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The Interpolating Sets for A^∞

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1. INTRODUCTION

Let A denote the algebra of functions analytic on the open unit disc $D = \{z : |z| < 1\}$ and continuous on $\bar{D}$, the closure of D. Let A^∞ denote the algebra of functions f such that f and all its derivatives belong to A. By identifying each function in A^∞ with its restriction to $\partial D = \{z : |z| = 1\}$, A^∞ may be regarded as a closed subalgebra of $C^\infty(\partial D)$, the space of complex-valued infinitely differentiable functions on ∂D. A closed subset E of ∂D is said to be a *strong interpolating set* for A^∞ if for every $\varphi \in C^\infty(\partial D)$ there exists $f \in A^\infty$ such that

$$\frac{d^j f}{d\theta^j}(e^{i\theta}) = \frac{d^j \varphi}{d\theta^j}(e^{i\theta}),$$

for all $j = 0, 1, 2,\ldots$ and all $e^{i\theta} \in E$. We give here a characterization of the strong interpolating sets for A^∞.

Our solution to this interpolation problem is in terms of the function

$$\rho(z) = \rho(z, E) = \inf\{|z - w| : w \in E\},$$

the Euclidean distance from z to E.

THEOREM 1.1. *A closed set $E \subset \partial D$ is a strong interpolating set for A^∞ if and only if there exist constants C_1 and C_2 such that*

$$\frac{1}{b-a}\int_a^b \log \frac{1}{\rho(e^{i\theta}, E)}\, d\theta \leqslant C_1 \log \frac{1}{b-a} + C_2 \tag{1.1}$$

for all $a < b$.

We remark that (1.1) is equivalent to the condition that the harmonic extension u of $-\log \rho(e^{i\theta}, E)$ into D satisfy

$$u(re^{i\theta}) = 0\left(\log \frac{1}{1-r}\right), \qquad r \to 1^-. \tag{1.2}$$

(See Lemma 3.3.)

The solution to the corresponding problem for A and $C(\partial D)$, the space of continuous functions on ∂D, was given by Carleson [2] and Rudin [11]. They showed that the closed sets $E \subset \partial D$ with Lebesque measure zero are the ones with the property that for every $\varphi \in C(\partial D)$ there exists $f \in A$ such that $f(e^{i\theta}) = \varphi(e^{i\theta})$ for all $e^{i\theta} \in E$.

The closed sets of measure zero in ∂D are also the (proper) boundary zero sets of the functions in the space A. Theorem 1.1 shows that the situation is different in A^∞. Namely, the (proper) boundary zero sets of the functions in the space A^∞ are precisely those closed subsets $E \subset \partial D$ for which $-\log \rho(e^{i\theta}, E)$ is integrable on ∂D ([1, 9, 13, 14]). Such subsets of ∂D are called Carleson sets. Clearly, (1.1) is a more restrictive condition on $-\log \rho(e^{i\theta}, E)$, so that in A^∞ not all Carleson sets are strong interpolating sets.

2. The Dual Problem

We are going to prove Theorem 1.1 by stating it in terms of functional analysis, formulating the equivalent dual problem, and then solving the dual problem. The proof is based on the one given by Glicksberg [3] for the analogous problem for A.

Let E be a closed subset of ∂D and let $I(E)$ denote the ideal of all functions in $C^\infty(\partial D)$ which vanish, together with all their derivatives, on E. Also, let $I_A(E) = I(E) \cap A^\infty$. Consider the quotient map from $C^\infty(\partial D)$ onto $C^\infty(\partial D)/I(E)$, and denote the restriction of this map to A^∞ by $\mathscr{R}$. The problem of characterizing the strong interpolating sets may be stated: for what closed sets $E \subset \partial D$ does $\mathscr{R}$ map A^∞ onto $C^\infty(\partial D)/I(E)$?

There are two obvious comments to be made. First, if $\mathscr{R}$ is onto, then $I_A(E) \neq \{0\}$. For, let $F \in A^\infty$ be such that $F(e^{i\theta}) - e^{-i\theta}$ belongs to $I(E)$. Then $G(z) = 1 - zF(z)$ belongs to $I_A(E)$ and obviously is not identically zero. Consequently, if $\mathscr{R}$ is onto, E must be the zero set of a nontrivial A^∞ function; that is, a Carleson set.

Second, note that if E is a proper closed subset of ∂D, then the range of $\mathscr{R}$ is dense in $C^\infty(\partial D)/I(E)$. To show this, is suffices to show that if K is a proper closed arc in ∂D, then the A^∞ functions restricted to K are dense in $C^\infty(K)$. By Runge's Theorem the functions $e^{in\theta}$, $n = 0, \pm 1, \pm 2,\ldots$, belong to the closure in $C^\infty(K)$ of the set of A^∞ functions restricted to K. The proof is finished by observing that the finite linear combinations of the functions $e^{in\theta}$, $n = 0, \pm 1, \pm 2, \ldots$, are dense in $C^\infty(K)$.

Let $\mathscr{D}'(\partial D)$ denote the dual space of $C^\infty(\partial D)$; that is, $\mathscr{D}'(\partial D)$ is the space of Schwartz distributions on ∂D. Let $I(E)^\perp$ (respectively $I_A(E)^\perp$) denote all the distributions $T \in \mathscr{D}'(\partial D)$ such that $T(f) = (f, T) = 0$ for all $f \in I(E)$ (respectively $I_A(E)$). Let $\mathscr{A}$ denote the analytic distributions; that is,

$$\mathscr{A} = \{T \in \mathscr{D}'(\partial D) : T(f) = 0 \text{ for all } f \in A^\infty\}.$$

An equivalent form of the interpolation problem is given by the following proposition.

Proposition 2.1. *Let E be a proper closed subset of ∂D. Then $\mathscr{R}$ is onto if and only if every $T \in I_A(E)^\perp$ can be decomposed into the sum of two distributions, $T = T_1 + T_2$, with $T_1 \in \mathscr{A}$ and $T_2 \in I_A(E)^\perp$. Moreover, whenever such a decomposition exists, it is unique.*

Proof. We have seen that $\mathscr{R} : A^\infty \to C^\infty/I(E)$ has dense range. Hence $\mathscr{R}$ is onto if and only if $\mathscr{R}$ has closed range. By the closed range theorem for Frechet spaces[7, p. 308], $\mathscr{R}$ has closed range if and only if the dual map $\mathscr{R}' : (C^\infty/I(E))' \to (A^\infty)'$ has closed range. Identifying $(C^\infty/I(E))'$ with $I(E)^\perp$ and $(A^\infty)'$ with $\mathscr{D}'/\mathscr{A}$ we conclude that $\mathscr{R}'$ has closed range if and only if $(I(E)^\perp + \mathscr{A})/\mathscr{A}$ is closed in $\mathscr{D}'/\mathscr{A}$; that is, when $I(E)^\perp + \mathscr{A}$ is closed in $\mathscr{D}'$. It is easily seen that $I_A(E)^\perp$ is the closure in $\mathscr{D}'$ of $I(E)^\perp + \mathscr{A}$. Thus $\mathscr{R}$ is onto if and only if $I(E)^\perp + \mathscr{A} = I_A(E)^\perp$.

Finally if $T \in I(E)^\perp \cap \mathscr{A}$ then set $F(z) = T(1/(z - s))$. As $T \in I(E)^\perp$, F is analytic off E. Also $T \in \mathscr{A}$ implies $F \equiv 0$ in $|z| > 1$. Hence $F \equiv 0$. So $T = 0$ and $I(E)^\perp + \mathscr{A}$ is a direct sum.

3. Proof of Theorem 1.1

We are going to prove Theorem 1 by making use of the criterion of Proposition 2.1. Half of this proof requires a close study of the distributions $T \in I_A(E)^\perp$ and makes use of results proved in [13].

Every $T \in \mathscr{D}'(\partial D)$ has a Fourier series $T(e^{i\theta}) = \sum_{n=-\infty}^{\infty} d_n e^{in\theta}$, convergent to T in the strong topology of $\mathscr{D}'(\partial D)$, where $d_n = (e^{-in\theta}, T)$. Let B' denote the closed subspace of $\mathscr{D}'(\partial D)$ consisting of those distributions with vanishing positive Fourier coefficients. The space $C^\infty(\partial D)$ is the topological direct sum of A^∞ and the closed subspace of $C^\infty(\partial D)$ functions with vanishing non-negative Fourier coefficients. Thus $\mathscr{D}'(\partial D)$ is the topological direct sum of B'

and the distributions with vanishing nonpositive Fourier coefficients,i.e., the analytic distributions. Thus B' may be indentified as the dual of A^∞. Note that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to A^∞ and $T(e^{i\theta}) = \sum_{n=0}^{\infty} b_n e^{-in\theta}$ belongs to B', then $T(f) = (f, T) = \sum_{n=0}^{\infty} a_n b_n$.

For each $T \in B'$, the transform $T(\zeta) = (f_\zeta , T)$, where $f_\zeta(z) = \zeta(\zeta - z)^{-1}$, is a function analytic in $(\mathbb{C} \cup \{\infty\}) \sim \bar{D}$ with $\log | T(\zeta)| = 0(\log(| \zeta | - 1)^{-1})$, $| \zeta | \to 1^+$. If T has Fourier series $\sum_{n=0}^{\infty} b_n e^{-in\theta}$, then $T(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^{-n}$ is the Laurent expansion of $T(\zeta)$ for $| \zeta | > 1$. Also, T may be recovered from $T(\zeta)$ as follows. The functions $T_r(e^{i\theta})$, $r > 1$, regarded as elements of B', converge to T in the weak topology of $\mathscr{D}'(\partial D)$, that is, for all $g \in C^\infty(\partial D)$,

$$T(g) = (g, T) = \lim_{r\to 1^+} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\theta})\, T(re^{i\theta})\, d\theta. \tag{3.1}$$

Moreover, if $T(\zeta)$ is analytic in $(\mathbb{C} \cup \{\infty\}) \sim \bar{D}$ and

$$\log | T(\zeta)| = 0(\log(| \zeta | - 1)^{-1}), \qquad | \zeta | \to 1^+,$$

then (3.1) defines a distribution $T \in B'$ with $T(\zeta)$ as its transform.

THEOREM 3.1. *Let E be a Carleson set. A function $T(\zeta)$, analytic for $| \zeta | > 1$, is the transform of a distribution $T \in B' \cap I_A(E)^\perp$ if and only if:*

(i) *$T(\zeta)$ can be continued analytically to $\mathbb{C} \sim E$;*

(ii) *there are positive constants C, K, N such that*

$$| T(\zeta)| \leqslant C\rho(\zeta, E)^{-N} + K, \qquad | \zeta | \geqslant 1;$$

(iii) $$\lim_{r\to 1^-} \int_{-\pi}^{\pi} \log^+ | T(re^{i\theta})|\, d\theta = \int_{-\pi}^{\pi} \log^+ | T(e^{i\theta})|\, d\theta;$$

and

(iv) $T(0) = 0$.

Proof. Suppose $T \in B' \cap I_A(E)^\perp$. Then (i) is Lemma 5.5 of [13], and (ii) is Lemma 5.10 of [13]. To prove (iii) and (iv), we use the representation for $T(z)$, $| z | < 1$, which was derived in the proof of Lemma 5.5 of [13]. It was shown there that if $f \in I_A(E)$ and $T \in I_A(E)^\perp$, then the distribution fT is analytic and in fact is an A^∞ function. Further, $T(z) = (fT)(z)/f(z)$ for $| z | < 1$. Choosing f to be an outer function in $I_A(E)$, it is clear that, as a function in the unit disc D, T has bounded characteristic and has no inner function in the denominator of its canonical factorization. Consequently, (iii) is satisfied([10, p. 82]). To verify (iv) observe that

$$T(0) = \frac{(fT)(0)}{f(0)} = \frac{\left(\sum_{n=0}^{\infty} a_n b_n\right)}{f(0)} = \frac{(f, T)}{f(0)} = 0.$$

where the a_n and b_n are the Fourier coefficients of f and T respectively.

Conversely, suppose (i)-(iv) hold. Condition (ii) implies that

$$| T(\zeta)| = 0((| \zeta | - 1)^{-N}), \qquad | \zeta | \to 1^+.$$

As remarked above, (3.1) defines $T \in B'$ with $T(\zeta)$ as its transform. For $f \in I_A(E)$, Taylor's formula with remainder shows that for any positive integer n, $|f(z)| = 0(\rho(z, E)^n)$, $\rho(z, E) \to 0$, $| z | \leqslant 1$. Hence, by (ii) and the bounded convergence theorem,

$$(f, T) = \lim_{r \to 1^+} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\, T(re^{i\theta})\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\, T(e^{i\theta})\, d\theta.$$

It is well known (see e.g., [10, p. 82]) that (iii) implies that $f(z)\, T(z)$, $| z | < 1$, has bounded characteristic and, in addition, has the factorization

$$f(z)\, T(z) = B(z)\, S(z)\, F(z)$$

where B is a Blaschke product, S is a singular inner function, and F is the outer function with boundary values $|F(e^{i\theta})| = |f(e^{i\theta})\, T(e^{i\theta})|$ a.e. Since $F(e^{i\theta})$ is essentially bounded, $f(z)\, T(z)$ is bounded for $| z | < 1$. Thus, by Cauchy's Theorem and (iv),

$$(f, T) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\, T(e^{i\theta})\, d\theta = f(0)\, T(0) = 0.$$

Hence $T \in I_A(E)^{\perp}$ and the proof is complete.

In order to use Proposition 2.1 and Theorem 3.1 to prove the necessity part of Theorem 1.1 we need the following.

PROPOSITION 3.2. *Let E be a Carleson set for which* (1.1) *fails. Then there exists a function $T(z)$ analytic on $\mathbb{C} \sim E$ satisfying* (i)-(iv) *of Theorem 3.1 and such that*

$$\log | T(z)| = 0\left(\log \frac{1}{1 - | z |}\right), \qquad | z | \to 1^-,$$

fails.

We use the following three Lemmas in proving this proposition. Let $P(r, t)$ denote the Poisson kernel; i.e.,

$$P(r, t) = \frac{1 - r^2}{1 - 2r \cos t + r^2}.$$

LEMMA 3.3. *Suppose g is nonnegative and integrable on ∂D. Let*

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, t)\, g(e^{i(\theta - t)})\, dt.$$

There exist positive constants C_1, C_2 such that for all $\delta > 0$,

$$\int_{|t| \leqslant \delta} g(e^{i(\theta - t)})\, dt \leqslant C_1 \delta \log \delta^{-1} + C_2 \delta, \tag{3.2}$$

if and only if there exist positive constants C_1', C_2' such that

$$u(re^{i\theta}) \leqslant C_1' \log \frac{1}{1 - r} + C_2'. \tag{3.3}$$

Proof. Suppose (3.2) holds and $r > 0$. Set $t_0 = 0$ and $t_n = 2^n(1 - r)$, $n = 1, 2, \ldots$. With N the least integer such that $t_N \geqslant \pi$, write the integral defining u as the sum of the integrals over the sets

$$E_j = \{t : t_{j-1} \leqslant |t| \leqslant t_j\}, \qquad j = 1, 2, \ldots, N.$$

For $t \in E_j$,

$$P(r, t) \leqslant \text{const } 4^{-j}(1 - r)^{-1}.$$

Thus

$$u(re^{i\theta}) \leqslant \text{const}(1 - r)^{-1} \sum_{j=0}^{N} 4^{-j} \int_{|t| \leqslant t_j} g(e^{i(\theta - t)})\, dt.$$

The inequality (3.3) then follows by using (3.2) to estimate the integrals in the last sum.

Conversely, suppose (3.3) holds. With r such that $\delta = (1 - r)/\sqrt{r}$, we have $P(r, t) \geqslant \frac{1}{2}(1 + r)/(1 - r)$ for $|t| \leqslant \delta$. Thus

$$u(re^{i\theta}) \geqslant \frac{1}{2\pi} \int_{|t| \leqslant \delta} P(r, t)\, g(e^{i(\theta - t)})\, dt,$$

$$\geqslant \frac{1 + r}{4\pi(1 - r)} \int_{|t| \leqslant \delta} g(e^{i(\theta - t)})\, dt.$$

Now (3.2) follows from this and (3.3).

LEMMA 3.4. *Let E be a closed subset of ∂D, and for $k = 2, 3$ let*

$$U_k = \left\{ z \in D : 1 - |z| > \frac{\rho(z, E)^k}{2^k} \right\}.$$

There exists $\psi \in C^\infty(\mathbb{C} \sim E)$ such that $0 \leq \psi(z) \leq 1$ for all $z \in \mathbb{C} \sim E$, $\psi(z) = 1$ for $z \in U_2$, $\psi(z) = 0$ for $z \notin U_3$, and

$$\left| \frac{\partial \psi}{\partial \bar{z}}(z) \right| \leq \text{const} \left(\frac{1}{\rho(z, E)} \right)^2$$

for all $z \in \mathbb{C} \sim E$.

Proof. If $z \in U_2$, then the distance from z to the complement of U_3 is clearly at least const $\rho(z, E)^2$. Thus, it is well-known that such a function ψ exists.

For the next lemma let Ω denote an open subset of $\mathbb{C}$ and let p be subharmonic on Ω.

LEMMA 3.5. *If $\varphi \in C^\infty(\Omega)$ with $|\varphi(z)| \leq C_1 \exp\{C_2 p(z)\}$, $z \in \Omega$, for some positive constants C_1, C_2, then there exists $v \in C^\infty(\Omega)$ such that $\partial v/\partial \bar{z} = \varphi$ on Ω and*

$$|v(z)| \leq \frac{C_1'(1 + |z|^2)^2}{\epsilon(z)} \exp(C_2 \tilde{p}(z)), \qquad z \in \Omega,$$

where C_1' is a positive constant,

$$\epsilon(z) = \min\{1, \tfrac{1}{2} \rho(z, \mathbb{C} \sim \Omega)\}, \qquad \textit{and} \qquad \tilde{p}(z) = \sup\{p(z + \zeta) : |\zeta| \leq \epsilon(z)\}.$$

Proof. This may be found in Kiselman [8]. It is a consequence of Hörmander's L^2 estimates for solutions to the equation $\partial v/\partial \bar{z} = \varphi$ [4, Theorem 2.2.1′] and the estimate

$$|u(z)| \leq \frac{1}{\epsilon} \left(\iint_{|\zeta| \leq \epsilon} |u(z + \zeta)|^2 \, dm(\zeta) \right)^{1/2} + \sup_{|\zeta| \leq \epsilon} \left| \frac{\partial u}{\partial \bar{\zeta}}(z + \zeta) \right|,$$

which is a special case of Lemma 4.4 of [6].

Proof of Proposition 3.2. Let

$$g(e^{i\theta}) = \log \frac{2}{\rho(e^{i\theta}, E)},$$

$$F(z) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} g(e^{it}) \, dt \right\},$$

and $u(z) = \log |F(z)|$. Then F is analytic for $|z| < 1$, and by Lemma 3.3,

$$u(z) = 0\left(\log \frac{1}{1 - |z|}\right), \qquad |z| \to 1^-,$$

fails.

Using the notation of Lemma 3.4, we claim that for some constant $C > 0$,

$$u(z) \leqslant \log \frac{1}{\rho(z, E)} + C, \qquad |z| < 1, \quad z \notin U_2. \tag{3.4}$$

To verify (3.4) write

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{S_1} P(r, \theta - t)\, g(e^{it})\, dt + \frac{1}{2\pi} \int_{S_2} P(r, \theta - t)\, g(e^{it})\, dt,$$

where

$$S_1 = \left\{ e^{it} \in \partial D : \pi > |e^{it} - e^{i\theta}| > \frac{1}{\sqrt{2}} \rho(z, E) \right\}$$

and

$$S_2 = \left\{ e^{it} \in \partial D : |e^{it} - e^{i\theta}| \leqslant \frac{1}{\sqrt{2}} \rho(z, E) \right\}.$$

For $e^{it} \in S_1$ and $re^{i\theta} \notin U_2$, $P(r, \theta - t) \leqslant 1$; therefore, the integral over S_1 is bounded by the $L^1(\partial D)$ norm of g. The integral over S_2 does not exceed $\max\{g(e^{it}) : t \in S_2\}$ which in turn does not exceed $\log \rho(re^{i\theta}, E)^{-1} + \text{const}$ for $re^{i\theta} \notin U_2$.

Now take ψ as in Lemma 3.4 and define

$$\varphi = \frac{\partial}{\partial \bar{z}} \psi F = F \frac{\partial \psi}{\partial \bar{z}}.$$

By (3.4),

$$|\varphi(z)| \leqslant C_1 \exp(5 \log \rho(z, E)^{-1}), \qquad z \in \mathfrak{C} \sim E.$$

Applying Lemma 3.5 with $\Omega = \mathfrak{C} \sim E$ and $p(z) = \log \rho(z, E)^{-1}$, there exists $v \in C^\infty(\mathfrak{C} \sim E)$ such that

$$\frac{\partial v}{\partial \bar{z}} = \varphi \qquad \text{and} \qquad |v(z)| \leqslant \frac{K(1 + |z|^2)^2}{\epsilon(z)} \rho(z, E)^{-5}, \qquad z \in \mathfrak{C} \sim E. \tag{3.5}$$

Set $T(z) = z(\psi F - v)$. It is clear that $T(z)$ satisfies (iv) and (i) of Theorem 3.1. Also, (3.5) implies (ii) since $\rho(z) \sim |z|$ and $\epsilon(z) = 1$ when $|z|$

is large. To show (iii) note first that $\epsilon(z) = \frac{1}{2}\rho(z)$ when $\rho(z) < 1$, so (3.5) and the integrability of $\log \rho(z, E)^{-1}$ imply that

$$\lim_{r \to 1^-} \int_{-\pi}^{\pi} \log^+ |v(re^{i\theta})|\, d\theta = \int_{-\pi}^{\pi} \log^+ |v(e^{i\theta})|\, d\theta.$$

Secondly, since F is an outer function in D,

$$\lim_{r \to 1^-} \int_{-\pi}^{\pi} \log^+ |F(re^{i\theta})|\, d\theta = \int_{-\pi}^{\pi} \log^+ |F(e^{i\theta})|\, d\theta.$$

These two facts clearly imply (iii). Thus $T(z)$ is the transform of a distribution in $B' \cap I_A(E)^{\perp}$. The failure of

$$\log |F(z)| = 0(\log(1 - |z|)^{-1}), \qquad |z| \to 1^-,$$

taken together with (3.4) implies the failure of

$$\log |\psi(z) F(z)| = 0(\log(1 - |z|)^{-1}), \qquad |z| \to 1^-.$$

Hence, (3.5) implies the failure of

$$\log | T(z)| = 0(\log(1 - |z|)^{-1}, \qquad |z| \to 1^-.$$

Proof of Theorem 1.1. First, suppose that (1.1) holds. We are going to show that there is a decomposition of the form described in Proposition 2.1 for every $T \in I_A(E)^{\perp}$. Then E is a strong interpolating set by that proposition. Since $\mathscr{D}'(\partial D) = B' \oplus \mathfrak{A}$, it is enough to prove this for $T \in B' \cap I_A(E)^{\perp}$.

Take $T \in B' \cap I_A(E)^{\perp}$. By (i) of Theorem 3.1, $T(\zeta)$ can be continued analytically to $\mathbb{C} \sim E$. Moreover, (iii) of Theorem 3.1 implies that for $|z| < 1$, $T(z)$ has bounded characteristic and has no singular factor in the denominator of its cannonical factorization. Thus

$$\log | T(re^{i\theta})| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) \log | T(e^{it})|\, dt, \qquad r < 1.$$

Then by (ii) of Theorem 3.1, there are positive constants C_1, C_2 such that

$$\log | T(z)| \leqslant C_1 u(z) + C_2, \qquad |z| < 1,$$

where

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) \log \frac{2}{\rho(e^{it}, E)}\, dt,$$

is as in Lemma 3.3. Since we are assuming (1.1), Lemma 3.3 implies

$$\log | T(z)| = 0\left(\log \frac{1}{1 - |z|}\right), \qquad |z| \to 1^-. \tag{3.6}$$

Because (3.6) holds,

$$T^-(g) = (g, T) = \lim_{r\to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\theta})\, T(re^{i\theta})\, d\theta, \qquad g \in C^\infty(\partial D),$$

defines an analytic distribution. We claim that $T = (T - T^-) + T^-$ is the desired decomposition. It is immediate from (3.1) and (3.6) that the distribution $T - T^-$ is supported on E. Thus, by Theorem XXXIII of [12], $T - T^- \in I(E)^\perp$.

Conversely, let E be a Carleson set but such that (1.1) fails. Then according to Proposition 3.2, there exists a function $T(\zeta)$ analytic in $\mathfrak{C} \sim E$ satisfying (i)-(iv) of Theorem 3.1 but such that

$$\log | T(\zeta)| = 0\left(\log \frac{1}{1 - |\zeta|}\right), \qquad |\zeta| \to 1^-,$$

fails. Now by Theorem 3.1, $T(\zeta)$ is the transform of a distribution $T \in B' \cap I_A(E)^\perp$. We claim that T admits no decomposition $T = T_1 + T_2$ with $T_1 \in \mathfrak{A}$ and $T_2 \in I(E)^\perp$. Suppose such a decomposition exists. Taking $f_\zeta(z) = \zeta(\zeta - z)^{-1}$, it is routine to verify that the function $T_2(\zeta) = (f_\zeta, T_2)$ is analytic on $\mathfrak{C} \sim E$. Since $T_1 \in \mathfrak{A}$,

$$T(\zeta) = (f_\zeta, T) = (f_\zeta, T_1 + T_2) = (f_\zeta, T_2) = T_2(\zeta)$$

for $|\zeta| > 1$. But $T(\zeta)$ is also analytic on $\mathfrak{C} \sim E$, so $T(\zeta) = T_2(\zeta)$ for $|\zeta| < 1$. Since $T \in B'$,

$$(f_\zeta, T_1) = (f_\zeta, T - T_2) = -(f_\zeta, T_2) = -T_2(\zeta) = -T(\zeta)$$

for $|\zeta| < 1$. Because T_1 is a distribution, it is easily seen that

$$\log |(f_\zeta, T_1)| = 0(\log(1 - |\zeta|)^{-1}), \qquad |\zeta| \to 1^-.$$

But this contradicts the failure of

$$\log | T(\zeta)| = 0(\log(1 - |\zeta|)^{-1}), \qquad |\zeta| \to 1^-.$$

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