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Renormalization of Some One-Space Dimensional Quantum Field Theories by Unitary Transformation*

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The one-space dimensional $\varphi^m\bar{\psi}\psi$ interaction is renormalized by unitary transformation. A uniformly convergent sequence of unitary operators is defined which transforms the sequence of cut-off Hamiltonians, arranged in order of increasing cut-off energy, to a sequence of operators converging strongly on a dense set of states.

The theories here considered were treated in Ref. [1] by K. Hepp, using the method of J. Glimm [2]. We here treat these theories by use of methods initiated in [3]. We achieve an improvement of the result in [3] by obtaining a uniformly convergent sequence of unitary operators. The renormalization here performed is different from that in [1] and [2] in as much as unitary transformations are employed, but it is similar in that a form of truncation is used.

$$H(\lambda) = H_0 + V(\lambda) + \Delta(\lambda). \tag{1}$$

We will construct unitary operators $U(\lambda)$ with the property that

$$\lim_{\lambda\to\infty} U^{-1}(\lambda)\, H(\lambda)\, U(\lambda) \mid \psi) = H \mid \psi), \tag{2}$$

for $\psi \in \mathscr{D}$ a dense domain to be defined; the limit taken in a strong sense, and H a symmetric operator.

Specifically, we consider the Hamiltonian

$$H(\lambda) = H_{0B} + H_{0F} + V(\lambda) + \Delta(\lambda), \tag{3}$$

with $\Delta(\lambda)$ the renormalization terms for

$$V(\lambda) = g \int dx : \bar{\psi}_\lambda(x)\, \psi_\lambda(x): : \varphi_\lambda{}^m(x): \tag{4}$$

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The theory is constructed from fields defined on a unit interval with periodic boundary conditions:

$$H_{0F} = \sum_p \omega_p(a_p{}^*a_p + b_p{}^*b_p), \tag{5}$$

$$H_{0B} = \sum_k \mu_k d_k{}^*d_k, \tag{6}$$

$$\varphi_\lambda(x) = \frac{1}{\sqrt{2}} \sum_k \frac{1}{\sqrt{\omega_k}} e^{ik\cdot x}(d_k + d_{-k}^*) \qquad \omega_k \leqslant \lambda. \tag{7}$$

As in [2], we define an operator Γ^N that acts linearly on sums of normal ordered products of creation and annihilation operators:

$$\Gamma^N(c_{p_1}^* \cdots c_{p_s}^* c_{p_{s+1}} \cdots c_{p_T})$$

$$= \begin{cases} p(N) \dfrac{1}{E_{p_1} + \cdots + E_{p_s}} c_{p_1}^* \cdots c_{p_T} p(N) & \text{if } \sum_1^s \omega_{p_i} > \sum_{s+1}^T E_{p_i} \\ p(N) \dfrac{-1}{E_{p_{s+1}} + \cdots + E_{p_T}} c_{p_1}^* \cdots c_{p_T} p(N) & \text{if } \sum_{s+1}^T E_{p_i} > \sum_1^s E_{p_i} \\ 0 & \text{if } \sum_{s+1}^T E_{p_i} = \sum_1^s E_{p_i} \end{cases} \tag{8}$$

c, c^* are understood as the a, b, d, a^*, b^*, d^*, and E_{p_i} as either ω_{p_i} or μ_{p_i}. $p(N)$ is the projection onto states with N or less particles in them. The use of $p(N)$ in the Γ^N operation is the only truncation in the procedure to be followed.

The unitary operators $U(\lambda)$ are constructed as a product of unitary operators.

$$U(\lambda) = \cdots e^{-A_N} \cdot e^{-A_{N-1}} \cdots e^{-A_1}. \tag{9}$$

Here A_N is constructed by applying $\Gamma^{((M+2)NR)}$ to the sum of those terms in $V(\lambda)$ whose $E_{\max}$ satisfies

$$\alpha N^s < E_{\max} \leqslant \alpha(N+1)^s, \tag{10}$$

$E_{\max}$ is the maximum energy of the creation and annihilation operators in a given term.

$$E_{\max}(c_{p_1}^* \cdots c_{p_s}^* c_{p_{s+1}} \cdots c_{p_T}) = \max_{1 \leqslant i \leqslant T}(E_{p_i}); \tag{11}$$

α, s, and R remain to be suitably chosen.

The L^2 norm of the kernel of A_N may be estimated as

$$\sim c\sqrt{\frac{1}{\alpha N^s} - \frac{1}{\alpha(N+1)^s}}. \tag{12}$$

Logarithmic powers are not relevant in this estimate, and with the projection operators $p((M+2)\,NR)$ appearing in A_N, we may estimate the operator norm of A_N

$$|A_N| \sim c[(M+2)\,NR]^{\frac{M+2}{2}} \sqrt{\frac{1}{\alpha N^s} - \frac{1}{\alpha(N+1)^s}}, \tag{13}$$

if we satisfy jointly

$$c^2[(M+2)\,R]^{M+2} = \alpha^{1-2\beta}, \tag{14}$$

and

$$s = \frac{M+3}{1-2\beta}, \tag{15}$$

and

$$1 = s\beta^2. \tag{16}$$

It then follows that

$$|A_N| \sim \frac{1}{\alpha^\beta} \frac{1}{N^{\beta s}} \left(\frac{\beta s}{N}\right), \tag{17}$$

(15) and (16) determine s and β, and (14) determines α. It is to be noted that R may be picked arbitrarily and only α effected.

Estimates are required for expressions such as

$$U^{-1}HU = H + \sum_R (A_R, H) + \sum_{R_1 \leqslant R_2} (!)(A_{R_1}, (A_{R_2}, H)) + \cdots, \tag{18}$$

where (!) indicates the required products of factorials when indices are repeated. Estimates roughly proceed as

$$\begin{aligned} &\left| \sum_{R_1 \leqslant R_2 < \cdots \leqslant R_s} (!)(A_{R_1}, (A_{R_2}, \ldots, (A_{R_s}, H) \cdots) \right| \\ &\leqslant \sum_{R_1 \leqslant R_2 \leqslant \cdots < R_s} (!)\, 2^s\, |A_{R_1}| \cdot |A_{R_2}| \cdots |A_{R_2}| \cdot |H| \\ &\leqslant |H| \cdot e^{2\Sigma |A_{R_i}|}. \end{aligned} \tag{19}$$

This estimate must be modified since H has unbounded L^2 kernels to exploit the cancellations between the first few commutators. Because of the projections in the A_i operators the cancellations not being exact, there are disconnected diagrams in the expansion (18). But if R is chosen large enough, this causes no problems. It is exactly these disconnected terms that give difficulty in the $[\varphi^4]_{2+1}$ model. The domain $\mathscr{D}$ is chosen to consist of the states containing finite numbers of particles and whose particle momenta all lie in a bounded region. On this domain, the series (18) can be shown to converge using the operator norms of the A_i above estimated, and estimates of the L^2 norm of the kernels of certain linear combinations of the first few commutators. It is not necessary either to count diagrams or to estimate the L^2 norm of arbitrarily complicated ones.

The computations here involved are simpler than those in [1] and [2]. Perhaps the unitary operator procedure will be useful for proving some properties of these models.

References

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