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Mutual Information for Stochastic Differential Equations*

TYRONE E. DUNCAN

Department of Computer, Information and Control Engineering, College of Engineering, University of Michigan, Ann Arbor, Michigan 48104

Mutual information is calculated for processes described by stochastic differential equations. The expression for the mutual information has an interpretation in filtering theory.

1. INTRODUCTION

In many problems in information theory it is necessary to calculate the mutual information between two processes. For some processes described by stochastic differential equations, we shall calculate the mutual information. The expression for the mutual information is obtained as a function of certain filtering errors which are related to some filtering problems for the original processes.

2. PRELIMINARIES

The processes that we shall consider will be described by the following stochastic differential equations:

$$dX_t = a(t, X_t, Y_t)\, dt + d\tilde{B}_t,$$

$$dY_t = c(t, X_t, Y_t)\, dt + dB_t, \tag{2}$$

where the functions a and c are continuous in t and globally Lipschitz continuous in the remaining variables $t \in [0, 1]$, $X_0 \equiv Y_0 \equiv 0$, and the processes B and $\tilde{B}$ are independent standard Brownian motions.

With the above assumptions on (1) and (2), the solutions exist, are unique, and are functionals of the past Brownian motions (*K.* Itô, 1961).

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For subsequent discussion it will be convenient to denote the measures for the processes B, $\tilde{B}$, X, Y and X and Y described by (1) and (2) as μ_B, $\mu_{\tilde{B}}$, μ_X, μ_Y, and μ_{XY}, respectively. It is known (Girsanov, 1960) that the measures μ_{XY} and $\mu_B\mu_{\tilde{B}}$ are mutually absolutely continuous, i.e., $\mu_{XY} \sim \mu_B\mu_{\tilde{B}}$.

We shall prove an additional result for the absolute continuity of certain measures which will be useful for calculating the mutual information between the processes X and Y.

LEMMA. *Let μ_B and μ_Y be the measures for the processes B and Y, respectively, given in* (2). *$\mu_Y \sim \mu_B$ and*

$$\frac{d\mu_Y}{d\mu_B} = \exp\left[\int \hat{c}_s \, dY_s - \tfrac{1}{2}\int \hat{c}_s^2 \, ds\right], \tag{3}$$

where

$$\hat{c}_s = E[c(s, X_s, Y_s) \mid Y_u ; 0 \leqslant u \leqslant s] \tag{4}$$

and Y in (3) *has the μ_B measure.*

Proof. Using a result of Girsanov (1960) we know that $\mu_{XY} \sim \mu_{\tilde{B}}\mu_B$ and

$$\frac{d\mu_{XY}}{d\mu_{\tilde{B}} \, d\mu_B} = \exp\left[\int a_s \, dX_s - \tfrac{1}{2}\int a_s^2 \, ds + \int c_s \, dY_s - \tfrac{1}{2}\int c_s^2 \, ds\right], \tag{5}$$

where $a_s = a(s, X_s, Y_s)$, $c_s = c(s, X_s, Y_s)$ and X and Y have the $\mu_{\tilde{B}}$ and μ_B measure, respectively.

We only need to show that when we reduce the Radon–Nikodym derivative (5) to $d\mu_Y/d\mu_B$, we have (3).

Let

$$\Lambda = d\mu_{XY}/d\mu_{\tilde{B}}d\mu_B , \tag{6}$$

and

$$\Psi_s = E[\Lambda_s \mid Y_u ; 0 \leqslant u \leqslant s]. \tag{7}$$

The functional Ψ has the following representation (Kunita and S. Watanabe, 1967; Hitsuda 1968; Duncan 1970a):

$$\Psi = \exp\left[\int \phi_s \, dY_s - \tfrac{1}{2}\int \phi_s^2 \, ds\right], \tag{8}$$

where ϕ_s is measurable with respect to $\mathscr{B}(Y_u ; 0 \leqslant u \leqslant s)$ (the sub-σ-field generated by $\{Y_u ; 0 \leqslant u \leqslant s\}$ a standard Brownian motion since the measure is μ_B) and

$$\int \phi_s^2 \, ds < \infty \qquad \text{a.s. } \mu_B . \tag{9}$$

Applying the formula for stochastic dfferentials (K. Itô, 1951) to Ψ, we have

$$\Psi_t = 1 + \int_0^t \phi_s \Psi_s \, dY_s . \tag{10}$$

We shall now relate ϕ to the terms in the exponential for Λ. Using the formula for stochastic differentials, the functional Λ can be written as

$$\Lambda_t = 1 + \int_0^t a_s \Lambda_s \, dX_s + \int_0^t c_s \Lambda_s \, dY_s . \tag{11}$$

Since Λ is a Radon–Nikodym derivative, the stochastic integrals in (11) are martingales and since X and Y in (11) are independent Brownian motions, we have

$$E\left[\int_0^t a_s \Lambda_s \, dX_s \mid Y_u ; 0 \leqslant u \leqslant t\right] = 0. \tag{12}$$

Consider now the second stochastic integral in (11). Clearly,

$$\int (c_s \Lambda_s)^2 \, ds < \infty \qquad \text{a.s. } \mu_{\tilde{B}} \mu_B .$$

Thus we can define the stochastic integral as a limit in L^1 of integrals of uniformly stepwise functions (K. Itô, 1951). Since the product $c\Lambda$ is almost surely sample function continuous we can select an arbitrary sequence of partitions of [0, 1] that become dense such that the integrals of the uniformly stepwise functions $\{\alpha_n\}$ defined as

$$\alpha_n(s) = c_{t_i^{(n)}} \Lambda_{t_i^{(n)}} \qquad t_i^{(n)} \leqslant s < t_{i+1}^{(n)} ,$$

where $\{t_i^{(n)}\}$ is a partition of [0, 1] with $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = 1$ converge in L^1 to

$$\int c_s \Lambda_s \, dY_s .$$

Fix $s \in [0, 1]$. By properties of the Radon–Nikodym derivative and conditional expectation we have

$$\int E[c_s \mid Y_u ; 0 \leqslant u \leqslant s] \, d\mu_{XY} = \int E[c_s \Lambda_s \mid Y_u ; 0 \leqslant u \leqslant s] \, d\mu_{\tilde{B}} \, d\mu_B$$

or

$$\int \hat{c}_s \, d\mu_Y = \int \hat{c}_s \Psi_s \, d\mu_B .$$

Since the sub-σ-fields $\mathscr{B}(Y_u\,; 0 \leqslant u \leqslant s)$ are continuous and

$$\int \hat{c}_s^2\, ds < \infty \qquad \text{a.s.},$$

we have for all $t \in (0, 1)$

$$\lim_{n\to\infty} E\left[\int_0^1 \alpha_n(s)\, dY_s \mid Y_u\,; 0 \leqslant u \leqslant t\right] = \int_0^t \hat{c}_s \Psi_s\, dY_s\,, \tag{13}$$

where the limit is taken with respect to the topology induced by convergence in probability. Since conditional expectation and L^1 limits commute, we also have

$$\lim_{n\to\infty} E\left[\int_0^1 \alpha_n(s)\, dY_s \mid Y_u\,; 0 \leqslant u \leqslant t\right]$$
$$= E\left[\int_0^t c_s \Lambda_s\, dY_s \mid Y_u\,; 0 \leqslant u \leqslant t\right]. \tag{14}$$

This limit uses the topology of L^1 convergence. Therefore, we have for all t

$$E\left[\int_0^t c_s \Lambda_s\, dY_s \mid Y_u\,; 0 \leqslant u \leqslant t\right] = \int_0^t \hat{c}_s \Psi_s\, dY_s \qquad \text{a.s. } \mu_B\,, \tag{15}$$

so that $\hat{c}_s$ is a member of the same equivalence class as ϕ_s. Thus

$$\Psi = \exp\left[\int \hat{c}_s\, dY_s - \tfrac{1}{2}\int \hat{c}_s^2\, ds\right]. \tag{16}$$

To show $\mu_B \ll \mu_Y$ the arguments proceed as above. ∎

To compute the mutual information we shall use the following result obtained by Gelfand and Yaglom (1957), Chiang (1958), and Perez (1957).

THEOREM 1. *Let ξ and η be two random vectors with joint probability measure $P_{\xi\eta}$ and marginal probability measures P_ξ and P_η, respectively. Assume that $P_{\xi\eta} \ll P_\xi P_\eta$. Then the mutual information between ξ and η is*

$$J(\xi, \eta) = \int \alpha(x, y) \log \alpha(x, y)\, dP_\xi(x)\, dP_\eta(y), \tag{17}$$

where

$$\alpha(x, y) = \frac{dP_{\xi\eta}(x, y)}{dP_\xi(x)\, dP_\eta(y)} \tag{18}$$

3. Main Result

We shall now obtain an expression for the mutual information between the processes X and Y.

THEOREM 2. *The mutual information* $J(X, Y)$ *between the processes* $\{X_u\,;\, 0 \leqslant u \leqslant 1\}$ *and* $\{Y_u\,;\, 0 \leqslant u \leqslant 1\}$ *described by* (1) *and* (2) *is*

$$J(X, Y) = \tfrac{1}{2}E\left[\int_0^1 (a(s, X_s, Y_s) - \hat{a}(s, X_s, Y_s))^2\, ds + \int_0^1 (c(s, X_s, Y_s) - \hat{c}(s, X_s, Y_s))^2\, ds\right], \tag{19}$$

where

$$\hat{a}(s, X_s, Y_s) = E[a(s, X_s, Y_s) \mid X_u\,;\, 0 \leqslant u \leqslant s], \tag{20}$$

$$\hat{c}(s, X_s, Y_s) = E[c(s, X_s, Y_s) \mid Y_u\,;\, 0 \leqslant u \leqslant s]. \tag{21}$$

Proof. To compute the mutual information $J(X, Y)$, using Theorem 1, we must determine the Radon–Nikodym derivative

$$\frac{d\mu_{XY}(x, y)}{d\mu_X(x)\, d\mu_Y(y)}.$$

Using Girsanov's result (1960) for absolute continuity for diffusion processes, we have

$$\frac{d\mu_{XY}}{d\mu_{\tilde{B}}\, d\mu_B} = \exp\left[\int a_s\, dX_s - \tfrac{1}{2}\int a_s^2\, ds + \int c_s\, dY_s - \tfrac{1}{2}\int c_s^2\, ds\right], \tag{22}$$

where, as in (5), X and Y are Brownian motions with measures $\mu_{\tilde{B}}$ and μ_B, respectively.

From the lemma, we have

$$\frac{d\mu_Y}{d\mu_B} = \exp\left[\int \hat{c}_s\, dY_s - \tfrac{1}{2}\int \hat{c}_s^2\, ds\right] \tag{23}$$

and

$$\frac{d\mu_X}{d\mu_{\tilde{B}}} = \exp\left[\int \hat{a}_s\, dX_s - \tfrac{1}{2}\int \hat{a}_s^2\, ds\right], \tag{24}$$

where

$$\hat{a}_s = E[a(s, X_s, Y_s) \mid X_u\,;\, 0 \leqslant u \leqslant s]. \tag{25}$$

Let

$$\Phi = d\mu_{XY}/d\mu_X d\mu_Y \,. \tag{26}$$

We have

$$J = \int \Phi \log \Phi \, d\mu_X \, d\mu_Y \,. \tag{27}$$

Using the chain rule for Radon–Nikodym derivatives, we have

$$\Phi = \frac{d\mu_{XY}}{d\mu_{\tilde{B}} \, d\mu_B} \frac{d\mu_B}{d\mu_Y} \frac{d\mu_{\tilde{B}}}{d\mu_X} \,. \tag{28}$$

By the transformation of measures, we have

$$\begin{aligned} \log \Phi = & \int (a_s - \hat{a}_s) \, dX_s + \int (c_s - \hat{c}_s) \, dY_s \\ & - \tfrac{1}{2} \int (a_s^2 + c_s^2) \, ds + \tfrac{1}{2} \int (\hat{a}_s^2 + \hat{c}_s^2) \, ds. \end{aligned} \tag{29}$$

Since we now have the measure μ_{XY}, we use (1) and (2) to rewrite $\log \Phi$ as

$$\begin{aligned} \log \Phi = & \int (a_s - \hat{a}_s)(a_s \, ds + d\tilde{B}_s) \\ & + \int (c_s - \hat{c}_s)(c_s \, ds + dB_s) \\ & - \tfrac{1}{2} \int (a_s^2 + c_s^2) \, ds + \tfrac{1}{2} \int (\hat{a}_s^2 + \hat{c}_s^2) \, ds. \end{aligned} \tag{30}$$

Since the stochastic integrals are martingales on μ_{XY}, we obtain

$$\begin{aligned} J(X, Y) &= E_{\mu_{XY}}[\log \Phi] \\ &= \tfrac{1}{2} E \left[\int_0^1 (a_s - \hat{a}_s)^2 \, ds + \int_0^1 (c_s - \hat{c}_s)^2 \, ds \right]. \quad \blacksquare \end{aligned} \tag{31}$$

Remark 1. In many applications the stochastic differential equation for X, the signal, does not depend on Y, the observations, and Theorem 2 reduces to various known results (Gelfand and Yaglom, 1957; Van Trees, 1968; Baker 1969; Duncan 1970b).

Remark 2. We can calculate mutual information when the diffusion coefficient for (1) is a function of t and X and the diffusion coefficient for (2) is a function of t and Y. If the former diffusion coefficient is a function of Y

or the latter is a function of X, then we have a singular situation which causes the mutual information to be infinite (Gelfand and Yaglom, 1957; Pinsker 1964).

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