A Note on the Minimum Effort Control Problem*

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1. INTRODUCTION

By a continuous linear system we shall mean a system with input $u$ and output $x$, governed for $t \geq t_0$ by the system of integral equations

$$x(t) = \Phi(t, t_0) x^0 + \int_{t_0}^{t} \Phi(t, s) B(s) u(s) ds.$$  (1)

Here $u(t)$ and $x(t)$ are (real or complex) vector functions with $m$ and $n$ components respectively and $\Phi(t, s)$ denotes the system transition matrix. In [1] Neustadt studied various "effort" functions $c(u)$ associated with such a system. In particular he showed that if the time $T$ is fixed and effort is defined by

$$c(u) = \int_0^T \sum_{j=1}^m |u_j(t)|^p dt^{1/p}, \quad 1 < p < \infty$$

then to each target state $x(T)$ there corresponds a unique minimum effort control $u^*(t)$ which transfers $x$ from $x^0$ to $x(T)$ in time $T$. The precise value $c(u^*)$ of the minimum effort was computed as well as the explicit form of the control vector $u^*(t)$.

In this note we will formulate and solve a generalization of Neustadt's problem. The result yields the existence and uniqueness of a minimum effort control and its precise form for a wide class of effort functions and includes the cases of discrete and composite (discrete-continuous) linear systems.

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2. The Minimum Effort Problem

To motivate what follows, let us consider the system of Eq. (1). For convenience we suppose the system is initially at rest so that \( x^0 = 0 \). Let \( B \) denote the cartesian product

\[
L_p(\tau) \times L_p(\tau) \times \cdots \times L_p(\tau), \quad \tau = [t_0, T], \quad 1 < p < \infty
\]

where \( L_p(\tau) \) consists, as usual, of those complex valued Lebesgue measurable functions on \( \tau \) whose \( p \)th power is integrable. Then to each \( u \in B \) there corresponds a unique \( x^* \) satisfying Equation (1). In particular, at time \( T \) we have

\[
x(T) = \Phi(T, t_0) \int_{t_0}^{T} \Phi(t_0, s) B(s) u(s) \, ds.
\]

With \( K \) denoting either the real or complex numbers, this leads us to define a transformation \( S \) from \( B \) to \( K^n \) by writing \( Su = x(T) \). It is easy to verify that \( S \) is linear. Moreover, with any choice of product norms on \( B \) and \( K^n \), \( S \) is bounded. Since it is clear that

\[
\| u \| = \left( \int_{t_0}^{T} \sum_{i=1}^{m} |u_i(t)|^p \, dt \right)^{1/p} \quad 1 < p < \infty
\]

defines a norm on \( B \), we see that a natural generalization of the control problem of Neustadt is the following.

**PROBLEM.** Let \( B \) and \( R \) be Banach spaces and \( T \) a bounded linear transformation from \( B \) into \( R \). For each \( \xi \) in the range of \( T \) find an element \( u \in B \) satisfying \( Tu = \xi \) which minimizes \( \| u \| \).

Consider the set \( T^{-1}(\xi) \) of all pre-images of \( \xi \) under \( T \). The solution to the general minimum effort problem must then answer the following questions: Does \( T^{-1}(\xi) \) contain an element of minimum norm? If so, is this element unique? Finally, if both these answers are yes, and if we write \( T^*\xi \) for the unique minimum pre-image of \( \xi \) under \( T \), what is the nature of the function \( T^* \) so defined, and more specifically, how can one compute its values?

Initially, we allow \( B \) to be an arbitrary (real or complex) Banach space. After having answered the first two questions we will see the need of requiring two additional properties of \( B \) (namely reflexivity and rotundity) to insure the existence of the minimum energy function \( T^* \) associated with \( T \). For convenience in studying \( T^* \) we will then impose a third restriction on \( B \) (smoothness). As regards \( T \), we require that it be onto \( R \). This amounts to assuming that \( T \) has a closed range and hence in particular, if \( T \) has a finite dimensional range, results in no loss of generality.
We begin with two examples which show that some additional restriction on $B$ is needed.

**Example 1.** Let $C$ denote the set of all real (or complex) valued continuous functions on the interval $0 \leq t \leq 1$ which vanish at $t = 0$. Then $C$ is a closed subspace of the usual Banach space of continuous functions on $[0, 1]$, and hence is a Banach space. Let $T$ be the bounded linear transformation from $C$ to $K$ defined by

$$ Tu = \int_0^1 u(t) \, dt. $$

Then it is easy to see that

1. $\inf \{ \| u \| : Tu = 1 \} = 1$.
2. $| Tu | < 1$ if $u \in C$ has norm 1.

It follows that the vector (number) 1 does not have a minimum pre-image under $T$.

**Example 2.** Let $D$ denote the plane equipped with the norm

$$ \| x \| = | x_1 | + | x_2 | \quad \text{if} \quad x = (x_1, x_2). $$

On $D$ we define the linear transformation $T$ by

$$ Tx = x_1 + x_2. $$

It is obvious that $\| T \| = 1$ and hence that any $x \in D$ satisfying $Tx = 1$ has norm $\geq 1$. It follows that both of the vectors $(0, 1)$ and $(1, 0)$ are minimum pre-images of 1 under $T$.

In short, the minimum effort function $T^*$ associated with $T$ can fail to exist by virtue of either a lack of or an overabundance of minimum pre-images. It is worth observing that the space $C$ above is not reflexive and the space $D$ has a "flat" unit ball (connect the points $(0, 1)$, $(1, 0)$, $(-1, 0)$, $(0, -1)$). We now proceed to remedy both these defects in $B$.

**Definition.** Let $U = \{ x : \| x \| \leq 1 \}$ be the unit ball in $B$ and $\partial U$ the boundary of $U$. $B$ is called **rotund** [2] or **strictly convex** [3] if one of the following equivalent conditions satisfied:

1. $\partial U$ contains no line segments.
2. $\| x_1 + x_2 \| = \| x_1 \| + \| x_2 \|$ implies $x_2 = \lambda x_1$ or $x_1 = \lambda x_2$ for some $\lambda \geq 0$.

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1. Observe that it follows from (2) that rotundity is preserved by any linear isometry.
(3) For each bounded linear functional ϕ on B there is at most one x ∈ U with ⟨x, ϕ⟩ = ϕ(x) = ||ϕ||.

(4) Each convex subset C of B has at most one minimum element (i.e., there is at most one vector x ∈ C satisfying ||x|| ≤ ||z|| for all z ∈ C.

The following lemma lists some examples of rotund Banach spaces.

**Lemma 1.**

1. Any Hilbert space is rotund.
2. The spaces ℓ_2, ℓ_p, are rotund for 1 < p < ∞.
3. If B_1, ..., B_n are rotund Banach spaces, then so is

   \[ B = B_1 × B_2 × ⋯ × B_n \]

   when the norm of x = (x_1, x_2, ..., x_n) in B is defined by either of

   \[ ||x|| = \left( \sum_i ||x_i||^p \right)^{1/p}, \quad 1 < p < \infty \]

   \[ ||x|| = \left( \sum_{i,j} a_{ij} ||x_i|| ||x_j|| \right)^{1/2} \]

   where \([a_{ij}]\) is a strictly positive \(n × n\) matrix each of whose entries is nonnegative.

**Proof.** The first assertion follows immediately from (2) of the above definition and the parallelogram law. The second is well-known and may be found in [4; p. 211] for example. The proof of (3) is straightforward but somewhat detailed and hence will be omitted.

Observe now that because T is linear and continuous the set \(T^{-1}(ξ)\) is convex and closed for each \(ξ ∈ R\). The following theorem therefore gives necessary and sufficient conditions on B for our first two questions to be answered affirmatively for every T on B.

**Theorem 1.** Let B be a Banach space. Then each closed convex set C in B has at least one (at most one) minimum element if and only if B is reflexive (rotund).

**Proof.** Property (4) above establishes half of the theorem. Suppose then that B is reflexive. Then U is weakly compact and consequently, if

\[ α = \inf \{||x|| : x ∈ C\}, \]

the sets

\[ C_n = \{z ∈ C : ||z|| ≤ α + 1/n\} \quad (n = 1, 2, ⋯) \]
form a decreasing sequence of non-empty, weakly compact subsets of \( B \) and therefore have nonempty intersection. The fact that reflexivity of \( B \) is also necessary was recently shown by Phelps [5].

Henceforth we assume that \( B \) is reflexive and rotund and focus attention on the function \( T^t \).

3. The Minimum Effort Function

We begin by examining a special case.

**Theorem 2.** If \( B = H \) is a Hilbert space, \( N \) is the null space of \( T \) and \( M = N^\perp \), then \( T^t = T_M^{-1} \) is the inverse of the restriction of \( T \) to \( M \).

**Proof.** The transformation \( T_M \) is \( 1-1 \), continuous, and onto the Banach space \( R \) and hence, by the Closed Graph Theorem, is invertible. Let \( \xi \) be a fixed vector in \( R \) and write \( u_\xi = T_M^{-1}(\xi) \). If \( u \in H \) is any pre-image of \( \xi \), then

\[
  u = (u - u_\xi) + u_\xi
\]

is the unique decomposition of \( u \) in \( N \oplus M \) and hence

\[
  ||u||^2 = ||u - u_\xi||^2 + ||u_\xi||^2 \geq ||u_\xi||^2
\]

The result follows from the definition of \( T^t \).

It is clear that the proof and even the statement of Theorem 2 makes no sense in \( B \). As a matter of fact, it turns out that the function \( T^t \) will not in general be linear, and different techniques are necessary.

If \( E \) is a Banach space then the Hahn-Banach theorem shows that to each non-zero \( x \) in \( E \) there corresponds at least one \( \varphi \in E^* \) such that

\[
  ||\varphi|| = 1, \quad \langle x, \varphi \rangle = ||x||
\]

If \( E \) is reflexive this result applied to \( E^* \) shows that to each \( \varphi \neq 0 \) in \( E^* \) there corresponds at least one \( x \in E \) such that

\[
  ||x|| = 1, \quad \langle x, \varphi \rangle = ||\varphi||
\]

To insure that for each \( \varphi \neq 0 \) in \( E^* \) the corresponding element \( x \) in \( E \) is unique it is sufficient (and in fact, necessary) that \( E \) be rotund. Thus if \( E \) is a rotund reflexive Banach space and \( \varphi \) is a continuous linear functional on \( E \), then \( \varphi \) is not only bounded on the unit ball of \( E \), but in fact attains its supremum, and does so uniquely.

The preceding remarks show that with a rotund reflexive Banach space \( B \)
we are justified in writing $\varphi$ for the unique vector in $B$ of norm 1 satisfying $\langle \varphi, \varphi \rangle = \| \varphi \|$ and in referring to $\varphi$ as the extremal of $\varphi$. We adopt the convention that the extremal 0 of the 0 functional is the 0 vector in $B$.

Now let $x \neq 0$ be a vector in $B$. Regarding $x$ as a linear functional on $B$ the Hahn-Banach produced $\varphi$ shows that $x$ attains its supremum on the unit ball of $B$, and that rotundity of $B^*$ is necessary and sufficient for $x$ to attain its supremum uniquely. Thus, requiring that both $B$ and $B^*$ be rotund (and reflexive) we can denote this unique $\varphi$ by $\bar{x}$ and speak of the extremal of $x$. Since the conjugate of any of the spaces of Lemma 1 is another of the same type it is clear that each of these is still a possible candidate for $B$.

A Banach space $E$ is called smooth if at each point of $\partial U$ there is exactly one supporting hyperplane of $U$. Day [2; p. 112] notes that the following properties are equivalent:

(1) $E$ is smooth.

(2) For each $x \in \partial U$ there is at most one $\varphi \in E$ such that $\| \varphi \| = 1$ and $\varphi(x) = 1$.

(3) The functional $x \rightarrow \| x \|$ has a Gateaux differential at each point of $\partial U$; that is,

$$\lim_{\epsilon \to 0} \frac{\| x + \epsilon h \| - \| x \|}{\epsilon}$$

exists for each $x \in \partial U$ and $h \in E$.

In addition, it is not difficult to see that for any Banach space $E$, $E$ is smooth (rotund) if $E^*$ is rotund (smooth). It follows from this that if $E$ is reflexive, $E^*$ is rotund if and only if $E$ is smooth. Accordingly, to enable the dual use of the term extremal in $B$, we henceforth require that $B$ be rotund, reflexive, and smooth. (This latter hypothesis will be seen to be dispensible.) We note the following properties of the extremal operation:

(i) $\bar{x} = x/\| x \|$ for $x \neq 0$ in $B$,

(ii) $\bar{\varphi} = \varphi/\| \varphi \|$ for $\varphi \neq 0$ in $B$,

(iii) $\lambda \bar{x} = (| \lambda | \| \lambda \|) \bar{x}$ any complex scalar $\lambda$.

The proof of the following theorem is straightforward.

**Theorem 2.** Let $x$ be given in $B$. The Gateaux derivative of the norm at $x$

$$G(x, h) = \lim_{\epsilon \to 0} \frac{\| x + \epsilon h \| - \| x \|}{\epsilon}$$

This shows that any isometric copy of a smooth space is smooth.
exists for each $h \in B$ and the mapping $h \mapsto G(x, h)$ defines a real linear functional on $B$ of norm 1 which assumes the value $\| x \|$ at $x$. Consequently, if $B$ is a real linear space this is the extremal $\bar{x}$ of $x$. In general this is the real part of the extremal of $x$:

$$G(x, h) = \text{Re} \langle h, \bar{x} \rangle \quad (\text{all } h \in B).$$

Recall that the conjugate $T^*$ of $T$ is the bounded linear transformation from $R^*$ to $B^*$ defined for $\varphi \in R^*$ by

$$\langle u, T^*\varphi \rangle = \langle Tu, \varphi \rangle \quad u \in B.$$

That is, $T^*\varphi$ is the linear functional on $B$ whose value at $u$ is the number $\langle Tu, \varphi \rangle$. The Hahn-Banach theorem shows that $\| T^* \| = \| T \|$. The fact that $T$ is onto $R$ shows that $T^*$ is one-to-one.

The next result deals with another special case.

**Lemma 2.** Suppose that for some $\xi \in R$ we have $\| T^*\xi \| = \| \xi \|$. Then $T^*\xi$ is given by the formula.

$$T^*\xi = \| \xi \| \overline{T^*\xi}.$$  

Here, if the norm on $R$ is not smooth, $\bar{\xi}$ is understood to be any extremal of $\xi$.

**Proof.** Without loss of generality we may assume that $\| T \| = 1$. Then $\| T^* \| = 1$ hence $\| T^*(\bar{\xi}) \| \leq 1$. This, together with

$$\langle T^*(\xi), T^*(\bar{\xi}) \rangle = \langle \xi, \bar{\xi} \rangle = \| \xi \| = \| T^*(\bar{\xi}) \|$$

shows that

$$T^*(\bar{\xi}) = \overline{T^*(\xi)}.$$  

Taking extremals we obtain the desired formula.

**Remark.** The formula in the preceding lemma yields $T^*(\xi)$ to within a positive constant in terms of the extremal operations on $R$ and $B$. It is generally an easy task to write an explicit formula for construction of extremals. For example, consider the product $B = B_1 \times B_2 \times \cdots B_n$ where the $B_i$ are rotund and $B$ is normed as in Lemma 1. Each bounded linear functional $\varphi$ on $B$ may be identified with an $n$-tuple $(\varphi_1, \varphi_2, \cdots, \varphi_n)$ where $\varphi_i \in B_i^*$. Let $\bar{\varphi}_i$ be the extremal of $\varphi_i$ in $B_i$. Then it is easy to verify that with the $p$-norm on $B$ the extremal $\bar{\varphi}$ of $\varphi$ is given by

$$\bar{\varphi} = (\alpha_1\bar{\varphi}_1, \alpha_2\bar{\varphi}_2, \cdots, \alpha_n\bar{\varphi}_n)$$

$$\alpha_i = \alpha^{-1} \| \varphi_i \|^{p-1}, \quad \alpha = \left( \sum \| \varphi_i \|^q \right)^{1/q}, \quad q = \frac{p}{p-1}.$$
Similarly, with the matrix norm on \( B \), \( \varphi \) has the form
\[
\bar{\varphi} = (\beta_1 \bar{\varphi}_1, \beta_2 \bar{\varphi}_2, \ldots, \beta_n \bar{\varphi}_n)
\]

\[
\beta_i = \beta^{-1} \sum_j b_{ij} \| \varphi_j \|, \quad \beta = \sum_{i,j} b_{ij} \| \varphi_i \| \| \varphi_j \|
\]

where \([b_{ij}]\) is the inverse of the matrix \([a_{ij}]\).

These formulas imply that the conjugate space \( B^* \) is (isometrically isomorphic to) the product \( B_1^* \times B_2^* \times \cdots \times B_n^* \) with the respective norms of \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \) given by
\[
\| \varphi \| = \left( \sum_i \| \varphi_i \|^q \right)^{1/q}, \quad \| \varphi \| = \sum_{i,j} b_{ij} \| \varphi_i \| \| \varphi_j \| .
\]

It follows that if each \( B_i \) is also reflexive and smooth, so that each \( x = (x_1, x_2, \ldots, x_n) \) in \( B \) has an extremal \( \bar{x} \) in \( B^* \), then
\[
\bar{x} = (\beta_1 \bar{x}_1, \beta_2 \bar{x}_2, \ldots, \beta_n \bar{x}_n)
\]

\[
\beta_i = (\beta^{-1} \| x_i \|)^{p-1}, \quad \beta = \left( \sum_i \| x_i \|^p \right)^{1/p}
\]

and
\[
\bar{x} = (\delta_1 \bar{x}_1, \delta_2 \bar{x}_2, \ldots, \delta_n \bar{x}_n)
\]

\[
\delta_i = \delta^{-1} \sum_j a_{ij} \| x_j \|, \quad \delta = \sum_{i,j} a_{ij} \| x_i \| \| x_j \| .
\]

In particular, if \( B = l_p, n \) is the space of complex \( n \)-tuple \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \) with the norm
\[
\| \xi \| = \left( \sum_i \| \xi_i \|^p \right)^{1/p}
\]

then
\[
\bar{\xi} = (\eta_1, \eta_2, \ldots, \eta_n)
\]

where
\[
\eta_i = \begin{cases} 
\frac{\| \xi_i \|}{\| \xi \|} \left( \| \xi_i \| \right)^{p-1} & \text{if } \xi_i \neq 0 \\
0 & \text{if } \xi_i = 0.
\end{cases}
\]

A precisely analogous formula holds in \( L_p \) (1 < \( p < \infty \)).

Observe also that if \( T \) arises from a linear system in the sense that for a system input \( u \), \( Tu \) is the value of the output state vector at some fixed
instant, then its range is finite dimensional so that $T^*$, being a linear transformation on a finite dimensional space, will be given by a matrix. Thus, evaluation of $T^*(\xi)$ is reduces to familiar computations. Finally, note that the preceding remarks in particular determine the extremal operations in (suitably normed) input spaces of the form

$$B = l_{p_1,n_1} \times l_{p_2,n_2} \times \cdots \times l_{p_k,n_k} \times L_{q_1} \times L_{q_2} \times \cdots \times L_{q_l}$$

where $1 \leq n_i \leq \infty$ and $1 < p_i, q_i < \infty$. In other words, $T$ may represent systems with digital and/or functional inputs.

**Lemma 3.** Let $C = T(U)$ be the image of the unit ball in $B$. Then $C$ is a convex, circled, weakly compact, neighborhood of 0 in $R$.

**Proof.** Since $T$ is linear, $C$ is convex and circled. The Opening Mapping Theorem shows that $T(U)$ contains a multiple of the unit ball in $R$, and hence is a neighborhood of 0. Finally, it is known [6; p. 115] that a continuous linear mapping from one Banach space into another remains continuous when both spaces are equipped with their weak topologies. Since $U$ is weakly compact in $B$ it follows that $T(U)$ is weakly compact in $R$.

It follows from Lemma 3 that $C$ is radial at 0. That is, for each $\xi \in R$ there is a scalar $\lambda > 0$ such that $\xi \in \lambda C$. Hence [7] the Minkowski functional $p$ given by

$$p(\xi) = \inf\{\lambda > 0 : \xi \in \lambda C\}$$

is defined and finite on all of $R$. Since $C$ is convex and circled the functional $p$ is subadditive and absolutely homogeneous.

$$p(\xi + \zeta) \leq p(\xi) + p(\zeta), \quad \xi, \zeta \in R$$

$$p(\lambda \xi) = |\lambda| p(\xi).$$

The next lemma lists a few facts we will need.

**Lemma 4.** (i) The interior of $C$ consists of those $\xi \in R$ for which $p(\xi) < 1$.

(ii) $\partial C = \{\xi \in R : p(\xi) = 1\}$ is the boundary of $C$.

(iii) $\partial C \subset T(\partial U)$.

**Proof.** The assertions (i) and (ii) are well known and follow directly from the definition of $p$. As for (iii), if $\xi \in \partial C$, then $\xi \in C$ and hence $\xi = Tu$ for some $u \in U$. Since by (ii), $p(\xi) = 1$, we have $\lambda^{-1} \xi \notin C$ for all $\lambda < 1$. But then $\lambda^{-1} u \notin U$ for all $\lambda < 1$. This means that $\|u\| \geq 1$, and since $u \in U$, that $\|u\| = 1$.

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\footnote{A set $C$ in a vector space $E$ is circled if $\lambda C \subset C$ for all $|\lambda| \leq 1$.}
REMARK. It is easy to construct examples to show that the reverse inclusion (iii) is not valid in general.

COROLLARY. The functional \( p \) is a norm on \( R \) equivalent to the given norm. In fact for some constant \( k > 0 \) we have

\[
\frac{1}{\| T \|} \| \xi \| \leq p(\xi) \leq k \| \xi \| \quad (\xi \in R).
\]

PROOF. Suppose \( \| \xi \| > 0 \) and let \( \lambda \) be any positive scalar with \( \xi \in \lambda C \). Then \( \frac{1}{\lambda} \xi \in T(U) \) and hence

\[
\| \frac{1}{\lambda} \xi \| \leq \| T \|.
\]

This implies that

\[
p(\xi) \geq \frac{\| \xi \|}{\| T \|}
\]

and hence that \( p \) is a norm on \( R \).

By Lemma 3, \( C = \{ \xi : p(\xi) \leq 1 \} \) is a neighborhood of \( 0 \) in \( R \) and hence there is an \( \epsilon > 0 \) such that \( p(\xi) \leq 1 \) if \( \| \xi \| \leq \epsilon \). Hence \( p(\xi) \leq (1/\epsilon) \| \xi \| \) for all \( \xi \in R \).

We are now able to obtain the promised characterization of \( T'(\xi) \). If \( N \) is a real linear functional on a real vector space \( E \) we will say that a subset \( C \) of \( E \) lies to the left of the hyperplane \( H = \{ \xi \in E : \langle \xi, N \rangle = \alpha \} \) provided that \( \langle \xi, N \rangle \leq \alpha \) for all \( \xi \in C \). \( H \) supports \( C \) if it meets \( C \) and if \( C \) lies entirely on one side of \( H \). A geometric form of the Hahn-Banach Theorem, valid in any topological vector space, asserts that a closed convex set with nonempty interior has a supporting hyperplane through each of its boundary points [8; p. 72].

**Theorem 4.** Let \( \xi_0 \neq 0 \) be a given vector in \( R \) and let \( \alpha = p(\xi_0)^{-1} \). Then there exists a unique vector \( N \) in the unit sphere of \( R^* \) such that

\[
T'(\xi_0) = p(\xi_0) T^* N.
\]

The functional \( N \) is uniquely determined by the conditions

(i) \( \| N \| = 1 \).

(ii) \( C \) lies to the left of the hyperplane \( H = \{ \xi \in R : \langle \xi, N \rangle = \alpha \langle \xi_0, N \rangle \} \). If \( B \) is a complex space this last requirement is to be interpreted as saying that

\[
\text{Re} \langle \xi, N \rangle \leq \text{Re} \langle \alpha \xi_0, N \rangle \quad \text{all} \quad \xi \in C.
\]
PROOF. Suppose first that $B$ is real. Since $C$ is closed, convex, and has nonempty interior it follows that there is a supporting hyperplane of $C$ at $\alpha \xi_0$ and hence a functional $N$ satisfying (i) and (ii). Note that since $0 \in C$, $N$ is nonnegative at $\alpha \xi_0$.

To prove the theorem it evidently suffices to prove:

(a) $\varphi \in R^*$ satisfies $T'(\xi_0) = \rho(\xi_0) T^*\varphi$ if and only if (ii) holds for $\varphi$.
(b) There is at most one $\varphi$ of norm 1 satisfying $T'(\xi_0) = \rho(\xi_0) T^*\varphi$.

The proof of (b) follows from the fact that the mapping $\varphi \to T^*\varphi$ is one-to-one from the unit sphere of $R^*$ into the unit sphere of $B$.

Suppose next that $T'(\xi_0) = \rho(\xi_0) T^*\varphi$ for some $\varphi \in R^*$. Then

$$\xi_0 = T'(\xi_0) = \alpha^{-1}T(T^*\varphi)$$

and hence

$$\langle \xi_0, \varphi \rangle = \langle T(T^*\varphi), \varphi \rangle = \langle T^*\varphi, T^*\varphi \rangle$$

$$= \| T^*\varphi \| = \| u, T^*\varphi \| = \langle Tu, \varphi \rangle$$

for all $u \in U$ and since $C = T(U)$ this shows that $\varphi$ satisfies (ii). (Note that since $\varphi$ is a real functional, the number $\langle u, T^*\varphi \rangle$ is real for any $u \in U$.)

Finally, suppose $\varphi \in R^*$ satisfies (ii). Since $\alpha \xi_0 \in \partial C$ there is a $u_0 \in \partial U$ with $Tu_0 = \alpha \xi_0$. Then

$$\langle u_0, T^*\varphi \rangle = \langle \alpha \xi_0, \varphi \rangle = \| \alpha \xi_0, \varphi \| = \| u_0, T^*\varphi \|.$$ 

Hence by definition of the norm of the functional $T^*\varphi$ on $B$ we have

$$\| T^*\varphi \| = \sup_{u \in U} | \langle u, T^*\varphi \rangle | \geq \langle u_0, T^*\varphi \rangle$$

and since $T^*\varphi \in U$,

$$\langle \xi_0, \varphi \rangle \geq \langle T(T^*\varphi), \varphi \rangle = \| T^*\varphi \|.$$ 

We conclude that $\langle u_0, T^*\varphi \rangle = \| T^*\varphi \|$ and hence that $u_0 = \overline{\alpha T^*\varphi}$. Thus the vector $\alpha^{-1}T^*\varphi$ is a pre-image (under $T$) of $\xi_0$ and to prove that this is $T'(\xi_0)$ it remains only to show that any $u \in B$ satisfying $Tu = \xi_0$ has a norm of at least $\alpha^{-1}$. This however follows from

$$\langle u, T^*\varphi \rangle = \langle \xi_0, \varphi \rangle = \alpha^{-1} \langle \alpha \xi_0, \varphi \rangle = \alpha^{-1} \| T^*\varphi \|$$

and the fact that

$$\| u \| = \sup_{f \in B^*} \frac{| \langle u, f \rangle |}{\| f \|}.$$ 

Suppose now that $B$ is a complex space. Then [7; p. 118] the boundary
point $\alpha \xi_0$ of $C$ can be separated from $C$ by a complex linear functional $N$ in the sense that

$$\text{Re} \left\langle \xi, N \right\rangle \leq \text{Re} \left\langle \alpha \xi_0, N \right\rangle \quad \forall \xi \in C.$$ 

The remainder of the argument now proceeds as before.

**Remark.** The unique vector $N$ in $R^+$ satisfying (i) and (ii) deserves, in a natural way, to be called the *outward normal* to $C$ at $\alpha \xi_0$. We have shown that there is an outward normal to $C$ at each of its boundary points.

Observe also that it follows from the theorem that $\| T'(\xi) \| = p(\xi)$. Since the latter function is a (uniformly) continuous function, we see that the minimum effort associated with each state $\xi \in R$ is a continuous function of $\xi$: if two vectors $\xi_1, \xi_2$ in $R$ are close, and if $u_1$ and $u_2$ are their minimum pre-images under $T$, then the norms of $u_1$ and $u_2$ are correspondingly close.

It is easy to show that in case $B = H$ is a Hilbert space, the formulas $T'(\xi) = T_{*1} \xi$ and $T'(\xi) = p(\xi) T^*N$ are consistent.

**Lemma 5.** For each $\xi \in R$, set $\| \xi \| = p(\xi)$. Then $\| \cdot \|$ is a norm on $R$, equivalent to the given norm. Let $R_1$ denote the space $R$ equipped with the norm $\| \cdot \|$. Then $R_1$ is rotund and smooth.

The proof of Lemma 5 is left to the reader. Now it follows from Lemma 5 and Theorem 4 that the definition $\| \xi \| = p(\xi)$ yields a norm on $R$ for which

$$\| \xi \| = \| T'(\xi) \|$$

holds identically in $\xi$. It therefore follows from Lemma 2 that

$$T'(\xi) = p(\xi) \overline{T^*(\xi')}, \quad \xi \in R$$

where $\xi'$ denotes the extremal of $\xi$ relative to the norm $\| \cdot \|$. That is, $\xi'$ is characterized by the equations

$$\sup_{p(\xi') = 1} \| \xi, \xi' \| = 1, \quad \langle \xi, \xi' \rangle = p(\xi).$$

Since by Lemma 1(c) applied to $R^*$,

$$T^*(\xi' / \| \xi' \|) = \overline{T^*(\xi')}$$

we have proven part of the following:

**Theorem 5.** Let $\xi$ be a fixed boundary point of $C$ and let $N$ be the outward normal to $C$ at $\xi$. Then

(i) $N = \xi' / \| \xi' \|$ where $\xi'$ is the extremal of $\xi$ relative to the norm $p(\xi)$ on $R$. 

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(ii) \( N \) is the unique vector \( \varphi \) in \( R^* \) of norm 1 satisfying \( \| T^* \varphi \| = \langle \xi, \varphi \rangle \).

(iii) \( N = \xi' / \| \xi' \| \) where \( \xi' \) is the bounded linear functional on \( R \) whose real part is defined for \( \zeta \in R \) by

\[
\text{Re} \langle \zeta, \xi' \rangle = \lim_{\epsilon \to 0} \frac{|\xi + \epsilon \xi' - \xi|}{\epsilon} = \lim_{\epsilon \to 0} \frac{\rho(\xi + \epsilon \xi') - \rho(\xi)}{\epsilon}.
\]

Proof. If \( \varphi \in R^* \) satisfies \( \| T^* \varphi \| = \langle \xi, \varphi \rangle \), then for any \( \zeta \in C \), we may choose \( u \in U \) so that \( Tu = \zeta \) to obtain

\[
\langle \zeta, \varphi \rangle = \langle Tu, \varphi \rangle = \langle u, T^* \varphi \rangle \leq \| T^* \varphi \| = \langle \xi, \varphi \rangle
\]

and hence, by Theorem 4, \( \varphi \) is a positive multiple of \( N \). This proves (ii).

Now consider (iii). We observe that since \( R \) is smooth its norm has a Gateaux derivative at each point on the boundary of its unit ball. That is,

\[
G(\xi, \zeta) = \lim_{\epsilon \to 0} \frac{|\xi + \epsilon \xi' - \xi|}{\epsilon}
\]

exists for each \( \xi \in \partial C \) and \( \zeta \) in \( R \). Assertion (iii) now follows from Theorem 2.

4. Discussion

It is clear from the preceding results that once one knows the set \( C \) relatively simple computations furnish (a) the minimum effort \( T \) needs to reach any given state \( \xi \) in \( R \) and (b) the precise pre-image \( T(\xi) \) of \( \xi \) whose effort is this minimum value. Indeed the boundary of the set \( \alpha C \) is a "level surface" consisting of those states \( \xi \in R \) which \( T \) can obtain with a minimum energy of precisely \( \alpha \), and the outward normals to \( C \) determine (to within a positive multiple) the class of minimum energy inputs. However, even in the relatively simple case in which \( B \) is finite dimensional, the equation \( C = T(U) \) is unsuitable for specifying \( C \). It is therefore, natural to seek a simpler way to determine \( C \). For example, if \( C \) is a multiple of the unit ball in \( R \) we need only one parameter to specify \( C \) completely; if \( C \) is an ellipsoid we need only to determine the size of its semiaxes, and so on. In any event the conditions of Theorem 5 are sufficient to compute \( N \) by iterative procedures if necessary.

References


