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Extensions of the Minimum Effort Control Problem*

W. A. PORTER†

AND

J. P. WILLIAMS

The Navigation and Control Systems Laboratory, Institute of Science and Technology, The University of Michigan, Ann Arbor, Michigan

Submitted by Lotfi Zadeh

1. INTRODUCTION

In an earlier paper [1] the following abstract minimum effort control problem was analysed.

PROBLEM I. Let B and R be Banach spaces and T a bounded linear transformation from B into R. For each ξ in the range of T find an element $u \in B$ satisfying $Tu = \xi$ while minimizing $\| u \|$.

For this problem to have a unique solution it was found to be both necessary and sufficient that B be both reflexive and rotund. Adopting these conditions attention is then focused on the development of a thorough characterization of the function $T^{\dagger}$ which sends every ξ in the range of T into its unique minimum norm pre-image $u_{\xi} \in B$.

The importance of this problem stems from a wide range of applications (see [2], [3], [4], and [7] for example) in automatic control. The present paper extends this initial problem in several ways, among which are the following:

PROBLEM II. Let B_0 and B_1 denote rotund reflexive Banach spaces and R any Banach space. T denotes a bounded linear transformation from B_0 onto R. F denotes a bounded linear transformation from B_0 into B_1. For an arbitrary fixed $\xi \in R$ find the $u_{\xi} \in B_0$ which minimizes the functional $J(u)$ over the set $T^{-1}(\xi) \subset B_0$ where;

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† Dr. Porter is also Associate Professor of Electrical Engineering at The University of Michigan.

(a) $J(u) = \| Fu \|$ $\qquad$ F is also one-to-one and onto

(b) $J(u) = \| u - \hat{u} \|$ $\qquad$ $\hat{u} \in B_0$ a fixed element

(c) $J(u) = \| Fu \|^2 + \| u \|^2$

(d) $J(u) = \| Fu - \hat{y} \|^2 + \| u - \hat{u} \|^2$ $\quad$ $(\hat{u}, \hat{y})$ an arbitrary tuplet of $B_0 \times B_1$.

In each of these cases, however, it is shown that the new versions of the problem are only apparently more general than the initial problem itself. In fact each version of Problem II may be reduced by simple and direct procedures to Problem I.

In the analysis of Problem II several facts, which shall now be summarized, will be useful. From [1] we have:

LEMMA 1. *Let $C = T(U) \subset R$ denote the image of the unit ball $U \subset B$. Then C is a convex, circled, weakly compact, neighborhood of $0 \in R$. Hence the Minkowski functional p*

$$p(\xi) = \inf\{\lambda > 0 : \xi \in \lambda C\}$$

is defined and finite on all of R and satisfies

$$p(\xi + \zeta) \leqslant p(\xi) + p(\zeta)$$
$$p(\lambda\xi) = | \lambda | \, p(\xi).$$

For each $\varphi \neq 0$ in the dual of a rotund reflexive Banach space B there exists a unique vector $\bar{\varphi}$ in B which satisfies $\langle \bar{\varphi}, \varphi \rangle = \| \varphi \|$. We shall refer to $\bar{\varphi}$ as the *extremal* of φ. With this convention the main results of the first reference is summarized in Theorem 1.

THEOREM 1. *Let $\xi \neq 0$ be an element of R. Then there exists a unique element N in the unit sphere of R^* such that*

$$T^{\dagger}(\xi) = p(\xi)\overline{T^* N}.$$

The functional N is uniquely determined by the conditions

(i) $\| N \| = 1$.

(ii) *C lies to the left of the hyperplane*

$$S = \{\zeta \in R : \langle \zeta, N \rangle = [p(\xi)]^{-1} \langle \xi, N \rangle\}$$

If B is a complex space this last requirement is replaced by

$$\operatorname{Re}(\langle \zeta, N \rangle) \leqslant \operatorname{Re}(\langle [p(\xi)]^{-1} \xi, N \rangle) \qquad \text{all} \quad \zeta \in C.$$

The vector N deserves in a natural way to be called the *outward normal* to C at $[p(\xi)]^{-1}\xi$. Our second theorem is proved in the appendix.

THEOREM 2. *Let B_0, B_1 be rotund, smooth, and reflexive Banach spaces and F a bounded linear transformation from B_1 into B_2. Let B denote the space B_0 equipped with the norm $|\ \ |$ given by*

$$| x | = (\| x \|^2 + \| Fx \|^2)^{1/2}.$$

Then B is rotund, smooth, and reflexive with $B^* = B_0^*$ and the extremal x' of an $x \in B$ is given by

$$x' = K(x)/| x |$$

where K is the one-to-one, norm preserving, and antihomogeneous operator (that is, $K(\lambda x) = \bar{\lambda} K(x)$) defined by

$$K(x) = \| x \| \bar{x} + \| Fx \| F^*(\overline{Fx}).$$

2. THE SOLUTION TO PROBLEM II

In this section four theorems will be presented which summarize the main results concerning the solution to Problem II. In each case it will be self-evident that Theorem 1 above is indeed the key to the solution. Consider first Problem II(b). With the function $T^{\dagger}$ defined as before the solution to this problem is given by

THEOREM 3. *Problem II(b) has the unique solution u_ξ given by*

$$u_\xi = T^{\dagger}(\xi - T\hat{u}) + \hat{u}.$$

PROOF: It is clear that u_ξ maps into ξ under T and if u is any pre-image of ξ under T, then $T(u - \hat{u}) = \xi - T\hat{u}$ implies, by definition of $T^{\dagger}$, that

$$\| u - \hat{u} \| \geqslant \| T^{\dagger}(\xi - T\hat{u}) \| = \| u_\xi - \hat{u} \|$$

It follows that u_ξ is a solution of I(a) and since this last inequality is strict unless $u - \hat{u} = T^{\dagger}(\xi - T\hat{u})$, we see that u_ξ is the *only* solution.

The next problem to be considered is II(c). If F is a mapping from B_0 into B_1 the subset $B(F)$ of $B_0 \times B_1$ defined by

$$B(F) = \{(u, v) : v = Fu, u \in B_0\}$$

is called the *graph* of F. If addition and scalar multiplication in $B_0 \times B_1$ are defined in the usual way then from [1, Lemma 1] it follows that $B_1 \times B_2$ is a rotund reflexive Banach space when endowed with the norm

$$\|(u, v)\|^2 = \| u \|^2 + \| v \|^2.$$

Moreover when F is linear and bounded $B(F)$ is a closed subspace of $B_1 \times B_2$ and hence also a rotund reflexive Banach space.

Let us now define the transformation $G : B(F) \to R$ by

$$G[(u, Fu)] = Tu, \qquad u \in B_0 . \tag{$\ddagger$}$$

G is evidently well-defined and linear. It is also easily shown that G is bounded. Observe now that Problem II(c) asks for a vector (u, Fu) in $B(F)$ such that $G(u, Fu) = \xi$ and such that

$$J(u) = \| u \|^2 + \| Fu \|^2 = \|(u, Fu)\|^2.$$

is a minimum. In other words, Problem II(c) reduces to Problem I with $G : B(F) \to R$ replacing $T : B_0 \to R$ as the operator of prime interest. We summarize these remarks in the following theorem.

THEOREM 4. *Problem II(c) has a unique solution u_ξ for each $\xi \in R$, namely, u_ξ is the abscissa of the vector $G^\dagger(\xi)$ in $B(F)$.*

REMARK 1. We can view this reduction in a slightly different manner which will prove useful.

Let us introduce a new norm $|\ |$ on B_1 by writing

$$| u | = (\| u \|^2 + \| Fu \|^2)^{1/2}$$

and let B denote B_1 equipped with the norm $|\ |$. Then B inherits rotundity reflexively and smoothness from B_1 and B_2 (see Theorem 2). Problem II(c) now asks to find the $u \in B$ for which $Tu = x$ and $| u |$ is a minimum and is thus recognized again as Problem I.

This latter viewpoint suggests immediately that Problem II(c) can be phrased in its most general form by equipping $B_1 \times B_2$ with any norm with respect to which it is reflexive and rotund (for instance see [1, Lemma 1]). Then if $J(u)$ consists of a norm on $B(F)$ induced by any eligible norm on $B_1 \times B_2$ the result is equivalent to Problem I.

For example, the minimum effort problem with $J(u)$ given by either of

$$J(u) = \| u \|^p + \| Fu \|^p \qquad (1 < p < \infty)$$

$$\begin{aligned} J(u) \quad & a_{11} \| u \|^2 + a_{12} \| u \| \, \| Fu \| \\ & + a_{21} \| u \| \, \| Fu \| + a_{22} \| Fu \|^2 \qquad a_{ij} \geqslant 0 \quad i, j = 1, 2 \end{aligned}$$

where the matrix $[a_{ij}]$ is strictly positive is equivalent to Problem I. Finally, it is clear that Problem I also contains the case in which $J(u)$ is of the form

$$J(u) = \sum_{i=0}^{n} \| F_i u \|^p \qquad (F_0 = I)$$

where $F_i : B_0 \to B_i$ are given transformations.

The proof of Theorem 4 suggests also the solution to Problem II(a).

THEOREM 5. *The unique solution of II(a) is given by*

$$u_\xi = p_F(\xi) F^{-1}(\overline{F^{*-1}(T^*\eta)})$$

where

$$p_F(\xi) = \inf\{\lambda > 0 : \xi \in \lambda C_F\}$$
$$C_F = \{Tu : \| Fu \| \leqslant 1\}.$$

and η is the unique vector in R^* satisfying

$$\text{(i) } \| \eta \| = 1$$
$$\text{(ii) } \langle \xi, \eta \rangle = \| F^{*-1} T^* \eta \|.$$

PROOF: Since F^{-1} is a bounded linear transformation from B_1 into B_0, it has a conjugate $(F^{-1})^*$ sending $B_0{}^*$ into $B_1{}^*$. For $u_2 \in B_1$, $f_2 \in B_1{}^*$ we have

$$\langle u_2, (F^{-1})^* F^* f_2 \rangle = \langle F^{-1} u_2, F^* f_2 \rangle = \langle F F^{-1} u_2, f_2 \rangle = \langle u_2, f_2 \rangle$$

hence $(F^{-1})^* F^* f_2 = f_2$. Similarly, for $u_1 \in B_0$, $f_1 \in B_0{}^*$

$$\langle u_1, F^*(F^{-1})^* f_1 \rangle = \langle F u_1, (F^{-1})^* f_1 \rangle = \langle F^{-1} F u_1, f_1 \rangle = \langle u_1, f_1 \rangle$$

hence $F^*(F^{-1})^* f_1 = f_1$. These equations show that F^* has a bounded inverse and that $(F^*)^{-1} = (F^{-1})^*$.

Now the unique solution u_ξ of II(a) is given by

$$u_\xi = p_F(\xi)(T^*\eta)'$$

where η is the unique vector of norm 1 in R^* satisfying

$$\langle \xi, \eta \rangle = | T^*\eta | = \sup_{\|Fu\|=1} |\langle u, T^*\eta \rangle|$$

and $(T^*\eta)'$ is the extremal of $T^*\eta$ with respect to the norm $| u | = \| Fu \|$ on B. Now with bars denoting the usual extremals in B_0 and B_1 respectively, the extremal u' of u in B is given by[1]

$$u' = F^*(\overline{Fu}).$$

It follows that

$$F^*(\overline{Fu_\xi}) = u_\xi = \left(\frac{u_\xi}{p_F(\xi)} \right)' = (T^*\eta)'' = \frac{T^*\eta}{| T^*\eta |}.$$

Hence

$$\overline{Fu_\xi} = | T^*\eta |^{-1} F^{*-1}(T^*\eta).$$

But then

$$\frac{Fu_\xi}{\| Fu_\xi \|} = \overline{\overline{Fu_\xi}} = \overline{(| T^*\eta |^{-1} F^{*-1}(T^*\eta))} = \overline{F^{*-1}(T^*\eta)}.$$

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[1] See the proof of Theorem 2 in the Appendix.

Since $\| Fu_\xi \| = | u_\xi | = | p_F(\xi)(T^*\eta)| = p_F(\xi)$, this completes the proof of the assertion concerning u_ξ.

The second part of the theorem follows from the obvious equality

$$| T^*\eta | = \sup_{u \in B_1} \frac{|\langle u, T^*\eta \rangle|}{\| Fu \|} = \sup_{v \in B_2} \frac{|\langle F^{-1}u, T^*\eta \rangle|}{\| v \|}.$$

Finally we consider Problem II(d) which brings the present line of development to fruition. To study Problem II(d) we introduce the graph $B(F)$ of F in the product space $B_1 \times B_2$ and the transformation $G : B(F) \to R$ as previously defined. Let $\hat{w} = (\hat{u}, \hat{y})$. Then with this change in notation Problem II(d) asks for a $w = (u, Fu)$ in $B(F)$ such that $G(w) = \xi$ and $\| w - \hat{w} \|^2$ is a minimum. Now if $\hat{w} \in B(F)$ we recognize this latter problem as precisely Problem II(b) (with $B(F)$, G, and w replacing B, T, and u respectively) and hence using the solution of that problem, we see that

$$(u_\xi, F_\xi) = G^\dagger(\xi - G\hat{w}) + (\hat{u}, F\hat{u})$$

defines the unique solution of II(d).

If $\hat{w}$ does not belong to $B(F)$ (i.e., if $\hat{y} \neq F\hat{u}$) then we cannot appeal to Problem II(b), but it is easy to see that II(d) still has a unique solution. For this, let M_ξ be the subset of $B_0 \times B_1$ defined by

$$\{(u, Fu) - (\hat{u}, \hat{y}) : G(u, Fu) = \xi\}.$$

Evidently M_ξ is closed and convex, and $B_1 \times B_2$ being rotund and reflexive, M_ξ has a unique element $(u_0, Fu_0) - (\hat{u}, \hat{y})$ of minimum norm. That is, there exists a unique vector u_0 in B_0 with $Tu_0 = \xi$ and

$$\begin{aligned} J(u_0) &= \|(u_0, Fu_0) - (\hat{u}, \hat{y})\|^2 \\ &= \min\{\|(u, Fu) - (\hat{u}, \hat{y})\|^2 : G(u, Fu) = \xi\} \\ &= \min\{J(u) : Tu = \xi\}. \end{aligned}$$

We have proved part of the following theorem:

THEOREM 6. *Problem II(d) always has a unique solution u_ξ. If $\hat{y} = F\hat{u}$ this solution is determined by*

$$(u_\xi, Fu_\xi) = G^\dagger(\xi - T\hat{u}) + (\hat{u}, \hat{y}).$$

If $B_1 = H_1$ and $B_2 = H_2$ are Hilbert spaces and if P is the orthogonal projection of $H_1 \times H_2$ onto the graph $H(F)$ of F, then u_ξ satisfies

$$(u_\xi, Fu_\xi) = G^\dagger(\xi - T\bar{u}) + (\bar{u}, F\bar{u})$$

where $(\bar{u}, F\bar{u}) = P(\hat{u}, \hat{y})$.

PROOF: It remains only to prove the assertion concerning the Hilbert space case. Let $w_0 = G^\dagger(\xi - T\bar{u}) + (\bar{u}, F\bar{u}) - (\hat{u}, \hat{y})$. Then $w_0 \in M_\xi$ and since $(I - P)(\hat{u}, \hat{y})$ is orthogonal to $H(F)$ we have

$$\begin{aligned} \| w_0 \|^2 &= \| G^\dagger(\xi - T\bar{u}) + (\bar{u}, F\bar{u}) - (\hat{u}, \hat{y})\|^2 \\ &= \| G^\dagger(\xi - T\bar{u})\|^2 + \|(\bar{u}, F\bar{u}) - (\hat{u}, \hat{y})\|^2 \end{aligned} \tag{*}$$

Also, if $u \in H_1$ satisfies $G(u, Fu) = \xi$, then the vector $(u, Fu) - (\bar{u}, F\bar{u})$ maps into $\xi - T\bar{u}$ under G and therefore

$$\|(u, Fu) - (\bar{u}, F\bar{u})\| \geqslant \|G^\dagger(\xi - T\bar{u})\|.$$

Hence $G(u, Fu) = \xi$ implies

$$\begin{aligned} \|(u, Fu) - (\hat{u}, \hat{y})\|^2 &= \|(u, Fu) - (\bar{u}, F\bar{u}) + (\bar{u}, F\bar{u}) - (\hat{u}, \hat{y})\|^2 \\ &= \|(u, Fu) - (\bar{u}, F\bar{u})\|^2 + \|(\bar{u}, F\bar{u}) - (\hat{u}, \hat{y})\|^2 \qquad (**) \\ &\geqslant \| G^\dagger(\xi - T\bar{u})\|^2 + \|(\bar{u}, F\bar{u}) - (\hat{u}, \hat{y})\|^2. \end{aligned}$$

It follows from Eqs. (*) and (**) that w_0 is the smallest element in M_ξ and hence that $w_0 = (u_\xi, Fu_\xi) - (\hat{u}, \hat{y})$. This completes the proof.

The Computation of $G^\dagger$

In Problems II(c) and II(d) the solution is found to require the use of the minimum effort function $G^\dagger$ which sends every $\xi \in R$ back into $(u_\xi, Fu_\xi) \in B(F)$, the unique minimum norm pre-image under G of ξ in the graph of F. Using the function $T^\dagger$ as a model it is easy to state the conditions necessary to specify $G^\dagger$. In the actual formulation of this function however, several associated problems occur which we shall now deal with.

In Hilbert spaces the solution to Problem II (that is Theorems 3, 4, 5, and 6) may be written much more explicitly. Let us first restrict attention to the computation of $G^\dagger$ when the Banach spaces B_0 and B_1 are replaced by the Hilbert spaces H_0 and H_1 respectively. With regard to the space $H_0 \times H_1$ let us first note that

$$\langle (u_1, v_1), (u_2, v_2) \rangle = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle$$

defines an inner product on $H_0 \times H_1$ which is complete with respect to the norm induced by this inner product. Hence it follows that $H_0 \times H_1$, and therefore also $H(F)$ the graph of F in $H_0 \times H_1$, is a Hilbert space.

The solution of Problem I was given by restricting the transformation T to the orthogonal complement of its null space. This restriction of T was nonsingular and its inverse maps any ξ in the range of T back into its pre-image with minimum norm. This process will work as well for the present situation and we shall proceed to locate the orthogonal complement of the null space of G. Notice first from Eq. (‡) that the null space of G, denoted by N_G, is given by the set

$$N_G = \{(u, Fu) : u \in N_T\}.$$

The complement of N_G is then determined by the following lemma:

LEMMA 2. *Let Q and M denote the orthogonal complements of the null spaces of the transformations G and T respectively. Then Q is given by*

$$Q = \{(u, Fu) : (I + F^*F)u \in M\}.$$

PROOF: The proof of Lemma 2 is given by the following chain of set equalities

$$\begin{aligned} Q &= \{(u, Fu) : (u, Fu) \perp (v, Fv) \text{ all } v \in N_T\} \\ &= \{(u, Fu) : \langle u, v\rangle + \langle Fu, Fv\rangle = 0, v \in N_T\} \\ &= \{(u, Fu) : \langle u, v\rangle + \langle F^*Fu, v\rangle = 0, v \in N_T\} \\ &= \{(u, Fu) : \langle (I + F^*F)u, v\rangle = 0, v \in N_T\} \\ &= \{(u, Fu) : (I + F^*F)u \in M\}. \end{aligned}$$

REMARK 2. It is occasionally convenient to deal with the set of abscissas of elements in Q. This set will be denoted by S. It follows from Lemma 2 that S is given by

$$S = (I + F^*F)^{-1}(M).$$

Here the fact that $(I + F^*F)$ is invertible has been used. This follows from the observation that since F^*F is positive, the spectrum of $I + F^*F$ lies on the real axis to the right of 1 and hence does not contain 0.

The solution to the Hilbert space version of Problem II(c) can now be concisely formulated:

THEOREM 7. *The unique solution u_ξ of Problem II(c) for Hilbert spaces is given by*

$$u = (I + F^*F)^{-1}T^{\dagger}\varphi$$

where φ is the unique vector in R satisfying

$$\xi = T(I + F^*F)^{-1}T^{\dagger}\varphi.$$

PROOF: Using the notation introduced above and appealing to the solution of Problem I we see that (u_ξ, Fu_ξ) is characterized by

$$\text{(i) } (u_\xi, Fu_\xi) \in Q$$

$$\text{(ii) } G(u_\xi, Fu_\xi) = \xi.$$

Now by the preceeding Lemma, $(u_\xi, Fu_\xi) \in Q$ if and only if $u_\xi = (I + F^*F)^{-1}v$ for some $v \in M$ and it is clear that v is uniquely determined by u_ξ. Since T is a one-to-one mapping from R onto M, $v = T^\dagger\varphi$ for a unique vector $\varphi \in R$. The theorem now follows the definition $G(u_\xi, Fu_\xi) = Tu_\xi$.

Theorem 6 has partially solved Problem II(d) for Hilbert spaces. To complete the picture we shall need the easily verified lemma.

LEMMA 3. *Let $(\bar{u}, F\bar{u})$ denote the orthogonal projection of $(u, y) \in H_0 \times H_1$ on the subspace $H(F)$. Then $\bar{u}$ is determined by*

$$\bar{u} = (I + F^*F)^{-1}[u + F^*y].$$

This lemma together with Theorem 6 and the characterization of $G^\dagger$ given above yield the theorem:

THEOREM 8. *The unique solution u_ξ of the Hilbert space version of II(d) is given by*

$$u_\xi = (I + F^*F)^{-1}(T^\dagger\eta + \hat{u} + F^*\hat{y})$$

where η is the unique vector in R satisfying

$$\xi = T(I + F^*F)^{-1}(T^\dagger\eta + \hat{u} + F^*y).$$

REMARK 3. It is clear that Problem II(d) contains Problems II(c) ($\hat{u} = \hat{y} = 0$), II(b) ($\hat{y} = 0, F = 0$), and II(a) ($\hat{u} = \hat{y} = 0, F = 0, u = F_0v$) as special cases. This is reflected in the fact that Theorem 8 reduces under the same conditions to Theorems 3, 4, 5, and 6 (Hilbert space case) respectively.

Let us now examine the general solution of II(c) in more detail. For this let B denote the space B_1 equipped with the new norm

$$| u | = (\| u \|^2 + \| Fu \|^2)^{1/2}.$$

As pointed out in Remark 1, Problem II(c) asks to find the unique vector $u = u_\xi$ in B for which $Tu = \xi$ and $| u |$ is a minimum. According to the solution of Problem I, there is a unique vector η in R^* of norm 1 for which

$$u_\xi = p_G(\xi)(T^*\eta)'.$$

Using Theorem 2 this is equivalent to

$$K\left(\frac{u_\xi}{p_G(\xi)}\right) = \left(\frac{u_\xi}{p_G(\xi)}\right)' = (T^*\eta)'' = \frac{T^*\eta}{| T^*\eta |}$$

that is,

$$u_\xi = K^{-1}T^*\left[\left(\frac{p_G(\xi)}{| T^*\eta |}\right)\eta\right]. \tag{‡‡}$$

The vector η is determined by the conditions

(i) $\| \eta \| = 1$

(ii) $\langle \xi, \eta \rangle = | T^*\eta |$

and finally, the Minkowski functional p_G associated with the set

$$C_G = \{Tu : | u | \leqslant 1\}$$

is given as usual by

$$p_G(\xi) = \inf_{\lambda}\{\lambda > 0 : \xi \in \lambda C_G\}.$$

REMARK 4. It is not difficult to show that the solution for Problem II(c) given by (††) and conditions (i) and (ii) includes the several previous results. Indeed if $F = 0$ then the right hand side of (††) reduces to $p(\xi)\overline{T^*\eta}$. If B_0 and B_1 are Hilbert spaces then $K(x)$ becomes $(I + F^*F)(x)$.

Aside from the expected difficulty in finding the set C_G, the introduction of the transformation F into Problem I brings about two added complications, namely, the inversion of the mapping K and the computation of the number $| T^*\eta |$:

$$| T^*\eta | = \sup_{|u|=1} |\langle u, T^*\eta \rangle| = \sup_{u \in B_1} \frac{|\langle u, T^*\eta \rangle|}{(\| u \|^2 + \| Fu \|^2)^{1/2}}.$$

As for the inversion of K it would seem that at best one can only hope for an iterative technique for computing K^{-1}. We will not pursue this problem any further and suggest only that several techniques are available for inverting a bounded one-to-one operator from one Banach space onto another (see Anselone [5] for example).

Let us consider the problem of computing the number $| T^*\eta |$ as a function of η. The definition above gives one technique. Two others are given in the following Lemma:

LEMMA 4. *For each η in R^* we have*

1. $| T^*\eta |^2 = \sup_{f_2 \in B_2^*} \| T^*\eta - F^*f_2 \|^2 + \| f_2 \|^2$

2. $| T^*\eta |^2 = \langle K^{-1}(T^*\eta), T^*\eta \rangle.$

PROOF: The second assertion follows from the fact that

$$K^{-1}(T^*\eta) = \frac{| T^*\eta |}{p_G(\xi)} u_\xi = | T^*\eta | (T^*\eta)'.$$

To prove the first assertion we revert back to the graph formulation of

Problem II(c). Here we consider the problem of finding the unique vector (u_ξ, Fu_ξ) in $B(F)$, the graph of F, such that

$$Tu = G(u, Fu) = \xi$$

and for which

$$\|(u, Fu)\| = (\| u \|^2 + \| Fu \|^2)^{1/2}$$

is a minimum. The solution is given by

$$(u_\xi, Fu_\xi) = \alpha\overline{G^*N}$$

where

$$\alpha = \inf\{\lambda > 0 : \xi \in \lambda C_G\}$$
$$C_G = \{G(u, Fu) : \|(u, Fu)\| \leqslant 1\}$$

and N is the unit (outward) normal to C_G at $\alpha^{-1}\xi$.

Now G, being a mapping from $B(F)$ onto R, will have a conjugate G^* mapping R^* into $B(F)^*$. Since $B(F)$ is a closed subspace of $B_1 \times B_2$, its conjugate space $B(F)^*$ may be identified with the quotient space[1] $B_1^* \times B_2^*/B(F)^0$, where $B(F)^0$, the annihilator of $B(F)$, is given by

$$B(F)^0 = \{(f_1, f_2) \in B_1^* \times B_2^* : \langle(u, Fu), (f_1, f_2)\rangle = 0, \text{ all } u \in B_1\}.$$

It is a straightforward task to show that

$$B(F^0) = \{(-F^*f_2, f_2) : f_2 \in B_2^*\}.$$

(Recall the analogous computation in the Hilbert space case above.) Thus the vectors in $B(F)^*$ are cosets of the form

$$\widehat{(f_1, f_2)} = (f_1, f_2) + B(F)^0 = \{(f_1 - F^*f_2', f_2 + f_2') : f_2' \in B_2^*\}$$

and the norm of such a coset is

$$\begin{aligned} \|\widehat{(f_1, f_2)}\| &= \inf_{f_2' \in B^*_2} \|(f_1 - F^*f_2', f_2 + f_2')\| \\ &= \inf_{f_2' \in B^*_2} (\|f_1 - F^*f_2'\|^2 + \|f_2 + f_2'\|^2)^{1/2}. \end{aligned}$$

For $(u, Fu) \in B(F)$ and $\eta \in R^*$ the computation

$$\begin{aligned} \langle G(u, Fu), \eta\rangle &= \langle Tu, \eta\rangle = \langle u, T^*\eta\rangle \\ &= \langle(u, Fu), (T^*\eta, 0)\rangle \\ &= \langle(u, Fu), (T^*\eta - F^*f_2', f_2')\rangle \qquad (f_2' \in B_2^*) \end{aligned}$$

[1] See [6, p. 116], for example.

shows that $G^*\eta$ is the coset $(\widehat{T^*\eta, 0})$. Hence we have

$$\| G^*\eta \| = \sup_{u \in B_1} \frac{|\langle u, T^*\eta \rangle|}{(\| u \|^2 + \| Fu \|^2)^{1/2}}$$

$$\| G^*\eta \| = \inf_{f'_2 \in B_2^*} (\| T^*\eta - F^*f_2' \|^2 + \| f_2' \|^2)^{1/2}$$

where the sup equation is the definition of the norm of a function on $B(F)$ and the inf equation arises from the identification of $B(F)^*$ with $B_1^* \times B_2^*/B(F)^0$. This completes the proof of 1.

We conclude this discussion of Problem II(c) by stating our results in the following theorem.

THEOREM 9. *The unique solution u_ξ of Problem II is given by*

$$u_\xi = K^{-1}T^* \left[\frac{p_G(\xi)}{| T^*\eta |} \right] \eta$$

where η is the unique vector in R^* satisfying

(i) $\| \eta \| = 1$

(ii) $\langle \xi, \eta \rangle = | T^*\eta |$

and K is the mapping from B_1 onto B_1^* defined by $K(x) = \| x \| \bar{x} + \| Fx \| F^*(\overline{Fx})$. The numbers $p_G(\xi)$ and $| T^*\eta |$ may be computed from either of the following:

$$p_G(\xi) = (\| u_\xi \|^2 + \| Fu_\xi \|^2)^{1/2}$$

$$p_G(\xi) = \inf\{\lambda > 0 : \xi \in \lambda C_G\}$$

$$| T^*\eta | = \sup_{u \in B_1} \frac{|\langle u, T^*\eta \rangle|}{(\| u \|^2 + \| Fu \|^2)^{1/2}}$$

$$| T^*\eta |^2 = \langle K^{-1}(T^*\eta), T^*\eta \rangle$$

$$| T^*\eta |^2 = \inf_{f_2 \in B^*_2} \| T^*\eta - F^*f_2 \|^2 + \| f_2 \|^2.$$

The set $C_G = \{Tu : \| u \|^2 + \| Fu \|^2 \leqslant 1\}$ is a convex, circled, weakly compact, neighborhood of 0 in R and has exactly one hyperplane of support through each of its boundary points.

APPENDIX

In this appendix we prove the following theorem.

THEOREM 2. *Let B_1, B_2 be rotund, smooth, and reflexive and let F be a bounded linear transformation from B_1 into B_2. Let B denote the space B_1 equipped with the norm*

$$|x| = (\|x\|^2 + \|Fx\|^2)^{1/2}$$

Then B is rotund smooth and reflexive with $B^* = B_1^*$ and the extremal x' of an x in B is given by

$$x' = \frac{\|x\|\bar{x} + \|Fx\| F^*(\overline{Fx})}{(\|x\|^2 + \|Fx\|^2)^{1/2}}$$

where the bars denote extremals in B_1 and B_2 respectively.

PROOF: Since F is bounded we have

$$\|x\|^2 \leqslant \|x\|^2 + \|Fx\|^2 = |x|^2 \leqslant (I + \|F\|^2)\|x\|^2$$

so that $|\ |$ is an equivalent norm on B_1. Hence B and B_1 have the same bounded linear functionals and, in particular, B is reflexive. If $B(F)$ denotes the graph of F in $B_1 \times B_2$ then $(x, Fx) \to x$ is a (linear) isometry on $B(F)$ onto B. Since $B_1 \times B_2$ (with the obvious norm) is rotund (Theorem 1), its isometric copy B is also rotund. Since smoothness is also preserved under isometries and any subspace of a smooth space is smooth it remains only to prove that $B_1 \times B_2$ is smooth.

Now the dual of $B_1 \times B_2$ may be identified with the space of pairs $(f_1, f_2) \in B_1^* \times B_2^*$ with the norm

$$\|(f_1, f_2)\| = (\|f_1\|^2 + \|f_2\|^2)^{1/2}.$$

Since B_1^* and B_2^* are rotund by hypothesis, an application of Lemma 1 of [1] shows that $(B_1 \times B_2)^*$ is rotund and this is equivalent with smoothness of $B_1 \times B_2$.

Finally with $K(x) = \|x\|\bar{x} + \|Fx\|F^*(\overline{Fx})$ observe that

$$\text{(i)}\quad \langle x, K(x)\rangle = \|x\|\langle x, \bar{x}\rangle + \|Fx\|\langle Fx, \overline{Fx}\rangle$$
$$= \|x\|^2 + \|Fx\| = |x|^2$$

and for any $y \in B$

$$\text{(ii)}\quad |\langle y, K(x)\rangle| = |\,\|x\|\langle y, \bar{x}\rangle + \|Fx\|\langle Fy, \overline{Fx}\rangle|$$
$$\leqslant \|x\| \cdot \|y\| + \|Fx\|\,\|Fy\|$$
$$\leqslant (\|x\|^2 + \|Fx\|^2)^{1/2}(\|y\|^2 + \|Fy\|^2)^{1/2} = |x|\,|y|.$$

In other words, the functional $\Phi(x) = K(x)/|x|$ has the value $|x|$ at x and is of norm 1 in B^*, and therefore is the extremal of x in B.

REMARK. It follows from the proof that $|K(x)| = |x|$ for each x in B and that K is onto B^*. The function K is also "antihomogeneous":

$$K(\lambda x) = |\lambda x|(\lambda x)' = |\lambda||x|\frac{|\lambda|}{\lambda}x' = \bar{\lambda}K(x).$$

Finally K is also $1-1$ for

$$|x|x' = K(x) = K(y) = |y|y'$$

implies

$$|x| = |y|$$

so that

$$x = |x|x'' = |x|(|x|x')' = |y|(|y|y')' = y.$$

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