Int. J. Heat Mass Transfer. Vol. 9, pp. 261–262. Pergamon Press 1966. Printed in Great Britain

SHORTER COMMUNICATION

POSSIBLE SIMILARITY SOLUTIONS OF THE LAMINAR NATURAL CONVECTION FLOW OF NON-NEWTONIAN FLUIDS

TSUNG YEN NA* and ARTHUR G. HANSEN†
The University of Michigan

(*Received* 27 *April* 1965 *and in revised form* 19 *July* 1965)

NOMENCLATURE

- a, thermal diffusivity;
- g, gravitational acceleration;
- T, temperature;
- u, velocity in the x-direction;
- v, velocity in the y-direction;
- x, coordinate axis parallel to the surface of the plate;
- y, coordinate axis perpendicular to the plate;
- θ, $T - T_\infty$;
- ψ, stream function ($u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$);
- η, similarity independent variable.

Subscripts
- w, conditions at wall;
- ∞, conditions at infinity.

Superscripts
- $'$, $''$, $'''$, first, second and third derivatives with respect to η.

ANALYSIS

A SIMILARITY analysis is made of the natural convection heat transfer of non-Newtonian fluids of the Ostwald–de Waele type over a non-isothermal vertical flat plate. The importance of such a flow system and the limitations on the power law model representing the natural convection flows of non-Newtonian fluids were discussed in detail by Acrivos [1]. It was also found by Acrivos that similarity solutions do not exist for the case of an isothermal plate. The basic equations for such a system are well-known and can be written as

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = m\frac{\partial}{\partial y}\left(\left|\frac{\partial^2\psi}{\partial y^2}\right|^{n-1}\frac{\partial^2\psi}{\partial y^2}\right) + g\beta\theta \quad (1)$$

and

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} = a\frac{\partial^2\theta}{\partial y^2} \quad (2)$$

with the boundary conditions

$$y = 0;\quad \frac{\partial\psi}{\partial y} = 0;\quad \frac{\partial\psi}{\partial x} = 0;\quad \theta = \theta_w(x)$$

$$y \to \infty;\quad \frac{\partial\psi}{\partial y} = 0;\quad \theta = 0$$

The application of the similarity analysis which has the goal of reducing equations (1) and (2) to ordinary differential equations will cause restrictions to be imposed on the function $\theta_w(x)$. The group-theoretic method is employed; the details of which are discussed in reference 2. Two distinct classes of transformations will be employed—a linear group of transformations and a spiral group.

1. *Linear group*

A one-parameter group of transformation is chosen of the form

$$x = A^{\alpha_1}\bar{x} \quad y = A^{\alpha_2}\bar{y},$$
$$\psi = A^{\alpha_3}\bar{\psi},\quad \theta = A^{\alpha_4}\bar{\theta}, \quad (3)$$

where α_1, α_2, α_3, α_4 and A are constants. We seek values of the α_i such that the form of the equations (1) and (2) is invariant under the transformation except for a multiplicative constant. Substituting these expressions into equations (1) and (2), equating the powers of A (to insure invariance of the equations under this group of transformations) and solving the resulting equations, one finds for $n \neq 1$,

$$\alpha_2 = \tfrac{1}{3}\alpha_1,\quad \alpha_3 = \tfrac{2}{3}\alpha_1,\quad \alpha_4 = -\tfrac{1}{3}\alpha_1 \quad (4)$$

The constant n of the power law model appears in a common coefficient of $(n - 1)$ and is cancelled. This step does not exist for the case of $n = 1$, i.e. Newtonian flow.

Next, we seek "absolute invariants" under this group of transformations. Absolute invariants are functions having the same form before and after the transformation. It is seen from equation (3) and (4) that

$$\frac{y}{x^{1/3}} = \frac{\bar{y}}{\bar{x}^{1/3}} \quad (5)$$

* Assistant Professor of Mechanical Engineering, University of Michigan, Dearborn Campus, Dearborn, Michigan.

† Professor of Mechanical Engineering, University of Michigan, Ann Arbor, Michigan.

This combination of variables is therefore invariant under this group of transformations and, consequently, is an "absolute invariant". We designate this functional form by

$$\eta = \frac{y}{x^{1/3}} \tag{6}$$

By the same reasoning, other absolute invariants are

$$F(\eta) = \frac{\psi}{x^{2/3}} \tag{7}$$

and

$$G(\eta) = \frac{\theta}{x^{-1/3}} \tag{8}$$

Applying the transformations, equations (6)–(8), the basic equations, equations (1) and (2), are transformed to the following ordinary differential equations

$$-\tfrac{1}{3}\left(\frac{dF}{d\eta}\right)^2 - \tfrac{2}{3}F\frac{d^2F}{d\eta^2} = m\frac{d}{d\eta}\left(\left|\frac{d^2F}{d\eta^2}\right|^{n-1}\frac{d^2F}{d\eta^2}\right) + g\beta G \tag{9}$$

$$-\tfrac{2}{3}F\frac{dG}{d\eta} - \tfrac{1}{3}G\frac{dF}{d\eta} = a\frac{d^2G}{d\eta^2} \tag{10}$$

The transformations, equations (6)–(8), show that similarity solutions exist only if

$$\theta_w(x) = C_1 x^{-1/3} \tag{11}$$

Only with these torms of $\theta_w(x)$ may constant boundary conditions be imposed on $(F\eta)$ and $G(\eta)$. The boundary conditions are transformed to

$$\eta = 0: \quad F = 0, \quad \frac{dF}{d\eta} = 0, \quad G = C_1$$

$$\eta \to \infty: \quad \frac{dF}{d\eta} = 0, \quad G = 0$$

This is, of course, necessary for the solution of ordinary differential equations (9) and (10). This confirms Acrivos' result that, for constant wall temperature, (θ_w = constant), similarity solutions do not exist [1].

For $n = 1$ (i.e. Newtonian flow), the absolute invariants are found to be

$$\eta = \frac{y}{x^{\alpha}} \tag{12}$$

$$F(\eta) = \frac{\psi}{x^{1-\alpha}} \tag{13}$$

and

$$G(\eta) = \frac{\theta}{x^{1-4\alpha}} \tag{14}$$

where $\alpha = a_2/a_1$ is an *arbitrary* constant. It follows that similarity solutions exist if

$$\theta_w(x) = C_2 x^m = C_2 x^{1-4\alpha} \tag{15}$$

where m is also an arbitrary constant. By replacing α by $(1 - m)/4$, the transformations, equations (12)–(14), become those of Sparrow and Gregg [3].

2. *Spiral group*

A one-parameter group of transformations is chosen in the form (see reference 2)

$$x = \bar{x} + \beta_1 b, \quad y = e^{\beta_2 b}\bar{y} \tag{16}$$

$$\psi = e^{\beta_3 b}\bar{\psi}, \qquad \theta = e^{\beta_4 b}\bar{\theta}$$

where β_1, β_2, β_3, β_4 and b are constants. By following the same steps as above, the following results are obtained. For $n \neq 1$,

$$\beta_2 = \beta_3 = \beta_4 = 0 \tag{17}$$

This means that the variables, x, y, ψ and θ, are themselves invariants under this group of transformations and therefore similarity solutions do not exist for this group. However, for $n = 1$, the absolute invariants are

$$\eta = \frac{y}{e^{\beta x}} \tag{18}$$

$$F(\eta) = \frac{\psi}{e^{-\beta x}} \tag{19}$$

and

$$G(\eta) = \frac{\theta}{e^{-4\beta x}} \tag{20}$$

Again, $\beta(=\beta_2/\beta_1)$ is an arbitrary constant. Examination of the boundary conditions show that similarity solutions exist only if

$$\theta_w(x) = c_3 e^{mx} \tag{21}$$

By replacing β by $(-m/4)$, the results of Sparrow and Gregg [3] are obtained.

REFERENCES

1. A. Acrivos, A theoretical analysis of laminar natural convection heat transfer to non-Newtonian fluids, *A.I.Ch.E.Jl* **6**, 584 (1960).
2. A. G. Hansen, *Similarity Analysis of Boundary Value Problems in Engineering*. Prentice-Hall, Englewood Cliffs, N.J. (1965).
3. E. M. Sparrow and J. L. Gregg, Similarity solutions for free convection from a non-isothermal vertical plate, *Trans. Am. Soc. Mech. Engrs* **80**, 379 (1958).