrete Ω-modules.

LEMMA 4.2. (i) *Let A be a discrete $\Omega[\![G]\!]$-module, then*

$$H^q(G, A) = \mathcal{E}xt_G{}^q(\Omega, A).$$

(ii) *Let A be a right pseudocompact $\Omega[\![G]\!]$-module, then*

$$H_q(G, A) = \mathcal{T}or_q{}^G(A, \Omega).$$

Proof. (i) Both $H^q(G, *)$ and $\mathcal{E}xt_G{}^q(\Omega, *)$ are right-derived functors of $A \rightsquigarrow A^G = \mathrm{Hom}_G(\Omega, A)$.

(ii) Both $H_q(G, *)$ and $\mathcal{T}or_q{}^G(A, \Omega)$ are left-derived functors of

$$A \rightsquigarrow A_G = A \otimes_G \Omega = A \otimes_G \frac{\Omega[\![G]\!]}{I(G)} = \frac{A}{AI(G)}.$$

COROLLARY 4.3. (i) *Let $G = \varprojlim G_i$ and let $A = \varinjlim A_i$ where G_i are profinite groups and A_i are discrete G_i-modules, then*

$$\varinjlim \mathcal{E}xt_{G_i}(\Omega, A_i) = \mathcal{E}xt_{G_i}(\Omega, A).$$

(ii) *If $B = \varprojlim B_i$ where B_i are pseudocompact G_i-modules, then*

$$\varprojlim \mathcal{T}or^{G_i}(B_i, \Omega) = \mathcal{T}or^G(B, \Omega).$$

Proof. The first assertion follows from Lemma 4.2 and from Proposition 8 of Reference [*14*]. The second is a consequence of the first and of Corollary 2.6.

Remarks. (1) If U runs through the open normal subgroups of G, then $H_q(G, A) = \varprojlim H_q(G/U, A/AI(U))$ by Corollary 4.3(ii). Thus our definition of homology groups for profinite groups agrees with the special case mentioned in Problem 4, pp. 1-55 of Reference [*14*].

(2) Corollary 4.3 is true for supplemented pseudocompact algebras as we see by using standard resolutions of Ω (cf. References [*4*] and [*10*]) but we shall not need this fact.

COROLLARY 4.4. $\mathrm{hd}_{\Omega[\![G]\!]}\Omega = \mathrm{cd}_\Omega G$.

Proof. A simple $\Omega[\![G]\!]$-module C is an $\Omega/\mathfrak{M}$-module for some open maximal ideal $\mathfrak{M}$ by Lemma 1.4. If the characteristic of $\Omega/\mathfrak{M}$ is 0, then C is divisible hence $H^n(G, C) = 0$ for all $n \geqslant 1$. The result now follows easily from Corollary 3.2, Lemma 4.2 and, Proposition 11 of Reference [*14*].

Lemma 4.5. *Let G be a profinite group and let H be a closed subgroup of G. Then $\Omega[\![G]\!]$ is a projective $\Omega[\![H]\!]$-module. In particular*

$$\operatorname{gl}\dim \Omega[\![H]\!] \leqslant \operatorname{gl}\dim \Omega[\![G]\!].$$

Proof. If the groups are finite, then $\Omega[\![G]\!]$ is a free $\Omega[\![H]\!]$-module generated by the cosets of G mod H. In general, let U be an open normal subgroup of G, then $\Omega[\![G/U]\!]$ is a projective $\Omega[\![H/U \cap H]\!]$-module. Since

$$\Omega[\![G]\!] = \varprojlim \Omega[\![G/U]\!] \qquad \text{and} \qquad \Omega[\![H]\!] = \varprojlim \Omega[\![H/U \cap H]\!],$$

the first assertion follows from Corollary 3.3. The second is an immediate consequence of the first and the definition of dimension.

Lemma 4.6. *Let A be a pseudocompact $\Omega[\![G]\!]$-module and let C be a discrete $\Omega[\![H]\!]$-module. Then there is a natural isomorphism:*

$$\mathscr{E}xt_G^n(A, \operatorname{Hom}_H(\Omega[\![G]\!], C) \simeq \mathscr{E}xt_H^n(A, C).$$

Proof. The case $n = 0$ is a special case of Lemma 2.4. Since $\Omega[\![G]\!]$ is a projective H-module, the functor $C \rightsquigarrow \operatorname{Hom}_H(\Omega[\![G]\!], C)$ is an exact functor which takes injective H-modules into injective G-modules (use Corollary 2.6 and Proposition 3.1). The general case follows upon using an injective resolution of C.

We need a generalization of the restriction and corestriction maps as defined in Reference [*14*]. We define a G-monomorphism:

$$i : C \to \operatorname{Hom}_H(\Omega[\![G]\!], C)$$

by $i(c) = x \cdot c$ for $c \in C$ and $x \in \Omega[\![G]\!]$. This induces the *restriction* homomorphism:

$$\operatorname{res}_H{}^G : \mathscr{E}xt_G(A, C) \xrightarrow{i_*} \mathscr{E}xt_G(A, \operatorname{Hom}_H(\Omega[\![G]\!], C)) = \mathscr{E}xt_H(A, C).$$

If the index of H in G is finite, we define a G-epimorphism:

$$\pi : \operatorname{Hom}_H(\Omega[\![G]\!], C) \to C$$

by

$$\pi(f) = \sum_{x \in G/H} x^{-1} f(x).$$

This induces the *corestriction* homomorphism

$$\operatorname{Cor}_G{}^H : \mathscr{E}xt_H(A, C) = \mathscr{E}xt_G(A, \operatorname{Hom}_H(\Omega[\![G]\!], C)) \xrightarrow{\pi^*} \mathscr{E}xt_G(A, C).$$

We verify that $\operatorname{Cor}_G{}^H \cdot \operatorname{Res}_H{}^G$ is multiplication by $(G : H)$ since this is obvious for $n = 0$ and we are dealing with maps of cohomological functors. Thus we may prove as usual the following result (cf. Proposition 9, p. I-11 of Reference [*14*]):

LEMMA 4.7. *Let H be a closed subgroup of the profinite group G. If $(G : H)$ is relatively prime to p in the supernatural sense, then* $\mathrm{Res}_H{}^G$ *is injective on the p-primary component of* $\mathcal{E}xt_G(A, C)$.

We write $H = \cap\, U$ where U runs through open subgroups of G. Since A and C are G-modules, we find that

$$\varinjlim \mathcal{E}xt_U(A, C) = \mathcal{E}xt_H(A, C)$$

by using a G-projective resolution of A. Our claim follows immediately.

PROPOSITION 4.8. *If* $n > \mathrm{gl}\dim \Omega$, *then* $\mathcal{E}xt_G{}^n(A, C)$ *is a torsion Abelian group annihilated by the order of G.*

Proof. We suppose first that G is finite. Then $\mathrm{res}_1{}^G = 0$ since $\mathcal{E}xt_\Omega{}^n(A, C) = 0$ by hypothesis. Since $\mathrm{Cor}_G{}^1 \cdot \mathrm{Res}_1{}^G$ is multiplication by $(G : 1)$ on $\mathcal{E}xt_G{}^n(A, C)$, the result is true in this case. Now suppose that G is arbitrary and that A is a G/U-module for some open normal subgroup U of G. The proof of Lemma 4.2 shows that

$$\mathcal{E}xt^q_{\Omega[G]}\left(\Omega\left[\frac{G}{U}\right], C\right) = H^q(U, C)$$

which is a torsion Abelian group annihilated by the order of U if $q \geqslant 1$ by Corollary 3 of Reference [*14*]. The proof in this case is completed by the spectral sequence of Lemma 3.6, which becomes

$$\mathcal{E}xt^p_{\Omega[G/U]}(A, H^q(U, C)) \Rightarrow \mathcal{E}xt^n_{\Omega[G]}(A, C).$$

In general, we write $A = \varprojlim A/I(U)\,A$ where U runs through the open normal subgroups of G and apply Lemma 3.4.

COROLLARY 4.9. *Let G be a profinite group and let G_p be a p-Sylow subgroup for each p. Then*

$$\mathrm{gl}\dim \Omega[\![G]\!] = \sup_{p\in r(\Omega)} \mathrm{gl}\dim \Omega[\![G_p]\!],$$

where the right-hand side is interpreted as $\mathrm{gl}\dim \Omega$ *if $r(\Omega)$ is empty.*

Proof. We may suppose, by Lemma 3.8, that Ω is a local ring with residue class field k of characteristic q with the agreement that $G_0 = 1$. Let $n = \mathrm{gl}\dim \Omega[\![G_q]\!]$, then $n \leqslant \mathrm{gl}\dim \Omega[\![G]\!]$ by Lemma 4.5. Let A and C be simple $\Omega[\![G]\!]$-modules, then A and C are in fact simple $k[\![G]\!]$-modules, by Lemma 1.4, and thus $\mathcal{E}xt^{n+1}_{\Omega[G]}(A, C)$ is a vector space over k. We distinguish two cases:

(a) if $q = 0$, then $\mathcal{E}xt^{n+1}_{\Omega[G]}(A, C)$ is a uniquely divisible Abelian group which must vanish by Proposition 4.8;

(b) if $q \neq 0$ then $\mathscr{E}xt^{n+1}_{\Omega[[G]]}(A, C)$ is a q-primary Abelian group which vanishes by Lemma 4.7, since $\mathscr{E}xt^{n+1}_{\Omega[[G_q]]}(A, C) = 0$ by hypothesis.

In either case the proof is completed by Theorem 3.5.

Proof of Theorem 4.1. We may suppose that G is a pro-p-group and that Ω is a local ring with residue class field k of characteristic p by Lemma 3.8 and Corollary 4.9. As we remarked in Example (2) at the beginning of this section, $\Omega[[G]]$ is a local pseudocompact algebra and the result follows from the conjunction of Corollaries 3.7 and 4.4.

5. Presentation of Groups and Cohomological Dimension

We preserve the notation introduced in Section 4 with the following simplification: *Ω will henceforth denote the ring $\hat{\mathbf{Z}}_p$ of p-adic integers.* We note that the dualizer of Ω (cf. Section 2) is $\mathbf{Q}_p/\hat{\mathbf{Z}}_p$ and that $\text{gl dim}\, \Omega = 1$.

Let G be a profinite group presented as $G = F/N$, where F is a free profinite group and N is a closed normal subgroup of F.[4] In fact, the only property of F which will be used is that $\text{cd}_p F = 1$ (cf. Section 3.4 of Reference [*14*]).

Lemma 5.1. *Let G be a profinite group and let $I(G)$ be the kernel of the natural augmentation $\epsilon : \Omega[[G]] \to \Omega$, then $\text{cd}_p G \leqslant r$ if and only if*

$$\text{hd}_{\Omega[[G]]} I(G) \leqslant r - 1.$$

Proof. This is immediate from Corollary 4.4 and the exact sequence

$$0 \to I(G) \to \Omega[[G]] \to \Omega \to 0. \tag{5.1.1}$$

Theorem 5.2.[5] *Let G be a profinite group and suppose $G = F/N$, where N is a closed subgroup of the group F. Let N_p be the p-Sylow subgroup of the compact Abelian group N/N', where N' denotes the commutator subgroup of N. Suppose* $\text{cd} F = 1$, e.g., *F is a free profinite group. Then*

$$\text{hd}_{\Omega[[G]]} N_p = \text{cd}_p G - 2$$

unless $\text{cd}_p G = 1$, *in which case N_p is a projective $\Omega[[G]]$-module.*

Proof. We note first that N_p is a compact $\Omega[[G]]$-module with the action of G induced via inner automorphisms of F. We recall that for any $\Omega[[N]]$-

[4] This is always possible by Theorem 1.3 of [*6*].

[5] The proofs given here apply equally in the discrete case. For instance Lyndon's identity theorem [*12*] shows that under his hypotheses N/N' is a free $\mathbf{Z}[G]$-module, hence it follows immediately that $\text{cd} G = 2$, as he shows by computation.

module A, $A \hat{\otimes}_{\Omega[N]} \Omega = A/AI(N)$. Since $\Omega[\![F]\!]$ is a projective $\Omega[\![N]\!]$-module, by Lemma 4.5, the beginning of the sequence of (5.1.1) is given by

$$0 \to \mathscr{T}\!or_1^N(\Omega, \Omega) \to \frac{I(F)}{I(F)\,I_N} \to \frac{\Omega[\![F]\!]}{\Omega[\![F]\!]\,I_N} \to \Omega \to 0,$$

where $AI_N = AI(N)$. But $\mathscr{T}\!or_1^N(\Omega, \Omega) = N_p'$ as $\Omega[\![G]\!]$-module since the proof in the discrete case (e.g., p. 190 of Reference [*4*]) will work with minor changes. We thus obtain the following exact sequence of G-modules:

$$0 \to N_p \xrightarrow{\delta} \frac{I(F)}{I(F)\,I_N} \to \frac{I(F)}{\Omega[\![F]\!]\,I_N} \to 0, \tag{5.2.1}$$

where δ is determined by $\delta(n) = 1 - n$. Let $\mathfrak{N}$ be the kernel of the natural epimorphism $\Omega[\![F]\!] \to \Omega[\![G]\!]$, then $\mathfrak{N} = \Omega[\![F]\!]\,I_N$ since N is normal and $I(G) = I(F)/\mathfrak{N}$. Thus (5.2.1) can be rewritten as follows:

$$0 \to N_p \to \frac{I(F)}{I(F)\,\mathfrak{N}} \to I(G) \to 0. \tag{5.2.2}$$

Since $\mathrm{cd}_p F = 1$, $I(F)$ is a projective $\Omega[\![F]\!]$-module, by Lemma 5.1, and thus $I(F)/I(F)\,\mathfrak{N}$ is a projective $\Omega[\![F]\!]/\mathfrak{N} = \Omega[\![G]\!]$-module. The conclusion follows from (5.2.2) and Lemma 5.1.

COROLLARY 5.3. *Let G be a pro-p-group and suppose $G = F/N$, where F is a free pro-p-group. Then the following are equivalent:*

(i) $\mathrm{cd}_p G \leqslant 2$;

(ii) *N_p is a free $\hat{\mathbf{Z}}_p[\![G]\!]$-module generated by the images of a minimal set of generators for the closed normal subgroup N of F.*

Proof. Since $\hat{\mathbf{Z}}_p[\![G]\!]$ is a local ring, the usual argument shows that every projective is free and a generating set is obtained by lifting a generating set of $N_p/N_p\mathfrak{M} = N/[N,F]\,N^p$, where $\mathfrak{M} = (p, I(G))$ is the maximal ideal of $\hat{\mathbf{Z}}_p[\![G]\!]$ (cf. Corollary 1, p. 393 of Reference [7]). But a set $\{n_i\}$ of elements of N generates N as a closed normal subgroup if and only if their images generate the compact vector space $N/[N,F]\,N^p$ over the field with p elements. The result now follows immediately from Theorem 5.2.

The *strict cohomological p-dimension of G*, written $\mathrm{scd}_p G$, is the smallest integer n such that the p-primary component of $H^{n+1}(G, A)$ vanishes for all discrete G-modules A. It is easy to see that $\mathrm{cd}_p G \leqslant \mathrm{scd}_p G \leqslant \mathrm{cd}_p G + 1$ (Proposition 13 of Reference [*14*]), but it is harder to decide between these two possibilities. Serre has given the following criterion (Corollary 4 of Reference [*14*]).

CRITERION 5.4. *Suppose that* $\mathrm{cd}_p G = n$ *is finite. Then* $\mathrm{scd}_p G = n$ *if and only if* $H^{n+1}(U, \mathbf{Z})$ *has trivial p-primary component for each open subgroup* U *of* G.

We immediately obtain the following result:

COROLLARY 5.5. *Suppose* $\mathrm{cd}_p G = n$ *is finite. Then the following are equivalent:*

(i) $\mathrm{scd}_p G = n$.

(ii) $H^n(U, \mathbf{Q}_p/\mathbf{Z}_p) = 0$ *for every open subgroup* U *of* G.

(iii) $\mathcal{T}or_n^{\Omega[U]}(\Omega, \Omega) = 0$ *for every open subgroup* U *of* G.

Proof. Since $\mathbf{Q}$ is divisible, we have that

$$H^{n+1}(U, \mathbf{Z})(p) = H^n(U, \mathbf{Q}/\mathbf{Z})(p) = H^n(U, \mathbf{Q}_p/\hat{\mathbf{Z}}_p)$$

for all $n \geqslant 1$. This proves the equivalence of (i) and (ii). The equivalence of (ii) and (iii) follows from Lemma 4.2 and Corollary 2.6, since $\mathbf{Q}_p/\hat{\mathbf{Z}}_p$ is the dualizer of Ω.

COROLLARY 5.6. *Let* $\mathrm{cd} G = 2$ *and suppose* G *is a pro-p-group with finitely many relations. Then the following are equivalent:*

(i) $\mathrm{scd} G = 2$.

(ii) *For each open subgroup* U *of* G, *the rank of the torsion group of* U/U' *is equal to the number of relations of* U.

Proof. We recall that the number of relations in U is given by the dimension of $H^2(U, F_p)$, where F_p is the finite field with p elements. We recall that $H^2(U, F_p)$ is the dual of $\mathcal{T}or_2^U(F_p, \Omega)$ by Corollary 2.6 and that $\mathcal{T}or_1^U(\Omega, \Omega) = U/U'$. The result follows immediately from Corollary 5.5 and the $\mathcal{T}or$ sequence of $0 \to \Omega \xrightarrow{p} \Omega \to F_p \to 0$; namely

$$\mathcal{T}or_2(\Omega, \Omega) \xrightarrow{p} \mathcal{T}or_2(\Omega, \Omega) \to \mathcal{T}or_2(F_p, \Omega) \to \frac{U}{U'} \xrightarrow{p} \frac{U}{U'},$$

since $\mathcal{T}or_2^U(\Omega, \Omega)$ is p-primary by Proposition 4.8, and $H^2(U, F_p)$ is finite because $H^2(G, F_p)$ is finite by hypothesis [in fact $H^2(G, A)$ is then finite for any finite A; hence, by Shapiro's lemma so is $H^2(U, F_p)$].

PROPOSITION 5.7. *Let* F *be a profinite group of cohomological p-dimension* 1 *and let* N *be a normal subgroup of* F. *Then the following are equivalent for* $G = F/N$.

(i) $\mathrm{scd}_p G = 2$;

(ii) *N_p is a projective $\Omega[\![G]\!]$-module and the p-Sylow subgroup of $N \cap V'/[N, V]$ is trivial for each open normal subgroup V of F containing N.*

Proof. As usual, we denote by $[A, B]$ the closed subgroup generated by all commutators $[a, b]$ with $a \in A$, $b \in B$.

Let U be an open subgroup of G and let V be the complete inverse image of U in F, so that $V/N = U$. The Hochschild-Serre spectral sequence gives rise to the following exact sequence (p. I-15 of Reference [*14*]):

$$0 \to H^1\left(U, \frac{\mathbf{Q}}{\mathbf{Z}}\right) \to H^1\left(V, \frac{\mathbf{Q}}{\mathbf{Z}}\right) \to H^1\left(N, \frac{\mathbf{Q}}{\mathbf{Z}}\right)^V \to H^2\left(U, \frac{\mathbf{Q}}{\mathbf{Z}}\right)$$
$$\to H^2\left(V, \frac{\mathbf{Q}}{\mathbf{Z}}\right) = 0,$$

where the last term vanishes since $\operatorname{cd} V = 1$. Since the groups act trivially on $\mathbf{Q}/\mathbf{Z}$, we may rewrite the sequence as follows:

$$0 \to \operatorname{Hom}\left(\frac{U}{[U, U]}, \frac{\mathbf{Q}}{\mathbf{Z}}\right) \to \operatorname{Hom}\left(\frac{V}{[V, V]}, \frac{\mathbf{Q}}{\mathbf{Z}}\right) \to \operatorname{Hom}\left(\frac{N}{[N, V]}, \frac{\mathbf{Q}}{\mathbf{Z}}\right)$$
$$\to H^2\left(U, \frac{\mathbf{Q}}{\mathbf{Z}}\right) \to 0.$$

Taking p-primary components and passing to the duals, we obtain the following isomorphisms by Corollary 2.6:

$$\mathcal{T}or_2^U(\Omega, \Omega) = H^2\left(U, \frac{\mathbf{Q}_p}{\mathbf{Z}_p}\right)^* = ((N \cap V')/[N, V])_p .$$

The proposition follows from Theorem 5.2 and Corollary 5.5.

6. Application to Class Formations

We consider a profinite group G and a G-module A such that $A = \bigcup_F A_F$, where $A_F = H^0(F, A)$ and F runs through the open subgroups of G. We note that if N is an open subgroup of G contained as a normal subgroup in H, then A_N is an H/N module. We say that (G, A) is a *class formation* if A satisfies the following two axioms for all such pairs $N \subset H$:

(i) $H^1(H/N, A_N) = 0$;

(ii) $H^2(H/N, A_N)$ is cyclic of order $[H : N]$.

It is well known that we can choose a *canonical class* $u_{H/N}$ generating $H^2(H/N, A_N)$ which behaves properly with respect to inf, res, and cor

([1], [13]). Tate's theorem asserts that the cup product with $u_{H/N}$ induces an isomorphism

$$- \cup u_{H/N} : \hat{H}^{q-2}(H/N, \mathbf{Z}) \rightarrow \hat{H}^{q}(H/N, A_N)$$

for each integer q, where $\hat{H}$ denote the Tate cohomology groups. In particular, for $q = 0$, we obtain the Artin reciprocity isomorphism

$$A_H/N_{H/N}(A_H) \cong (H/N)/(H/N)',$$

where the prime indicates the commutator subgroup and $N_{H/N}$ denotes the norm map; passing to the limit as N shrinks to 1, we define a homomorphism

$$\omega_H : A_H \rightarrow H/H'$$

which shall be called the *reciprocity map* for the formation. If H is a normal subgroup of K, then ω_H is a homomorphism of K/H modules. Let C_H be the kernel of ω_H and let D_H be its co-kernel; then the exact sequence

$$0 \rightarrow C_H \rightarrow A_H \xrightarrow{\omega_H} H/H' \rightarrow D_H \rightarrow 0$$

of K/H modules induces homomorphisms

$$d_q : \hat{H}^{q-2}(K/H, D_H) \rightarrow \hat{H}^{q}(K/H, C_H)$$

as the composition of two co-boundary maps.

The main result of this section is the following.

THEOREM 6.1. *The following are equivalent for a class formation* (G, A).

(i) $\operatorname{scd}_p G = 2$;

(ii) *For each integer q and each pair $H \subset K$ of open subgroups of G such that H is normal in K, d_q induces an isomorphism onto on the p-primary components of the respective cohomology groups.*

This theorem is immediately applicable to all classical class formations:

(i) In local class field, the reciprocity map is injective and its co-kernel is $\hat{\mathbf{Z}}/\mathbf{Z}$, where $\hat{\mathbf{Z}}$ is the total completion of the integers. Since $\hat{\mathbf{Z}}/\mathbf{Z}$ is uniquely divisible, it is cohomologically trivial (cf. Chap. 14 of Reference [13]).

(ii) For function fields of one variable over a finite field, the situation is as in (i) (cf. Chap. 8 of Reference [1]).

(iii) For number fields, the reciprocity map is surjective and the kernel is the connected component of the identity in the idele class groups. The cohomology groups of the kernel are thus elementary 2-groups which are trivial if and only if the ground field is totally imaginary (cf. Chap. 9 of Reference [1]).

(iv) For a field complete under a discrete rank-one valuation with algebraically closed residue class field, the reciprocity map is an isomorphism (cf. Reference [*15*]).

For any field k, let G_k denote the Galois group of its separable closure. We have thus proved the following consequence of Theorem 6.1.

COROLLARY 6.2. *In Cases* (i)–(iv) *we have* $\operatorname{scd}_p G_k = 2$, *unless* $p = 2$ *and* k *is a number field which is* not *totally imaginary* (*in which case complex conjugation is an element of order* 2 *making* $\operatorname{cd}_2 G_k$ *infinite!*).

Tate has announced this result in cases (i)–(iii) as a consequence of his duality theorems ([*16*]). A proof in the p-adic case may be found in Proposition 15 of [*14*] and the weaker result $\operatorname{cd}_p G_k = 2$ is proved as Proposition 13 of [*14*].

LEMMA 6.3. *Let T be a finite group and let*

$$0 \to A \to B \xrightarrow{w} C \to D \to 0$$

be an exact sequence of T-modules; we have an induced homomorphism

$$d_q : \hat{H}^{q-2}(T, D) \to \hat{H}^q(T, A)$$

defined as the composition of two co-boundary maps. The following are equivalent:

(i) *For each q, d_q is an isomorphism on the p-primary components;*

(ii) *For each q, $w^* : \hat{H}^q(T, B) \to \hat{H}^q(T, C)$ is an isomorphism on the p-primary components.*

Proof. One splits up the exact sequence into two short exact sequences. The result follows from easy but lengthy diagram chasing along the associated cohomology sequences.

We shall say that a profinite group G is *p-malleable*[6] if for each open normal subgroup H contained as a normal subgroup in K, the cup product

$$- \cup \zeta_{K/H} : \hat{H}^{q-2}(K/H, \mathbf{Z}) \to \hat{H}^q(K/H, H/H')$$

induces an isomorphism on the p-primary components for all integers q, where $\zeta_{K/H}$ is the 2-cohomology class of the extension

$$1 \to H/H' \to K/H' \to K/H \to 1.$$

[6] This concept was introduced by Kawada [9] whose paper inspired some of the proofs of the last two sections of mine.

The theorem of Weil–Šafarevič [*1*] asserts that the diagram

$$\begin{array}{ccc} \hat{H}^{q-2}(K/H, \mathbf{Z}) & \xrightarrow{-\cup u_{K/H}} & H^q(K/H, A_H) \\ & \searrow^{-\cup \zeta_{K/H}} & \downarrow{\scriptstyle \omega_H{}^q} \\ & & H^q(K/H, H/H') \end{array} \tag{6.3.1}$$

is commutative. Thus the second condition of Theorem 6.1 is shown by Lemma 6.3 to be equivalent to the group-theoretic assertion that G be p-malleable. To complete the proof of Theorem 6.1 it thus suffices to verify the following.

THEOREM 6.4. *The profinite group G is p-malleable if and only if* $\mathrm{scd}_p G = 2$.

The proof is broken up into a sequence of lemmas. As in Section 5, we denote by H_p the p-Sylow subgroup of H/H' and we observe that $\hat{H}^q(K/H, H_p)$ is the p-primary component of $H^q(K/H, H/H')$. Since $\mathbf{Q}$ and the p-adic numbers $\hat{\mathbf{Q}}_p$ are uniquely divisible, while $\hat{\mathbf{Q}}_p/\hat{\mathbf{Z}}_p$ is the p-primary component of $\mathbf{Q}/\mathbf{Z}$, we conclude that $\hat{H}^q(K/H, \hat{\mathbf{Z}}_p)$ is the p-primary component of $\hat{H}^q(K/H, \mathbf{Z})$. This shows that G is p-malleable if and only if

$$- \cup \zeta_{K/H} : H^{q-2}(K/H, \hat{\mathbf{Z}}_p) \to H^q(K/H, H_p) \tag{6.4.1}$$

is an isomorphism for all q.

LEMMA 6.5. *Let $I(G)$ be the augmentation ideal of $\hat{\mathbf{Z}}_p[\![G]\!]$. Then G is p-malleable if and only if*

$$\hat{H}^q(K/H, I(G)/I(G)I_H) = 0$$

for all q and all pairs of open subgroups H and K, with H normal in K.

Proof (Kawada): We use the exact sequence of K/H-modules introduced in the proof of 5.2; namely

$$0 \to H_p \to I(G)/I(G)I_H \to \hat{\mathbf{Z}}_p[\![G]\!]/\hat{\mathbf{Z}}_p[\![G]\!]I_H \to \hat{\mathbf{Z}}_p \to 0,$$

where AI_H is the closed submodule of A generated by $\{a(1-h) \mid a \in A, h \in H\}$. By Lemma 4.5, $\hat{\mathbf{Z}}_p[\![G]\!]$ is a projective K-module, hence the third term is a projective K/H-module and *a fortiori* is cohomologically trivial. We conclude from Lemma 6.3 that $I(G)/I(G)I_H$ is cohomologically trivial if and only if

$$d_q : \hat{H}^{q-2}(K/H, \hat{\mathbf{Z}}_p) \to \hat{H}^q(K/H, H_p)$$

is an isomorphism for all q. In view of the remarks above, the proof will be complete as soon as we know that d_q is induced by the cup product with $\zeta_{K/H}$. This fact is verified by explicit calculation in the proof of Theorem 1 of Reference [9].

COROLLARY 6.6. *If* $\mathrm{cd}_p G = 1$, *then* G *is p-malleable.*

Proof. By Lemma 5.1, $I(G)$ is a projective G-module and *a fortiori* is a projective K-module. Thus $I(G)/I(G)I_H$ is a projective module over $\hat{\mathbf{Z}}_p[K/H] = \hat{\mathbf{Z}}_p[\![K]\!]/\hat{\mathbf{Z}}_p[\![K]\!]I_H$ for each normal subgroup H of K. Therefore G is p-malleable by Lemma 6.5.

LEMMA 6.7. *Let* A *be a pseudocompact* $\hat{\mathbf{Z}}_p[\![G]\!]$*-module satisfying the following property. For each open normal subgroup* H *of* G, A/AI_H *is a cohomologically trivial* G/H*-module. Then* $\mathrm{hd}_{\hat{\mathbf{Z}}_p[\![G]\!]} A \leqslant 1$.

Proof. It follows from Lemma 4.2 that

$$\mathscr{T}\!or_n^{K/H}(A/AI_H, \hat{\mathbf{Z}}_p) = 0 \qquad \text{for all} \quad n \geqslant 1$$

and all open normal subgroups H of G. Thus

$$\mathscr{T}\!or_n^{K}(A, \hat{\mathbf{Z}}_p) = 0 \tag{6.7.1}$$

for all K and all $n \geqslant 1$ by Corollary 4.3(ii). Let $0 \to B \to F \to A \to 0$ be an exact sequence of pseudocompact $\hat{\mathbf{Z}}_p[\![G]\!]$-modules with F free. In view of (6.7.1), we obtain the following exact sequence upon tensoring with $\hat{\mathbf{Z}}_p$ over $\hat{\mathbf{Z}}_p[\![K]\!]$,

$$0 \to B/BI_K \to F/FI_K \to A/AI_K \to 0.$$

Since A/AI_K is a cohomologically trivial G/K-module, we have

$$\mathrm{hd}_{\hat{\mathbf{Z}}_p[G/K]} A/AI_K \leqslant 1$$

by Theorem 8, p. 152 of Reference [13]. Because F/FI_K is a free G/K-module, we conclude that B/BI_K is a projective G/K-module. Hence $B = \lim B/BI_K$ is a projective module over $\hat{\mathbf{Z}}_p[\![G]\!] = \lim \hat{\mathbf{Z}}_p[G/K]$ by Corollary 3.3.

COROLLARY 6.8. *If* G *is p-malleable, then* $\mathrm{scd}_p G = 2$.

Proof. It follows immediately from Lemmas 6.5, 6.7, and 5.1 that $\mathrm{cd}_p \leqslant 2$. For each open subgroup K of G, the $\mathscr{T}\!or$ sequence of

$$0 \to I(G) \to \hat{\mathbf{Z}}_p[\![G]\!] \to \hat{\mathbf{Z}}_p \to 0$$

shows that

$$\mathscr{T}\!or_2^{K}(\hat{\mathbf{Z}}_p, \hat{\mathbf{Z}}_p) = \mathscr{T}\!or_1^{K}(I(G), \hat{\mathbf{Z}}_p) = 0$$

by (6.7.1), since $\hat{\mathbf{Z}}_p[\![G]\!]$ is a free K-module. Hence Corollary 5.5 implies our claim.

LEMMA 6.9. *Let F be p-malleable and let $G = F/N$ for some closed normal subgroup N. Then the following are equivalent.*

(i) *G is p-malleable.*

(ii) *The p-primary component of $\hat{H}^q(V/U, N/U' \cap N)$ is trivial for all q and all open subgroups U of F containing N and contained as normal subgroups in V.*

Proof. Let $H \subset K \subset G$ be as before and let U and V be the complete inverse images of H and K in F. Thus $U/N = H$, $V/N = K$ and $V/U = K/H$ so that the following diagram is commutative.

$$\begin{array}{ccccccccc} 1 & \to & U/U' & \to & V/U' & \to & V/U & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow\cong & & \\ 1 & \to & H/H' & \to & K/H' & \to & K/H & \to & 1 \end{array}$$

In particular, the 2-cohomology class of the bottom extension is induced by that of the top extension via the natural map

$$\hat{H}^2(V/U, U/U') \to \hat{H}^2(K/H, H/H').$$

The result is now an immediate consequence of the definition of p-malleable groups applied to the cohomology sequence of the exact sequence

$$1 \to N/U' \cap N \to U/U' \to H/H' \to 1$$

of K/H-modules.

Proof of Theorem 6.4. The necessity has been proved in Corollary 6.8. We suppose therefore that $\operatorname{scd}_p G = 2$. We may write $G = F/N$ with F a free profinite group. Since $\operatorname{cd}_p F = 1$, we know from Corollary 6.6 that F is p-malleable; hence it suffices to verify condition (ii) of Lemma 6.9. This is a consequence of Proposition 5.7. In fact, let H, K, U, V be as in the proof of Lemma 6.9. Then N_p is a projective $\hat{\mathbf{Z}}_p[\![K]\!]$-module and therefore

$$N_p/N_pI_H = N_p/N_pI_U = N_p/[N_p, U] = (N/U' \cap N)_p$$

is a projective K/H-module, hence is cohomologically trivial. In particular, the p-primary component of $\hat{H}^q(V/U, N/U' \cap N)$ is trivial and Lemma 6.9 applies.

In conclusion, we would like to mention an open problem. Let K be an algebraic number field and let S be a set of finite primes of K. Let K_S be the maximal extension of K unramified outside of S and let G_S be the Galois group of K_S over K (cf. Reference [*2*]). *What are necessary and sufficient conditions on S to conclude that* $\operatorname{scd}_p G_S = 2$? If $p = 2$, we must assume that K is totally imaginary as we saw earlier. Tate has asserted that a sufficient condition is that S contain all primes above p [*16*].[7] On the other hand, it is not hard to show from Theorem 6.1 that $\operatorname{scd}_p G_S = 2$ if S has a sufficiently large Dirichlet density, whether or not S contains the primes above p.

Appendix

We have collected here, for the convenience of the reader, the technical results on limits which are needed in the body of the paper. We fix the following notation: I is a directed partially ordered set; $(\Lambda_i, \lambda_{ij})$ is an inverse system of pseudocompact Ω-algebras; (A_i, α_{ij}) and (B_i, β_{ij}) are inverse systems of pseudocompact Λ_i-modules while (C_i, γ_{ij}) is a direct system of discrete Λ_i-modules satisfying the usual rules. For instance,

$$\lambda_{ij}(l_j)\, \alpha_{ij}(a_j) = \alpha_{ij}(l_j a_j), \qquad l_j \gamma_{ij}(c_i) = \gamma_{ij}(\lambda_{ij}(l_j)\, c_i)$$

for $i \leqslant j$, $l_j \in \Lambda_j$, $a_j \in A_j$ and $c_i \in C_j$. We write $\Lambda = \varprojlim \Lambda_i$ and $\lambda_i : \Lambda \to \Lambda_i$ for the natural projection and similarly for the other limits. Thus A and B are pseudocompact Λ-modules and C is a discrete Λ-module.

Lemma A.1. *Let U be an open subset of A_i containing $\alpha_i(A)$. Then there is some $j > i$ such that $\alpha_{ij}(A_j) \subset U$.*

Proof. We may suppose without loss of generality that U is an open submodule of A_i so that A_i/U has finite length. Passing to the limit over the cofinal set $j > i$, the exact sequence

$$0 \to \alpha_{ij}^{-1}(U) \to A_j \xrightarrow{\alpha_{ij}} \frac{\alpha_{ij}(A_j) + U}{U} \to 0$$

shows that $\bigcap_{j \geq i} (\alpha_{ij}(A_j) + U)/U = 0$, since inverse limits preserve exactness and the first term is $\varprojlim \alpha_{ij}^{-1}(U) = \alpha_i^{-1}(U) = A$ by hypothesis. Thus $\alpha_{ij}(A_j) \subset U$ for some $j \geqslant i$ since the intersection is over submodules of the module A_i/U of finite length.

Corollary A.2. *If the maps α_{ij} are epimorphisms, so are the limit maps $\alpha_i : A \to A_i$.*

[7] This is still wishful thinking. The author has shown that this is intimately connected with the p-adic regulator problem of Leopold.

Proof. If $\alpha_i(A) \neq A$, we can find an open submodule U containing the closed submodule $\alpha_i(A)$ and properly contained in A. Then Lemma A.1 shows that α_{ij} is not an epimorphism for some $j > i$.

LEMMA A.3. *The natural morphism*

$$\varinjlim \operatorname{Hom}_{\Lambda_i}(A_i, C_i) \to \operatorname{Hom}_\Lambda(A, C)$$

is a monomorphism which is an isomorphism if the α_{ij} and λ_{ij} epimorphisms.

Proof. We have natural maps

$$h_{ij} : \operatorname{Hom}_{\Lambda_i}(A_i, C_i) \to \operatorname{Hom}_{\Lambda_j}(A_j, B_j),$$

defined by $h_{ij}(f_i) = \gamma_{ij} f_i \alpha_{ij}$ for $i \leqslant j$, forming a direct system. Similarly, we have maps

$$\varphi_i : \operatorname{Hom}_{\Lambda_i}(A_i, C_i) \to \operatorname{Hom}_\Lambda(A, C),$$

defined by $\varphi_i(f_i) = \gamma_i f_i \alpha_i$, hence there is a canonical map:

$$\varphi : \varinjlim \operatorname{Hom}_{\Lambda_i}(A_i, C_i) \to \operatorname{Hom}_\Lambda(A, C).$$

Let $\varphi(f) = 0$, then $f = h_i(f_i)$ with f_i in $\operatorname{Hom}_{\Lambda_i}(A_i, C_i)$ for some i and

$$0 = \varphi(f)(A) = \varphi(h_i(f_i))(A) = \varphi_i(f_i)(A) = \gamma_i f_i \alpha_i(A).$$

Since A_i is pseudocompact, C_i is discrete and f_i is continuous, we conclude that $f_i\alpha_i(A)$ has finite length, hence there is a $j > i$ such that $\gamma_{ij} f_i \alpha_i(A) = 0$. Since 0 is open in C_j, $\alpha_i(A)$ is contained in the open submodule $\ker \gamma_{ij} f_i$. By Lemma A.1, there is a $k > j$ such that

$$\alpha_{ik}(A_k) \subseteq \ker \gamma_{ij} f_i \subseteq \ker \gamma_{ik} f_i .$$

enHce

$$\gamma_{ik} f_i \alpha_{ik} = h_{ik}(f_i) = 0,$$

i.e., $f = 0$ and thus φ is injective.

To show that φ is surjective, let $f : A \to C$ be a Λ-morphism. Then $\ker f$ is an open submodule of A and $\alpha_i^{-1}(U_i) \subset \ker f$ for some i and for some submodule U_i of A. Since $f(A)$ is of finite length, $f(A) = \gamma_k(D_k)$ for some k and some submodule D_k of C_k of finite length. We may choose $l > i$ and $l > k$ large enough to kill the kernel of $\gamma_k : D_k \to C$. We then define a map $g : A_l \to C_l$ by $g(a_l) = c_l$ where c_l is the unique element of $\gamma_{kl}(D_k)$ such that $f(a) = \gamma_l(c_l)$ with $\alpha_l(a) = a$ (recall that α_l is an epimorphism by Corollary A.2). The map is a well-defined element of $\operatorname{Hom}_{\Lambda_l}(A_l, C_l)$, since $\ker \alpha_l \subset \ker f$ by construction, and $\varphi(h_l(g)) = f$.

LEMMA A.4. *If the maps α_{ij}, β_{ij} and λ_{ij} are epimorphisms then the natural morphism*

$$\varphi : A \hat{\otimes}_\Lambda B \to \varprojlim A_i \hat{\otimes}_{\Lambda_i} B_i$$

is an isomorphism.

Proof. The morphism $\lambda_{ij} : \Lambda_j \to \Lambda_i$ gives A_i and B_i the structure of Λ_j-modules (which are pseudocompact since they are pseudocompact as Ω-modules!). We obtain maps

$$h_{ij} : A_j \hat{\otimes}_{\Lambda_j} B_j \to A_i \hat{\otimes}_{\Lambda_i} B_i$$

by composing the natural maps

$$A_j \hat{\otimes}_{\Lambda_j} B_j \to A_i \hat{\otimes}_{\Lambda_j} B_i \to A_i \hat{\otimes}_{\Lambda_i} B_i .$$

Thus we obtain an inverse system and $\varprojlim A_i \hat{\otimes}_{\Lambda_i} B_i$ is well defined. Similarly, we have maps $t_i : A \hat{\otimes}_\Lambda B \to A_i \hat{\otimes}_{\Lambda_i} B_i$ forming an inverse system which induces the morphism $\varphi : A \hat{\otimes}_\Lambda B \to \varprojlim A_i \hat{\otimes}_{\Lambda_i} B_i$.

Under our hypothesis α_i, β_i, and λ_i are epimorphisms by Corollary A.2. Let U and V be open submodules of A and B, respectively, then we may find a j and open submodules U_j and V_j of A_j and B_j such that $\alpha_j^{-1}(U_j) \subset U$ and $\beta_j^{-1}(V_j) \subset V$. In particular, we conclude that $\ker \alpha_j \subset U$ and $\ker \beta_j \subset V$. Thus we may define a map $f_j : A_j \times B_j \to A/U \otimes B/V$ by $f_j(a_j, b_j) = x_j \otimes y_j$, where x_j is the coset mod U of an element t_j such that $\alpha_j(t_j) = a_j$ and similarly for y_j. Then f_j is well defined and is in fact a bihomomorphism, since λ_j is surjective. We thus have a morphism

$$g_j : A_j \hat{\otimes}_{\Lambda_j} B_j \to \frac{A}{U} \otimes_\Lambda \frac{B}{V}$$

which may be combined with the natural projection

$$\varprojlim A_i \hat{\otimes}_{\Lambda_j} B_i \to A_j \hat{\otimes}_{\Lambda_j} B_j$$

to yield morphisms

$$\theta_{U,V} : \varprojlim A_i \hat{\otimes}_{\Lambda_i} B_i \to \frac{A}{U} \otimes_\Lambda \frac{B}{V}$$

which form an inverse system and hence define a morphism:

$$\theta : \varprojlim A_j \hat{\otimes}_{\Lambda_j} B_j \to \varprojlim \frac{A}{U} \hat{\otimes}_\Lambda \frac{B}{V} = A \hat{\otimes}_\Lambda B$$

which is the inverse of φ.

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