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STABILITY OF INTERCONNECTED SYSTEMS

by

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for

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PREFACE

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CHAPTER 1

INTRODUCTION

As engineers and scientists have become interested in increasingly complex dynamic systems (physical, biological, economic, etc.) involving many degrees of freedom and many nonlinear relations, the importance of the so-called "stability problems" has become increasingly evident. At the same time, the area encompassed by the term stability has continually broadened to include a larger and larger group of problems. Of the many reasons for this increasing interest in stability problems, there are two that are of particular significance in engineering applications.

The first reason for this rise of interest in stability follows from the older, parochial view of stability where a system is stable if the perturbing effects of small disturbances or parameter variations are also small. This might be called stability in the narrow sense. Certain features of the mathematical modeling techniques commonly used in engineering are responsible for the importance of this narrow concept of stability. The engineer must be continually aware of the approximations implicit in all mathematical models. At best, a useful mathematical model can represent only a few of the features of the system under study. Some features must be neglected and the analyst would like to be certain that these features are indeed negligible; that is, he would like some assurance that his approximations will not lead to erroneous results. Similarily, the designer must choose tolerances that are both reasonably loose to reduce cost and sufficiently tight to insure stability of operation. He must also insure that small random disturbances do not produce uncontrollable or unstable operation. At this level, both the analyst and the designer are concerned with stability in the narrow sense. These examples lie in what is sometimes called the area of structural stability, an area that has received considerable emphasis in the Soviet literature [Refs. 7, 18, 28].

The second reason for the rise of interest in stability is related to the increasing interest in the "qualitative approach" and the significance of stability problems in the qualitative theory of differential equations [Refs. 34, 35]. One of the fundamental problems in the theory of differential equations is that of describing the properties of the solutions from the form of the equations. This is also a fundamental problem in the applications where differential equations are frequently used as mathematical models. In the simplest cases, the properties of the solutions are obtained by simply integrating (solving) the differential equations. Unfortunately, this procedure is rarely possible in problems of practical interest and one is forced to accept a qualitative description of the solutions as a compromise with the formidable difficulties of exact analysis. Some of the central problems of the qualitative theory involve the determination of such properties as asymptotic behavior, boundedness, relationships between neighboring solutions, and effects of disturbances that are not small. These, and similar properties, are frequently classed as stability properties, and are part of what might be called the stability problem in the broad sense [Refs. 4, 6, 41].

One of the effects of this broader view of the stability problem is the enlargement of the scope of the term "stability." In the narrow view, stability implies an invariance of behavior in the face of perturbations or parameter changes. In this case, a few relatively simple definitions of stability are appropriate for a large class of problems. On the other hand, with the development of the qualitative theory a more pragmatic definition of stability appears [cf. Ref. 16]. There is now a growing tendency to assign the term stable to any desirable behavior (solution) and the term unstable to any undesirable behavior [Ref. 41]. The uninitiated is soon dismayed to find that there are now a considerable number of different stability definitions which appear to be unrelated though there are connections between them [Ref. 6].

Once the stability problem is clearly defined, it is necessary to obtain analytic techniques for determining whether the desired stability actually exists. In this area there are many special results, but the most promising general techniques are found in the group of ideas known as Lyapunov's Second (or Direct) Method [Ref. 27]. While originally developed for stability analysis in the narrow sense, Lyapunov's Second Method (abbreviated, LSM) is, in fact, a powerful tool applicable to most of the problems in the broader sense of stability and a wide class of problems in the qualitative theory [Ref. 16]. In its simplest form LSM involves the construction of a generalized metric on the solution space and the analysis of solution behavior in terms of this metric. Once the proper metric has been chosen, the general form of the solutions and their stability properties are readily determined. This approach is both conceptually satisfying and mathematically rigorous, but in its application there remains one practical difficulty of major importance--the selection of the proper metric (commonly called a "Lyapunov function"). This research attempts to reduce the difficulty of this selection in a particular class of problems; namely, the analysis of the stability of interconnected systems.

In the majority of situations, the application of Lyapunov's Second Method (mainly a problem of finding the proper metric) becomes more difficult as the system dimensionality (number of energy storages or degrees of freedom) increases. While many of the current papers on the generation of Lyapunov functions imply that the procedures described will apply to n^{th}-order systems [Refs. 13, 18, 29, 38], a test on problems of high dimension frequently reveals that only the third- or fourth-order problems worked as examples can be treated with reasonable facility. This limitation, coupled with the fact that many of the higher dimensional systems encountered in practice are actually, or can be considered as, a composite or interconnection of several lower order systems, suggests that one consider a piece-by-piece stability analysis; that is, a separation of the composite system into simpler subsystems to which LSM can be easily applied. The results of this piece-by-piece analysis might then be used to infer stability properties of the original composite system. This thesis presents the results of the application of this piece-by-piece concept of stability analysis. The problem to be considered may be formally stated as follows:

> Obtain information about the stability of composite systems from a study of the properties of their subsystems and the topology of their interconnections.

The successful solution of this problem leads to a circumvention of the formidable difficulties involved in the direct determination of the stability of high-order nonlinear systems.

CHAPTER 2

BACKGROUND AND SUMMARY

This chapter serves the two-fold purpose of (1) putting the stability problem in its proper historical and technical background, and (2) summarizing the new concepts and results to be presented in greater detail in Chapter 3. A limited amount of mathematical notation will be introduced as needed throughout this chapter. However, the emphasis will be on clarity of exposition rather than on careful or rigorous mathematical analysis. For a thorough treatment, the reader is referred to the bibliography and to the following chapters of this thesis. The notation introduced in this chapter is consistent with that used throughout the remainder of this thesis. A detailed discussion of mathematical notation is given in Section 3.1.

Let E^n be an n-dimensional vector space of vectors $x = \text{col}[x_1, x_2, \ldots, x_n]$ with the inner product $(x, y) = \sum_{i=1}^{n} x_i y_i$ and norm $\|x\| = (x, x)^{\frac{1}{2}}$. The transpose of a vector is indicated with a prime as x'. In general, $\dot{x} = f(x, t)$ is a vector differential equation (x and f are n-vectors) with a vector solution $x(t;x_0, t_0)$ passing through the point $x = x_0$ at $t = t_0$. A solution path in E^n is sometimes called a motion referring to the motion of the point x in this space (state space) as t increases. The solution $x(t) \equiv 0$ is called the null solution.

2.1 Historical Background

As scientists study increasingly complex systems, the use of mathematical models becomes increasingly necessary. If the model provides an adequate description of the system, it is possible to predict system behavior from manipulation of the model. However, it was noticed long ago that certain forms of behavior predicted by usually valid models were never observed in the actual system.

A common class of models for many systems are ordinary differential equations. In a large number of practical problems a state variable description [Ref. 46] of the system under study leads to a vector differential equation model of the form

$$\dot{y} = f(y, t) . \tag{2.1}$$

A point $\bar{y}$ in E^n where $f(\bar{y}, t) = 0$ for all t is called, for obvious reasons, an equilibrium point of the model and (for a valid model) corresponds to an equilibrium state for the original system. Early in the study of mathematical models for mechanical systems it was noted that certain equilibrium states predicted by an otherwise valid model were never observed in the physical system being modeled [Ref. 7]. A typical example is a small sphere rolling on the outside of a larger, fixed sphere in a uniform downward gravitational field. According to most simple models, there is an equilibrium point at the top of the large sphere. These models predict that if the small sphere is placed at this point, it will remain there. On the other hand, this equilibrium state is never observed in practice.

Experiences such as this suggested that there must be some additional property associated with those equilibrium points of the model which corresponded to observable equilibrium states. Eventually, it was recognized that this additional property was stability: the observed equilibrium positions were stable and the unobserved equilibrium positions were unstable. It was also recognized that the prediction of these unstable equilibrium states was due to an improper idealization in the mathematical model; i. e., certain disturbances were neglected that were, in this case, not negligible. This was the beginning of stability theory in the narrow sense. In the example of the small sphere rolling on the outside of the larger sphere, the equilibrium point at the top of the large sphere is unstable; that is, small deviations in the initial location or small vibrations neglected in the model produce large deviations in later positions. Since these small deviations or vibrations can never be eliminated in the real world (as opposed to the mathematical world), the event of the small sphere remaining at or very close to the top of the large sphere is never observed. (This is actually an example of a strong form of instability sometimes called complete instability [Ref. 12].)

Because of these experiences, some of the first systematic studies of stability emphasized stability of equilibrium positions. One common stability definition was what is now called Lyapunov stability or stability in the sense of Lyapunov [Ref. 6]. According to this definition, an equilibrium state $\overline{y}$ in an n-dimensional state space E^n is considered stable if any motion $y(t)$ can be kept in an ϵ-neighborhood of the equilibrium $\bar{y}$, for all future time, simply by starting it in some δ-neighborhood of the equilibrium That is, if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|y(t_o) - \bar{y}\| < \delta$ implies $\|y(t) - \bar{y}\| < \epsilon$ for all $t \geq t_o$ (see Fig. 1). An equilibrium point that is not stable is said to be unstable. This is the well-known ball on a smooth plane concept of stability commonly used even today. Even though it presently bears his name, this definition was actually considered long before Lyapunov's time in connection with the stability of equilibrium points of conservative systems [Ref. 6]. In 1788, Lagrange gave one of the early stability theorems when he proved that for a conservative system an equilibrium position, where the potential function has an isolated minimum, is stable (in the above sense) [Ref. 20].

In practice, the concept of Lyapunov stability is found to be less important than asymptotic stability. Asymptotic stability requires Lyapunov stability plus convergence to the equilibrium of all solutions starting sufficiently close; that is, for some $\delta_o > 0$, it holds that $\|y_o - \bar{y}\| < \delta$ implies that $\lim_{t \to \infty} \|y(t;y_o, t_o) - \bar{y}\| = 0$. This is the type of stability exhibited by a ball at the bottom of a hemispherical cup.

A related problem is the stability of a motion rather than an equilibrium point. An unstable motion predicted by a mathematical model will also be unobservable in the physical system. The same reasoning applies as in the case of the unstable equilibrium point, and a stable motion is defined as a generalization of the stable equilibrium

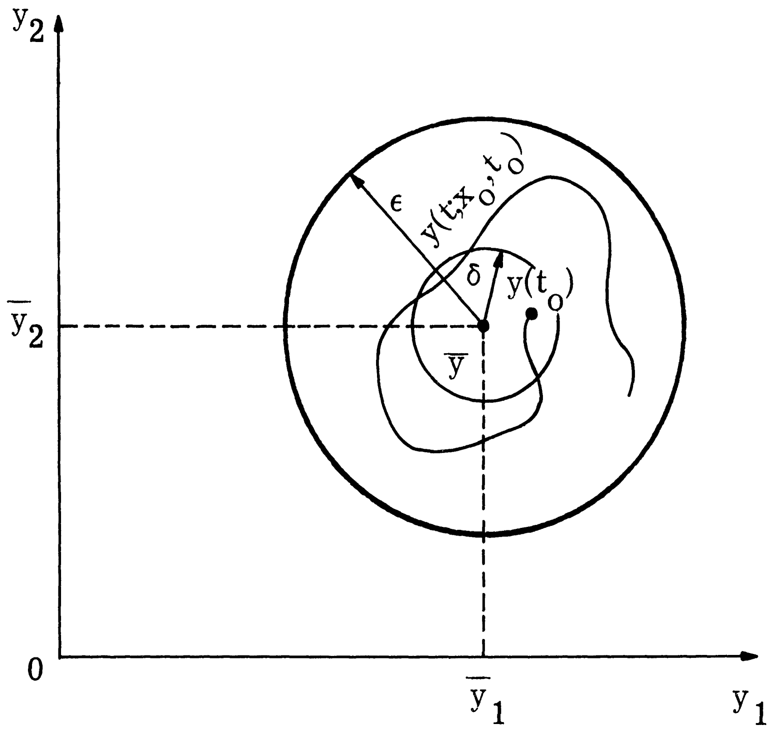

Fig. 1. Stability of an equilibrium point.

point concept. For each a in a set A let $y(t;a)$ define a motion (a curve parameterized by time) in E^n. A particular motion $y(t;\bar{a})$ corresponding to a particular $\bar{a}$ in A is said to be stable in the sense of Lyapunov if all neighboring motions can be kept in a tube of radius ϵ centered on $y(t;\bar{a})$ by starting them at time t_o in a disc of radius δ centered on $y(t_o;\bar{a})$. That is, the motion $y(t;\bar{a})$ is Lyapunov stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|y(t_o;\bar{a}) - y(t_o;a)\| < \delta$ implies $\|y(t;\bar{a}) - y(t;a)\| < \epsilon$ for all $t \geq t_o$. In addition, the motion $y(t;\bar{a})$ is asymptotically stable if the motions sufficiently close at t_o converge to $y(t;\bar{a})$ as t increases. That is, if for some $\delta_o > 0$ it holds that $\|y(t_o;\bar{a}) - y(t_o;a)\| < \delta_o$ implies $\lim_{t \to \infty} \|y(t;\bar{a}) - y(t;a)\| = 0$, [Ref. 12]. Once again, a motion that is not stable is termed unstable. In problems where the mathematical models are differential equations, the parameters, a, are usually the initial condition vectors, y_o. Such a situation is illustrated in Fig. 2.

In 1892, A. M. Lyapunov showed that all problems of stability of an equilibrium point and stability of motion can be reduced to a single problem of the stability of the null solution (an equilibrium point at the origin) of a special equation called the equation of perturbed motion [Ref. 27]. Moreover, he suggested an ingenious method for solving this particular problem--a method of such versatility that its full potential has not yet been realized.

Lyapunov considered only two types of stability: what is presently called Lyapunov stability and asymptotic stability. It is worthwhile to note the similarity between these concepts of stability and the very fundamental mathematical concepts of continuity and convergence.

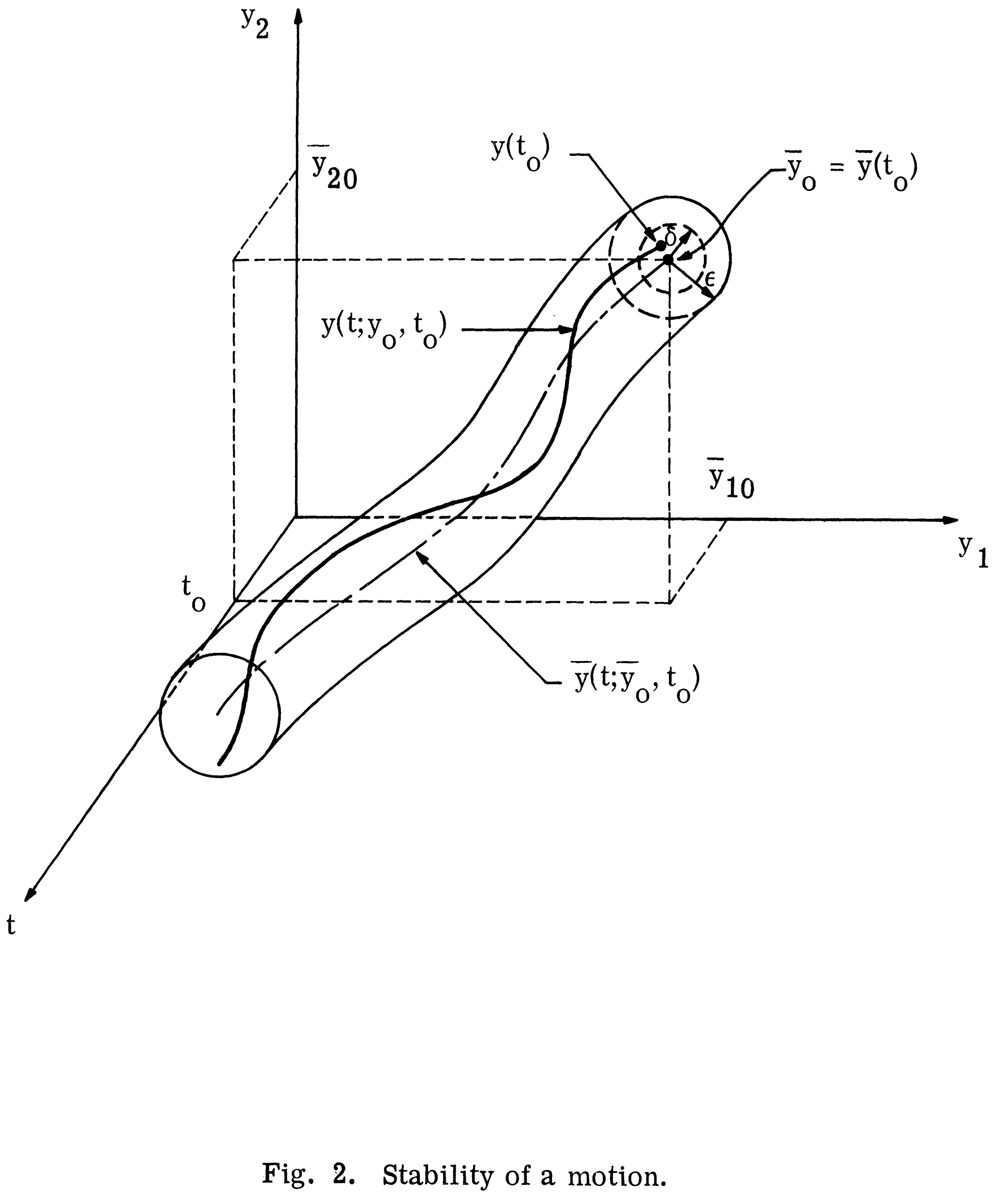

Fig. 2. Stability of a motion.

Stability of an equilibrium point is actually a statement of the continuity at that equilibrium point, of the solutions in the initial condition y_o and the uniformity of this continuity with respect to time. Similarily, asymptotic stability includes a statement of the convergence of the solutions to the equilibrium point. The fact that these two definitions of stability (and their variations to be described below) are so successful in such a wide variety of situations is undoubtedly due in part to this close relation to the very fundamental concepts of continuity and convergence.

The works of Lyapunov [Ref. 27] and some of the older Soviet authors such as Chetayev [Ref. 7] emphasized the narrow viewpoint in stability theory. With the rise of the qualitative approach and the emphasis of the broader viewpoint, new stability definitions began to appear. The Lyapunov stability and asymptotic stability definitions were extended in conformity with the various continuity and convergence concepts of analysis. The earlier results were equi-stability, uniform stability, uniform asymptotic stability, etc. [Refs. 16, 31]. More recently, the concepts of boundedness [Ref. 45], stability-in-the-large [Ref. 18], regions of asymptotic stability [Ref. 21], practical stability [Ref. 22], and ultimate stability [Ref. 23], have also been introduced by authors interested in studying particular qualitative properties.[1]

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1 Many of these concepts represent attempts to remedy the fact that asymptotic stability in itself states only that there is a nonzero region of attraction (region of initial states for which the perturbed motions will converge to the unperturbed motions). In the majority of practical problems, an estimate of the size of this region (the

It is remarkable that the basic techniques developed by Lyapunov in 1892 are still applicable to a majority of these new stability concepts introduced in the qualitative approach. The concept of "bounded input gives bounded output" stability introduced by James, Nichols and Phillips [Ref. 14] is an example of an apparently different stability involving an input-output relation. Yet even this type of stability can be treated by LSM (see Section 5.1, below).

In summary then, the stability problem as considered here is basically a problem in mathematics with important engineering implications. It is, in the most general form, a problem of ascertaining certain significant features of the solutions of a given mathematical model. In this thesis the mathematical models are assumed to be ordinary differential equations. The engineer can contribute to this basically mathematical problem by (1) suggesting solution features of particular significance, or (2) suggesting methods of attack motivated by physical experience. This thesis attempts to contribute in the second area.

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region of asymptotic stability) or an assurance of a region of at least a certain minimum size is required [Ref. 22]. One of the most common remedies is to seek asymptotic stability-in-the-large (abbreviated ASL) where the region of attraction is the whole space. The practical importance of this property has motivated a considerable number of attempts to obtain conditions guaranteeing asymptotic stability-in-the-large in different types of systems [Refs. 2, 3, 25].

2.2 A Coordinate Transformation

In his now famous dissertation of 1892 [Ref. 27], Lyapunov introduced a simple method for reducing a broad class of problems in stability of equilibrium points and stability of motion to one basic problem. The procedure is the following. First, consider a differential equation

$$\dot{y} = g(y, t) \qquad (2.2)$$

and assume that $y = \bar{y}$ is an equilibrium point; that is, assume that $g(0, t) = 0$ for all t. Now let $x(t) = y(t) - \bar{y}$ so that

$$\dot{x} = \dot{y} - 0 = g(x + \bar{y}, t) - g(\bar{y}, t) = g(x + \bar{y}, t) \triangleq f(x, t). \qquad (2.3)$$

The new equation

$$\dot{x} = f(x, t) , \qquad (2.4)$$

where $f(x, t)$ is defined in (2.3), has an equilibrium point at the origin since $f(0, t) = g(0 + \bar{y}, t) = 0$ and this equilibrium point will have the same stability properties as the equilibrium point $\bar{y}$ of (2.2). Next, let $\bar{y}(t)$ be a known solution of (2.2) and choose $x(t) = y(t) - \bar{y}(t)$. Now

$$\dot{x} = \dot{y} - \dot{\bar{y}} = g(x + \bar{y}, t) - g(\bar{y}, t) \triangleq f(x, t) , \qquad (2.5)$$

and the new equation

$$\dot{x} = f(x, t) , \qquad (2.6)$$

where $f(x, t)$ is now defined in (2.5), also has an equilibrium point at the origin since $f(0, t) = g(0 + \bar{y}, t) = 0$. Moreover, the stability properties of this equilibrium point of (2.6) indicate the stability properties of the motion $\bar{y}(t)$. The same notation is used for the right-hand sides of

(2.4) and (2.6) to emphasize the fact that both problems have been reduced to the same form: a differential equation with an equilibrium point at the origin.

Lyapunov calls $\bar{y}$ or $\bar{y}(t)$ the unperturbed motion and (2.4) or (2.6) the equation of perturbed motion (it is actually an equation describing the perturbation as a function of time). In this terminology, a neighboring solution to $\bar{y}$ or $\bar{y}(t)$ such as $y(t)$ is called a perturbed motion. The transformation introduced by Lyapunov reduces all problems concerning stability of motions or stability of equilibrium points of differential equations to a single problem concerning the stability of the equilibrium point $x = 0$ (the null solution) of the equation of perturbed motion

$$\dot{x} = f(x,t) \qquad f(0,t) = 0 . \tag{2.7}$$

Because of this transformation, a major portion of the current literature on stability theory considers only the problem of stability of the null solution of (2.7). It is assumed that all other problems can be reduced to this form. <u>For similar reasons, this thesis will consider only the problem of determining the stability of the null solution of an equation in the form of (2.7).</u>

Having made the above simplification, Lyapunov went on to describe methods of solving the problem of stability of the null solution of the equation of perturbed motion. He considered all techniques for solving this stability problem as divided into two classes. Those that require a determination of the solutions of the equation of perturbed motion are put in the first class which includes the techniques of linear approximation, direct integration, series solution, etc. The second class contains only methods which do not require direct determination of the solution but only a determination of its properties from the form of the differential equation (qualitative methods). Lyapunov's first method was then a technique, basically a series solution technique, illustrating the first class. While not without merit, this first method (for a discussion of the first method, see Cesari, Ref. 6), does not appear to have the versatility of his second method, a technique he included as an illustration of the second class. Lyapunov's Second Method (now sometimes called Lyapunov's Direct Method), has, in recent years, proven to be a tool of amazing versatility.

2.3 Lyapunov's Second Method

In its original form LSM involves the use of a positive definite function (sometimes called a Lyapunov function[2]) as a generalized metric: a measure of the distance from the origin of points in the state space. A continuous, real valued function $v(x)$ on E^n with continuous first partial derivatives is said to be positive definite [positive semi-definite] if $v(0) = 0$ and $v(x) > 0$, [$v(x) \geq 0$] for all $x \neq 0$. Similarly,

[2] This terminology is not uniform Many authors call a positive definite scalar function a Lyapunov function only if its total derivative meets certain requirements. In this thesis, unless otherwise noted, the terms Lyapunov function and positive-definite function are interchangable.

a continuous, real valued function v(x) on E^n with continuous first partial derivatives is said to be negative definite [negative semi-definite] if $v(0) = 0$ and $v(x) < 0$ [$v(x) \leq 0$] for all $x \neq 0$. The application of LSM amounts to the choosing of a particular positive definite function v(x) to establish a generalized metric on the state space and then testing its time derivative along the solution paths of the equation to determine whether the solutions are diverging from or converging to the origin. For example, consider the equation of perturbed motion,

$$\dot{x} = f(x) \tag{2.8}$$

with $f(0) = 0$. Choose a positive definite function v(x) and then[3]

$$\dot{v}(x) = \frac{dv(x)}{dt} = \sum_i \frac{\partial v}{\partial x_i} \frac{dx_i}{dt} . \tag{2.9}$$

Now along the solutions of (2.8)

$$\frac{dx_i}{dt} = f_i(x) , \tag{2.10}$$

and thus the time derivative of v along the solutions of (2.8) is

$$\dot{v}(x) = \sum_i \frac{\partial v}{\partial x_i} f_i(x) . \tag{2.11}$$

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3 If the right-hand side of (2.8) is time dependent, it may be necessary to choose a time dependent Lyapunov function v(x, t). The terms positive definite, etc. then require slightly different definitions which are given in Section 3.5. In this case

$$\dot{v}(x,t) = \sum_i \frac{\partial v}{\partial x_i} f_i(x,t) + \frac{\partial v}{\partial t} .$$

Equation (2.11) is frequently called the total derivative of v with respect to (2.8). The important fact is that $\mathring{v}(x)$ is determined directly from the right-hand side of (2.8). No reduction or solution is necessary. It can now be shown that if $\mathring{v}(x)$ is negative semi-definite, then the null solution of (2.8) is stable in the sense of Lyapunov. If $\mathring{v}(x)$ is negative definite, then the null solution is asymptotically stable. An even stronger form of stability can be assured if v(x) is chosen so that $\lim_{\|x\| \to \infty} v(x) = \infty$. In this case when $\mathring{v}(x)$ is negative definite the null solution is ASL. These are special cases of the theorems discussed in detail in Section 4.2 (also see Ref. 12).

A geometric interpretation of these results in the case $n = 2$ is given in Fig. 3. Since $v(0) = 0$ and v(x) is continuous with $v(x) > 0$ for $x \neq 0$, the v = constant curves for small values of the constant must be closed curves encircling the origin. The solution path, coming in from the left (Fig. 3), apparently crosses the v(x) = constant curves in such a way that $\mathring{v}(x)$ will be negative. If $\mathring{v}(x)$ is negative for all $x \neq 0$, then the solution must be proceeding into smaller and smaller v = constant regions and thus approaching the origin. This suggests asymptotic stability. Similarily, if $\mathring{v}$ is only negative semi-definite, then the solution must be at least staying inside the v = constant contours in which it started. Otherwise, $\mathring{v}$ would at some time be positive. This suggests Lyapunov stability. If v(x) has been chosen so that $\lim_{\|x\| \to \infty} v(x) = \infty$, then the v = constant curves are closed curves encircling the origin for all values of the constant. In this case, these curves

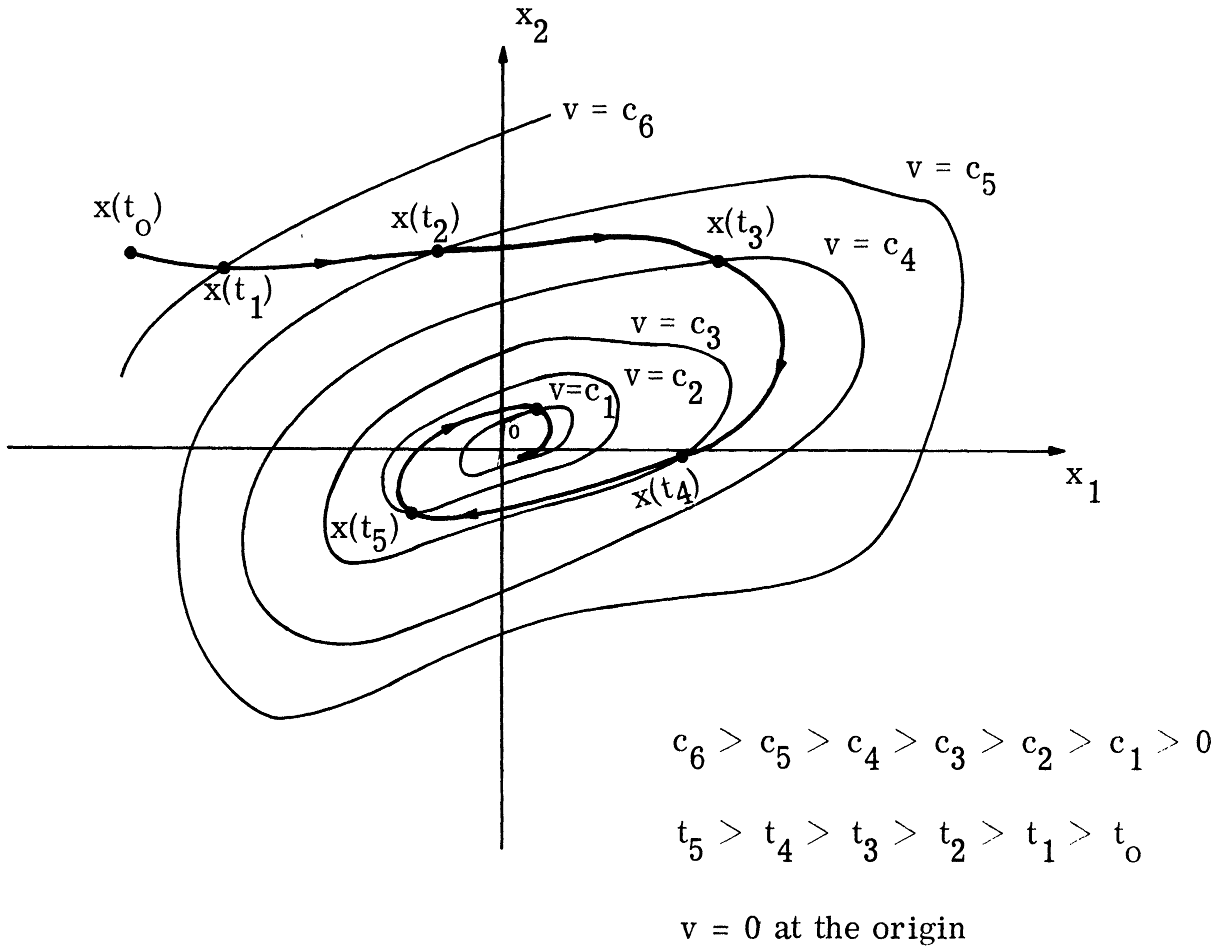


Fig. 3. A geometric interpretation of LSM when n = 2.

"gradually" fill the entire plane as the constant increases. If $\dot{v}(x)$ is now negative for all $x \neq 0$, solutions starting at <u>any</u> point in the plane must converge to the origin; this is ASL. These simple geometric concepts are really the heart of LSM. More sophisticated procedures are necessary in the application of these concepts to complex problems, but the basic idea of using a generalized metric and studying the derivatives of the generalized distance along the solutions carries through in every case.

Once the above example is understood, the advantages of LSM become obvious. The uses of this technique are limited only by the imagination of the user and his ability to analyze the results. Stability, location of solution curves, approximation of solutions and many other problems in the qualitative theory are all readily visualized as problems in the selection of Lyapunov functions and the studying of their derivatives. Unfortunately, this selection problem is quite difficult and the results obtained are often strongly dependent on the Lyapunov function chosen. While there are many results showing that the necessary Lyapunov functions exist [Ref. 12], there are few results showing how to find them. One of the reasons for this problem can be seen in Fig. 3. It is clear that a different choice of Lyapunov function, say, where the v = constant curves are circles, would not have resulted in $\dot{v}$ being negative definite along the solution path even though it did, in fact, converge to the origin. This is a common problem that arises because the theorems provide only sufficient conditions and thus negative results are inconclusive.

In Fig. 3, as is often the case, it is necessary to choose the Lyapunov function so that the v = constant curves are in some sense similar to the solution paths. Since the difficulty in doing this increases with the order and complexity of the differential equation, the problem of choosing Lyapunov functions is a major stumbling block in the application of LSM to complex high order systems. One of the aims of this thesis is a simplification of this problem for a special class of systems.

2.4 Review of Current Literature

There are many methods for analysis of the basic stability problem, namely, the problem of the stability of the null solution of the equation of perturbed motion. However, many of these procedures are special techniques developed for a special class of problems. For instance, for linear constant coefficient systems there are the eigenvalue location techniques such as the Routh-Hurwitz criteria [Refs. 6, 17, 43]. For general linear systems there are the techniques of functional analysis. For second-order equation there are the phase-plane techniques. When considering only local stability (such as Lyapunov stability), there are a number of results obtained through the use of integral equations. While each of these procedures has its advantages, the problem considered in this thesis, a general study of interconnected linear and nonlinear systems, requires a unified approach to the stability analysis of each "piece." Moreover, it is important to be able to consider stability in the broad sense (such as asymptotic stability-in-the-large) since local stability is of doubtful value in practical applications. In view of these requirements, LSM has been selected as the basic tool for the analysis which follows.

As noted above, the application of LSM in stability analysis involves the frequently tricky problem of choosing the proper generalized metric or Lyapunov function. Since the results often depend on the particular Lyapunov function chosen, this choice is frequently a very difficult problem requiring, at present, a considerable amount of ingenuity, intuition, experience or whatever else the analyst can call on for aid. Moreover, the difficulty of this choice is, in general, strongly dependent on the order of the differential equation under analysis with further complication arising when stronger forms of stability are sought. In this thesis asymptotic stability-in-the-large is of major interest.

There are a number of rules and procedures for the construction of Lyapunov functions available in the current literature. Most of these procedures are, in theory, applicable to the construction of Lyapunov functions for systems of arbitrary order. However, the user of these procedures finds that while the results obtained for second-, third-, and even fourth-order systems are frequently quite impressive, there are formidable computational problems encountered when attempting to handle, say, tenth-order systems. This order limitation on presently available procedures has led to the separation concepts (the idea of separating a high order system into several lower order systems) to be introduced below. However, before going on to this new approach, it is worthwhile to emphasize the problems encountered in treating high order systems by considering the currently available procedures in more detail.

First, consider the linear, constant-coefficient differential equation. This is one of the few cases where there are clearly defined procedures for problems of any order. The following theorem is due to Lyapunov [Ref. 27].

Theorem 2.1: The equilibrium solution $x = 0$ of the differential equation

$$\dot{x} = A x \tag{2.12}$$

is asymptotically stable (in-the-large, of course, since the equation is linear) if and only if given any symmetric, positive definite matrix[4] Q, there exists a symmetric positive definite matrix P such that

$$A' P + P A = -Q \ . \tag{2.13}$$

The proof of sufficiency is obvious if one uses the Lyapunov function $v(x) = x'Px$ so that $\nabla v = 2Px$ and thus,[5]

[4] A symmetric matrix S is said to be positive definite if the function $v(x) = x'Sx$ is positive definite (see Section 2.3).

[5] The vector ∇v is defined as $\nabla v = \mathrm{col}[\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n}]$. It is useful in working with Lyapunov functions since

$$\dot{v}(x) = \sum_i \frac{\partial v}{\partial x_i} f_i(x) = (\nabla v, f(x)) \ .$$

$$\dot{v}(x) = (\nabla v, Ax) = 2x' P Ax = x'(P A + A'P) x . \qquad (2.14)$$

The simplicity of this result has challenged a number of authors to find similar results for nonlinear systems. Krasovskii [Ref. 19] gives the following theorem for the general nonlinear differential equation

$$\dot{x} = f(x, t) \qquad \text{with } f(0, t) = 0 \qquad (2.15)$$

where f is continuous on E^n and has uniformly bounded first partial derivatives in every bounded neighborhood of the origin.

<u>Theorem 2. 2:</u> Let J(x, t) be the Jacobian matrix

$$J(x, t) = \left[\frac{\partial f_i(x, t)}{\partial x_j}\right] \qquad (2.16)$$

of (2. 15) and P be a positive definite symmetric constant matrix. Then the null solution x = 0 of (2. 15) is asymptotically stable-in-the-large if

$$J'(x, t) P + P J(x, t) = -Q(x, t) \qquad (2.17)$$

and the eigenvalues $\lambda_i(x, t)$, $(i = 1, \ldots, n)$, of Q(x, t) are greater than some constant $\gamma > 0$ for all x and t.

The proof of this theorem in the autonomous case (where f is independent of t) is easily completed with the Lyapunov function $v(x) = f' P f$ [Ref. 12]. In the general case, a more involved argument using the Lyapunov function $v(x) = x'P x$ is required [Ref. 13].

Another variation on the linear, constant-coefficient procedure for nonlinear autonomous equations is suggested by Ingwerson [Ref. 13] who chooses a positive definite symmetric matrix C and solves the equation

$$J'(x)\,P(x) + P(x)\,J(x) = -C \tag{2.18}$$

for P(x). This general P(x) is then modified by dropping the dependence on all but the variables x_i and x_j in the ij^{th} term. Then p_{ij}, the ij^{th} term of P(x), is a function only of x_i and x_j. The components of the gradient vector are then computed from this new matrix, $P(x_i, x_j)$, as

$$\frac{\partial v}{\partial x_i} = \int_0^{x_1} P_{i1}(x_i, x_1)\,dx_1 + \int_0^{x_2} P_{i2}(x_i, x_2)\,dx_2 + \ldots + \int_0^{x_n} P_{in}(x_i, x_n)\,dx_n \tag{2.19}$$

and v(x) is computed from ∇v as a standard line integral.

Schultz and Gibson [Ref. 38] have proposed a related procedure they call the Variable Gradient Method. This involves choosing ∇v directly in a special form that ensures that it is the gradient of some scalar v(x). The constants in ∇v can then be adjusted to make $\dot{v}(x) = (\nabla v, f(x))$ negative definite. Stability information is obtained from a study of v(x) which is again obtained by a line integration on ∇v.

It is clear that all of these procedures involve formidable difficulties when applied to high order systems. Just the solution of the algebraic problems in (2.17) and (2.18) becomes extremely difficult since numerical techniques cannot be used. While Schultz and Gibson avoid this problem, they have another problem in checking that the ∇v chosen is a proper gradient of some scalar. Even if this were not a problem, the interpretation and determination of positive definite properties in the resulting Lyapunov functions becomes very difficult for high order systems.

A different approach is taken by Zubov [Ref. 47]. For the autonomous system $\dot{x} = f(x)$ with $f(0) = 0$, he obtains a Lyapunov function by solving the partial differential equation

$$\sum_{i=1}^{n} \frac{\partial v(x)}{\partial x_i} f_i(x) - \phi(x)[1 + v(x)]\,[1 + \|f(x)\|^2]^{\frac{1}{2}} = 0 . \qquad (2.20)$$

The function $\phi(x)$ must be positive definite on E^n, but is otherwise arbitrary. In addition, the solution must be such that $-1 < v(x) < 0$ for all $x \neq 0$ (Zubov works with negative definite Lyapunov functions). This approach at least suggests a systematic procedure for determining $v(x)$, but, as yet, no one has been able to utilize this fact to significant advantage in higher order systems [Refs. 29, 42].

An important step in the direction of simplifying the use of LSM has recently been developed by Sell [Ref. 39]. He notes that in certain problems it is possible to show that particular types of stability exist if and only if every Lyapunov function in a given class has a corresponding property. This eliminates the problem of choosing a Lyapunov function and replaces it with the (hopefully easier) problem of analyzing the properties of some standard Lyapunov function along the solutions of the equation through a study of $\dot{v}$.

Unfortunately, Sell's work deals mainly with local stability problems. It remains to be seen whether similar procedures can be applied to generalized stability concepts such as ASL.

Problems specifically related to automatic control have also received a considerable amount of attention. The best known of these are probably the problems of Lur'e [Ref. 25]. Lur'e's problem of direct control is concerned with a system modeled by

$$\begin{cases} \dot{x} = Ax + pf(\sigma) \\ \sigma = b'x \end{cases} \tag{2.21}$$

where A is a constant matrix, p and b are constant vectors, x is the system state vector and $f(\sigma)$ is a scalar function. In addition, it is assumed that

$$f(0) = 0 \text{ and } \sigma f(\sigma) > 0 \text{ for all } \sigma . \tag{2.22}$$

This may be visualized as a single-input, single-output linear plant with a single nonlinear controller as shown in Fig. 4.

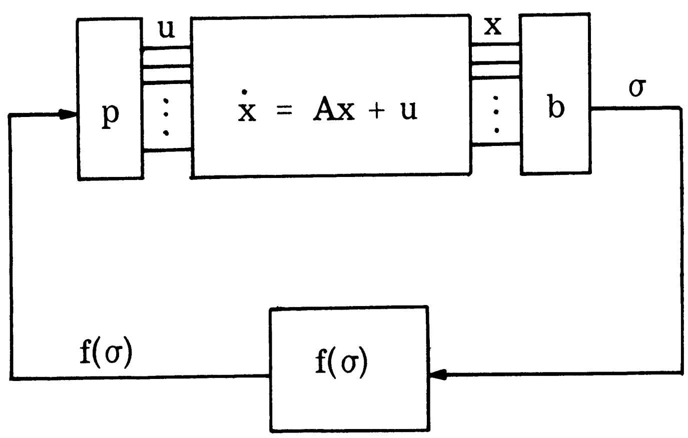


Fig. 4. Lur'e's problem of direct control.

Lur'e considers the so-called absolute stability problem of determining the element of b (with A and p given) so that the null solution of (2.21) is ASL for all continuous f such that (2.22) are satisfied.

This problem has received a great deal of attention from both Soviet and western authors [see discussion in Ref. 12]. A common approach is to use LSM with the Lyapunov function

$$v(x) = x' P x + \beta \int_{0}^{\sigma} f(\theta)\, d\theta \qquad (2.23)$$

where P is a constant, positive definite, symmetric matrix and β is a constant. The major problem in this approach is the determination of definiteness criteria for the scalar functions involved. For higher order systems, say of order greater than sixth, this becomes very difficult [Refs. 12, 26]. The method can, in theory, be extended in several ways such as the inclusion of several nonlinear elements, but computational problems are then even more difficult [Ref. 12]. For the problem with only one nonlinear element, a new approach utilizing the frequency domain behavior of the linear part (see Fig. 4), may simplify some of the computational difficulties [Ref. 36].

In another control problem, Aizerman [Ref. 1] considers a system modeled by

$$\mathring{x} = A x + f(x) \qquad (2.24)$$

where A is a constant matrix

$$f(x) = \begin{bmatrix} f_1(x_j) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.25}$$

$f(0) = 0$ and x_j is one of the components of x. It is assumed that there is a range of some parameter a, say $\alpha < a < \beta$, such that the linear system obtained from (2. 24) by replacing $f_1(x_j)$ by ax_j is asymptotically stable. In his famous conjecture [Ref. 2] Aizerman suggested that the nonlinear system (2. 24) might be ASL for an $f_1(x_j)$ such that

$$\alpha x_j^2 < x_j f_1(x_j) < \beta x_j^2 \quad \text{for } x \neq 0 . \tag{2.26}$$

It has since been shown that this is not generally true even for third-order systems [Ref. 26]. However, it is still quite important to ascertain <u>when</u> Aizerman's Conjecture does hold. This problem and its generalizations have attracted a great deal of attention in the control systems literature [Refs. 12, 18, 33] (see also a similar conjecture by Kalman, Ref. 15, and the discussion in Ref. 16), but complete results are currently available only for second-order systems [Ref. 33]. A few special third- and fourth- order cases have also been considered (see summary in Ref. 12). (In Section 6. 3, the separation concepts developed in this thesis are used to verify a generalization of Aizerman's Conjecture in an n^{th}-order problem with n nonlinear elements.)

In the recent mathematics literature a new viewpoint related to LSM has developed using the concept of the Auxiliary Equation [Refs. 5, 9, 10, 40]. In using the auxiliary equation to study the n^{th}-order equation (2.15), one seeks a positive definite function $v(x, t)$ and a scalar function $\omega(v, t)$ with $\omega(0, t) = 0$ such that along the solutions of (2.15)

$$\dot{v}(x, t) \leq \omega(v(x, t), t) \quad . \tag{2.27}$$

It can then be shown that, with certain reasonable restrictions, the stability properties of the first-order auxiliary equation

$$\dot{r} = \omega(r, t) \tag{2.28}$$

correspond to the stability properties of the original system. The importance of this approach can be appreciated when it is realized that this reduces the problem of stability of the equilibrium of the n^{th}-order system to the problem of stability of the equilibrium of a first-order system. However, once again, there are still formidable problems involved in selecting a v and an $\omega(v, t)$ such that the relation $\dot{v} \leq \omega(v, t)$ can be realized and as might be suspected, the difficulty of these problems also increases with the order of the original system. Both Rosen [Ref. 37] and Sell [Ref. 40] have attempted to solve this problem by using the standard Lyapunov function $v(x) = \|x\|$ (this function does not have continuous derivatives so a modification of some of the original theorems is necessary). This allows Rosen to reduce certain stability problems to nonlinear programming problems.

In this thesis auxiliary equations are used, but the separation concepts reduce the problem of finding one suitable auxiliary equation for the entire system to the simpler problem of finding several auxiliary equations for the "pieces" of the system; i. e., a vector auxiliary equation. Thus, the choice of a single Lyapunov function is circumvented by reducing the problem of stability of the n^{th}-order equation to the problem of stability of a system of several simpler first-order equations The form of this vector auxiliary equation depends on the topological interconnection features of the overall system. Apparently such an approach has not been considered in the literature to date. Thus, there are no previous developments of this problem to be reviewed.

2.5 Detailed Summary of Results

This section gives a detailed summary of the results to follow. It is intended first to provide an overall picture of the research for those who are not interested in the finer details and second, to help the prospective reader of the finer details obtain the necessary perspective. References will be omitted throughout this section since they are given in the detailed discussion which follows in Chapter 3.

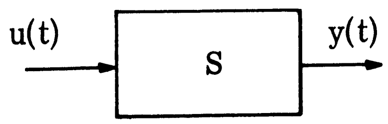
An important concept in the development is that of the composite system. A complex system obtained by interconnecting a set of simpler subsystems which will be called transfer systems because they will normally be input-output devices. The composite system may be, say, a servo system and the transfer systems will be its parts: the motors, amplifiers and transducers <u>A basic assumption is that all transfer systems to be studied can be modeled by ordinary differential equations.</u> A general transfer system model is

$$\begin{cases} \dot{x} = f(x, t, u(t)) \\ y(t) = h(x(t), t) \end{cases} \qquad (2.29)$$

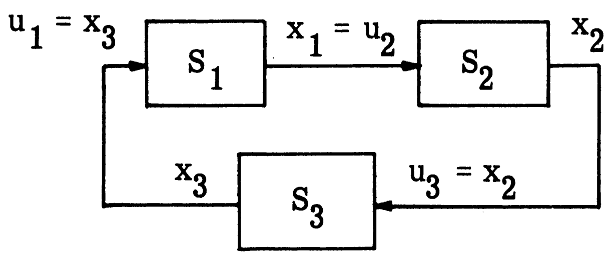
where x is the transfer system state vector, u(t) is the vector (or scalar) input and y(t) is the vector (or scalar) output. In a composite system the transfer systems are interconnected so that the outputs of some are the inputs of others. Thus the input, $u_i(t)$, for the i^{th} transfer system in some composite system is a sum of the other outputs, say

$$u_i(t) = \sum_j B_{ij} y_j(t) \;, \qquad (2.30)$$

where the B_{ij}'s are constant matrices. An example is shown in the block diagrams of Fig. 5.



(a) A transfer system.



(b) A simple composite system made up of three transfer systems, S_1, S_2, and S_3.

Fig. 5. Illustration of the concept of a composite system.

Note that this type of interconnection (as given in (2.30)) implies the usual system theory assumption that the transfer system models are not affected by the interconnection structure. That is, there is no "loading" of one transfer system by another.

With transfer systems clearly defined, it is possible to go on to study some of their properties using LSM. Of particular interest is the case where a transfer system has a region in its state space that is ASL. Here, the definitions of Lyapunov stability, asymptotic stability, and asymptotic stability-in-the-large have been generalized to include stability of regions. A region M, which is a subset of E^n, is said to be stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that if the distance from $x_o = x(t_o)$ to M is less that δ, then the distance from $x(t;x_o, t_o)$ to M will be less than ϵ for all $t \geq t_o$. Asymptotic stability of M requires that M be stable and also that for all x_o less than some distance $\delta_o > 0$ from M, the distance $x(t;x_o, t_o)$ from M goes to zero as $t \to \infty$. For asymptotic stability-in-the-large, this must occur for all δ_o taken arbitrarily large. These are basically generalizations of the usual definitions of the stability of the point $x = 0$. If the set M is put equal to $\{0\}$, the usual definitions result.

With the aid of these new stability definitions it is possible to define what will be called a "gain" for transfer systems. However, it is important to realize that this new gain is basically different from the gain considered in the frequency analysis of single-input, single-output transfer systems modeled by linear differential equations with constant coefficients. For such systems, this new gain is similar to the maximum magnitude of the transfer function $H(j\omega)$. For example, consider the case of a system modeled by the first-order (one pole) linear equation

$$\begin{cases} \mathring{x} = -ax + bu(t) \\ y = x \end{cases}$$

where x is a scalar. The transfer function is

$$H(s) = \frac{b}{s+a}$$

and the maximum magnitude of the gain is the d-c gain $\frac{b}{a}$. In this particular case, the gain obtained by estimating the size of stable regions in the state space is also $\frac{b}{a}$ (see Example 5. 1). In higher order linear constant-coefficient systems the gain obtained from these state space techniques will be greater than or equal to the maximum magnitude of $H(j\omega)$. For other systems (multiple-input, multiple-output, nonlinear, time-varying, etc.), this gain can best be visualized by considering the ratio A/B where

A = the size of the region (the norm of the largest vector in this region) in state space to which all of the solution trajectories converge,

B = the size (norm) of the input.

The gain is an upper bound on this ratio that is valid for all continuous and bounded inputs. For a transfer system to have a gain of this type, it must have regions in its state space that are ASL when the input has a certain size (norm) and the size of these regions should depend on the size of the input. Transfer systems having these properties are said to be in Class G and the gain of a transfer system in Class G can be estimated using LSM and the concepts of regions that are ASL.

A particular important subclass of Class G is those transfer systems with models of the form

$$\begin{cases} \dot{x} = f(x,t) + D\,u(t) & (2.31) \\ y = Hx & (2.32) \end{cases}$$

where H and D are constant matrices and the null solution of the unforced model

$$\dot{x} = f(x,t) \qquad f(0,t) = 0 \tag{2.33}$$

is exponentially stable-in-the-large (abbreviated ESL). This means that there are positive constants k_1 and k_2 such that for any $x_0 \in E^n$ the solutions $x(t;x_0,t_0)$ of (2.33) satisfy the inequality

$$\|x(t;x_0,t_0)\| \leq k_1 \|x_0\| \exp\{-k_2(t-t_0)\} \ . \tag{2.34}$$

This subclass of Class G is denoted Class E. Class E is particularly appealing for several reasons. First, Class E includes systems modeled by (2.31) and (2.32) when:

(a) $f(x,t) = Ax$ where A is a stable constant matrix (all eigenvalues of A have negative real parts),

(b) $f(x,t) = A(t)x$ for a large class of variable matrices $A(t)$,

(c) $f(x,t)$ is a member of an important class of nonlinear functions.

This last statement in case (c) is purposely vague since a precise characterization of the class of nonlinear functions for which (2. 33) is ESL is not available. However, it is easily shown (see Section 5. 2) that all $f(x, t)$ where $f(0, t) = 0$ and

$$x'f(x, t) \leq -c\|x\|^2 < 0 \quad \text{for } x \neq 0$$

lead to ESL in (2. 33). The essence of this restriction is best seen in the first-order, time-invariant case where

$$x f(x) \leq -c x^2 < 0 \quad \text{for } x \neq 0 .$$

Graphically, this means that $f(x)$ lies in the shaded sectors of Fig. 6. In higher dimensional situations the nonlinear characteristic is restricted in a similar fashion. While such characteristics are not completely general, they do represent a class of considerable importance. One obvious requirement is that $f(x, t)$ have a nonzero slope at the origin. This type of nonlinearity is commonly encountered in applications where linear behavior is observed in a small neighborhood of some equilibrium but nonlinear effects become important when larger displacements are encountered.

A second appealing feature of Class E is the fact that for any transfer system in Class E there is a Lyapunov function $v(x, t)$ such that

$$c_1\|x\|^2 \leq v(x, t) \leq c_2\|x\|^2 , \tag{2.35}$$

and its total derivative with respect to (2. 31) is such that

$$\dot{v} \leq -\alpha v + \gamma\|u\|^2 , \tag{2.36}$$

where α, γ, c_1, and c_2 are all positive constants.

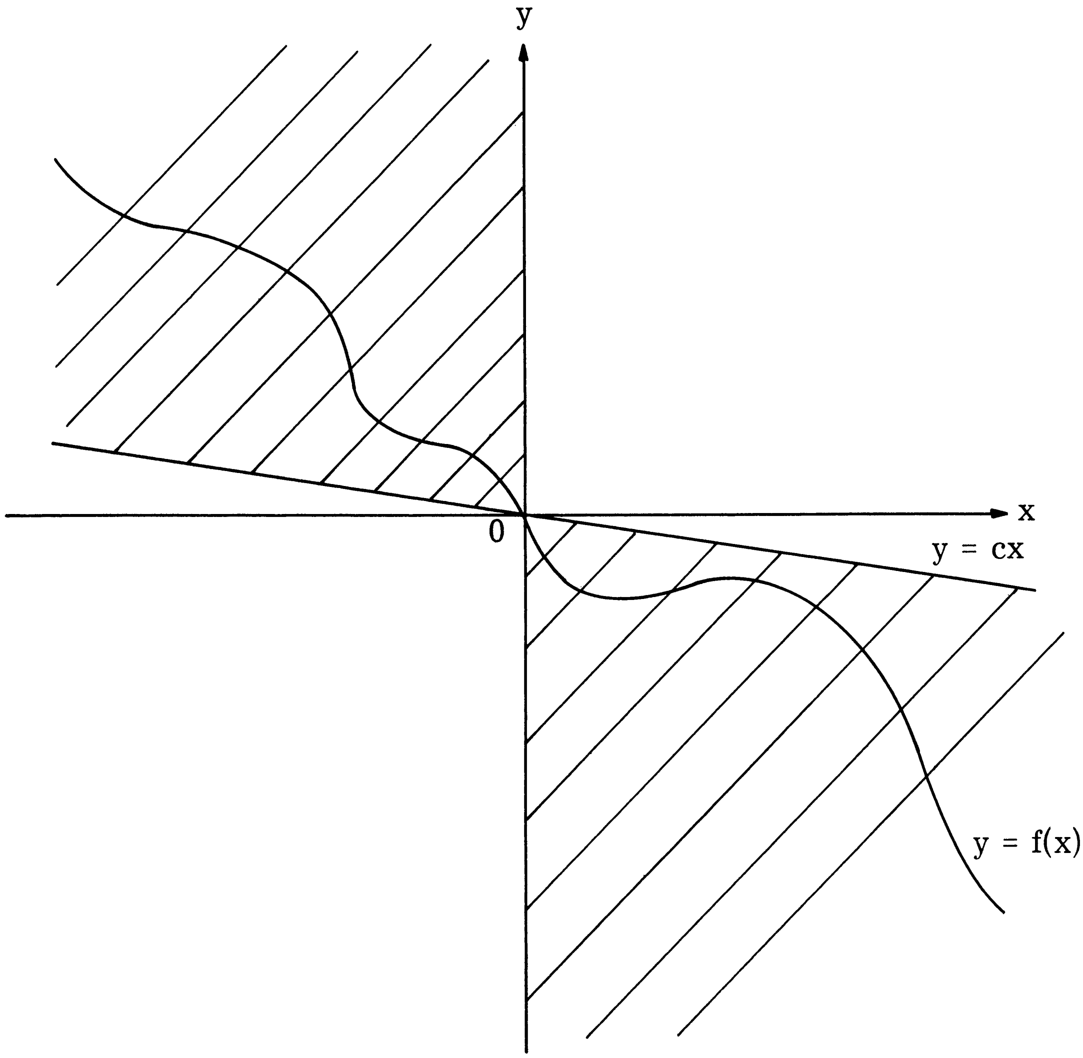


Fig. 6. A nonlinear characteristic leading to exponential stability-in-the-large.

This second feature is very important because it opens the door to the interconnection of the auxiliary equations [(2. 36) with inequality replaced by equality] for the transfer systems in the same way that the transfer systems are interconnected in the composite system. To illustrate this interconnection, consider the simple composite system shown in Fig. 5(b). Assume that each of the three transfer systems is in Class E and has a Lyapunov function

$$c_{i1}\|x_i\|^2 \le v_i(x_i, t) \le c_{i2}\|x_i\|^2 , \quad i = 1, 2, 3 \tag{2. 37}$$

whose total derivative is

$$\dot{v}_i \le -\alpha v_i + \gamma_i\|u_i\|^2 \quad . \quad i = 1, 2, 3 \tag{2. 38}$$

When the transfer systems are interconnected

$$\|u_1\|^2 = \|x_3\|^2 \le \frac{1}{c_{31}} v_3 ,$$

$$\|u_2\|^2 = \|x_1\|^2 \le \frac{1}{c_{11}} v_1 ,$$

$$\|u_3\|^2 = \|x_2\|^2 \le \frac{1}{c_{21}} v_2 . \tag{2. 39}$$

Using (2. 39) in (2. 38) with $\beta_1 = \frac{1}{c_{31}}\gamma_1$, $\beta_2 = \frac{1}{c_{11}}\gamma_2$, and $\beta_3 = \frac{1}{c_{21}}\gamma_3$, there results a system of differential inequalities,

$$\begin{cases} \dot{v}_1 \le -\alpha_1 v_1 + \beta_1 v_3 \\ \dot{v}_2 \le -\alpha_2 v_2 + \beta_2 v_1 \\ \dot{v}_3 \le -\alpha_3 v_3 + \beta_3 v_2 . \end{cases} \tag{2. 40}$$

When the inequalities in (2. 40) are replaced by equalities and v_i's are replaced by r_i's, there results the vector auxiliary system

$$\begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{bmatrix} = \begin{bmatrix} -\alpha_1 & 0 & \beta_1 \\ \beta_2 & -\alpha_2 & 0 \\ 0 & \beta_3 & -\alpha_3 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{2. 41}$$

or, in vector notation

$$\dot{r} = Ar \tag{2. 42}$$

where A is obviously defined from (2. 41). This is a generalization of the auxiliary equation described in Section 2. 4. Instead of reducing the original system model (the composite system model of order $n_1+n_2+n_3$ where n_i is the order of the i^{th} transfer system) to a first-order auxiliary equation for stability analysis, this procedure reduces it to a third-order system of auxiliary equations (in general, the auxiliary system will have an order equal to the number of transfer systems in the composite system). It is now easy to show that under certain reasonable restrictions the stability properties of the composite system, a high order nonlinear system, are the same as the stability properties of the third order linear system (2. 41). Moreover, it has been necessary to find Lyapunov functions only for the lower order transfer systems. The construction of a single Lyapunov function for the high order composite system as required in all previous applications of LSM has been avoided!

This procedure is, of course, generalized to a treatment of a composite system made up of an arbitrary number of transfer systems with arbitrary interconnections. The only restriction is that each composite system must be in Class E. Several examples illustrating this procedure are discussed near the end of Chapter 6.

CHAPTER 3

PRELIMINARY CONCEPTS

With the material in Chapter 2 to provide a background, an analysis of the basic problem can begin. Section 3. 1 deals with notation while Section 3. 2 introduces concepts pertaining to composite systems and their internal structure.

3. 1 Notation

The vector notation used is similar to that employed by Hahn [Ref. 12] or Cesari [Ref. 6]. (For an introduction to vector notation used with ordinary differential equations, see Coddington and Levinson, Ref. 8.) Let E^n denote the n-dimensional Euclidean space of n vectors, $x = \mathrm{col}(x_1, x_2, \ldots, x_n)$, where the x_i's $(i = 1, \ldots, n)$ are real numbers or real valued functions on the interval $T = [0, \infty)$ of the real line. The transpose of x is denoted by x' and for all x and y in E^n the inner product is defined as $(x, y) = x'y = \sum_{i=1}^{n} x_i y_i$. The norm of a vector in E^n is the Euclidean norm $\|x\| = (x, x)^{\frac{1}{2}}$ and if P is an m x n matrix of real elements, then $\|P\| = \min\{\alpha \mid \alpha\|x\| \geq \|Px\| \text{ for all } x \in E^n\}$. A useful metric on E^n is $d(x, y) = \|x-y\|$ and when limits and continuity are mentioned the implied topology is taken with respect to this metric. For any subset A of E^n the distance from x to A is $d(x, A) = \inf_{y \in A} d(x, y)$ and for any $\epsilon > 0$, $S_\epsilon(A) = \{x \mid d(x, A) < \epsilon\}$. A subset A of E^n is said to be bounded if there is a finite $\epsilon > 0$ such that $A \subseteq S_\epsilon(0)$.

The notation $A \subseteq B$ means A is a subset of B, while $A \subset B$ means A is a proper subset of B. If $R \subseteq E^n$, then R^c is the complement of R in E^n, $\overline{R}$ is the closure of R and B(R) is the boundary of R. If $M \subset E^n$ and $0 \in M$, then $s[M] = \sup_{y \in M} \|y\|$ is the "size" of M. If M_1 and M_2 are two subsets of E^n, containing the zero vector, then M_1 is smaller than M_2 if $s[M_1] < s[M_2]$ and the cartesian product $M_1 \times M_2$ is defined in the usual manner as $M_1 \times M_2 = \{ (m_1, m_2) \mid m_1 \in M_1 \text{ and } m_2 \in M_2 \}$. If f(x) is defined on $R \subseteq E^n$ and $B \subset R$, then $f(B) \triangleq \{ f(x) \mid x \in B \}$. If v(x) is a positive definite scalar function (defined in Section 4. 2), then $R_h = \{ x \mid v(x) < h \}$. For any clearly defined $t_i \in T$ the set $[t_i, \infty)$ will be denoted by T_i [i. e., if $t_o \in T$, then T_o is the set $[t_o, \infty)$].

The differential equation $\dot{x} = f(x, t)$ is normally an n^{th}-order vector differential equation with x(t) and f(x, t) denoting n-dimensional vector valued functions defined on T and $E^n \times T$, respectively. A solution to this differential equation is a function $x(t;x_o, t_o)$ such that $t_o \in T$, $x(t_o;x_o, t_o) = x_o$, and $\frac{d}{dt}[x(t;x_o, t_o)] = f(x(t;x_o, t_o), t)$ for all $t \in T_o$. It is generally assumed that all differential equations satisfy conditions sufficient to guarantee the existence, uniqueness, and continuity of all solutions in t, x_o, and t_o (continuity from the inside is implied at the boundary of any closed region).[6]

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[6] A variety of such sufficient conditions are available in the literature [Refs. 6, 40], but no specific conditions are assumed here. Necessary and sufficient conditions are not available.

Finally, let C be the normed linear space of continuous, bounded n-dimensional vector functions on T with norm $\|x(t)\|_c = \sup_{t \in T} \|x(t)\|$. Similarily, let $C_o \subset C$ such that $x \in C_o$ implies $\lim_{t \to \infty} x(t) = 0$.

3.2 Composite Systems, Transfer Systems, and Models

As mentioned in the introduction, a composite system is an interconnection of simpler subsystems. The basic building blocks of the composite systems considered in this report are called transfer systems.

Def. 3.1: A transfer system is any input-output device whose terminal variables may be characterized by relations of the form

$$\begin{cases} \dot{x} = f(x, t, u(t)) & (3.1) \\ y(t) = h(x(t), t) & (3.2) \end{cases}$$

where x(t) is an n-dimensional state vector, u(t) is a p-dimensional input vector and y(t) is a q-dimensional output vector.

Def. 3.2: The terminal relations (3.1) and (3.2) characterizing the transfer system are called the transfer system model.

Example 3.1: Consider the electric network shown in Fig. 7.

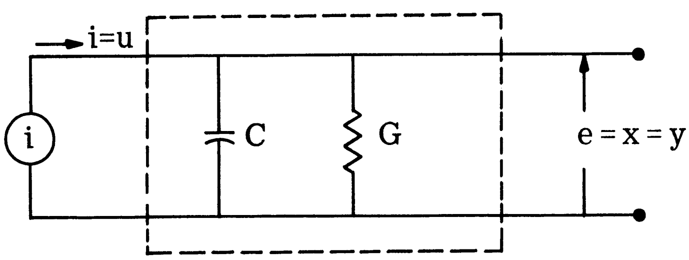

Fig. 7. Network for Example 3.1.

The dotted box contains a transfer system. When the input u(t) is a current u(t) = i(t), the state x(t) is the voltage across the capacitor x(t) = e(t), and the output v(t) is this same open circuit voltage y(t) = e(t), then the transfer system model is

$$\begin{cases} \dot{x} = -\frac{G}{C}x + \frac{1}{C}u(t) , \\ y(t) = x(t) . \end{cases}$$

A composite system can now be defined as an interconnection of transfer systems.

Def. 3.3: Consider a set of m transfer systems, S_i, $i = 1, \ldots, m$. A composite system is an interconnection of these transfer systems so that for the i^{th} transfer system the (vector) input u_i is given as

$$u_i = \sum_{j=1}^{m} B_{ij} y_j + G_i u \quad i = 1, \ldots, m$$

where y_j is the (vector output of the j^{th} transfer system, u is an external (vector) input to the composite system and B_{ij}, G_i are constant matrices. (Note that only linear interconnections are allowed.)

The partitioned matrix

$$B = \left[\begin{array}{c|c|c|c} B_{11} & B_{12} & \cdots & B_{1m} \\ \hline B_{21} & B_{22} & \cdots & B_{2m} \\ \hline \cdot & \cdot & \cdots & \cdot \\ \hline B_{m1} & B_{m2} & \cdots & B_{mm} \end{array}\right] \qquad (3.3)$$

where the submatrices B_{ij}, $(i, j = 1, \ldots, m)$ are the same as those used in Def. 3.3, will be termed the composite system interconnection matrix or simply the interconnection matrix since it indicates the type of interconnections present in the composite system.

If the individual transfer systems are n_i^{th}-order systems modeled by

$$\begin{cases} \dot{x}_i = f_i(x_i, t, u_i(t)) , \\ y_i = h_i(x_i, t) , \end{cases}$$

with state vectors $x_i = \text{col}[x_{i1}, \ldots, x_{in}]$ for $i = 1, \ldots, m$, then the composite system will be an $n = \sum_{i=1}^{m} n_i^{th}$-order system with state vector $x = \text{col}[x_{11}, \ldots, x_{in_1}, x_{21}, \ldots, x_{2n_2}, \ldots, x_{m1}, \ldots, x_{mn_m}]$ and the composite system model will be

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_m \end{bmatrix} = \begin{bmatrix} f_1(x_1, t, \sum_{j=1}^{m} B_{1j} h_j(x_j, t) + G_1 u) \\ f_2(x_2, t, \sum_{j=1}^{m} B_{2j} h_j(x_j, t) + G_2 u) \\ \vdots \\ f_m(x_m, t, \sum_{j=1}^{m} B_{mj} h_j(x_j, t) + G_m u) \end{bmatrix} = f(x, t, u) , \\ y(t) = h(x, t) , \end{cases} \qquad (3.4)$$

where y(t) is the output of the composite system. It should be pointed out that this sort of interconnection implies the usual system theory assumption [Ref. 44] that the <u>individual transfer system models are not affected by the various types of interconnections; that is, there is no</u>

"loading" effect of one system on another. This in itself greatly simplifies the mechanisms through which instability of the composite system can occur. The external input u is included to emphasize the fact that the composite system itself might be a transfer system in a larger composite.

If the individual tranfer system models have the form

$$\begin{cases} \dot{x}_i = f_i(x_i, t) + D_i u_i & (3.5) \\ y_i = H_i x_i & (3.6) \end{cases}$$

for $i = 1, \ldots, m$, where D_i and H_i are matrices and the interconnections are such that

$$u_i = \sum_j B_{ij} y_j + G_i u$$

then since $y_j = H_j x_j$,

$$D_i u_i = \sum_j D_i B_{ij} H_j x_j + D_i G_i u ,$$

and the composite system model takes on the particularly simple form

$$\begin{cases} \dot{x}_1 = f_1(x_1, t) + C_{11}x_1 + C_{12}x_2 + C_{13}x_3 + \ldots + C_{1m}x_m + K_1 u \\ \dot{x}_2 = f_2(x_2, t) + C_{21}x_1 + C_{22}x_2 + C_{23}x_3 + \ldots + C_{2m}x_m + K_2 u \\ \vdots \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dot{x}_m = f_m(x_m, t) + C_{m1}x_1 + C_{m2}x_2 + C_{m3}x_3 + \ldots + C_{mm}x_m + K_m u \\ y = h(x, t) , \end{cases} \qquad (3.7)$$

where $C_{ij} = D_i B_{ij} H_j$ and $K_i = D_i G_i$. Equation (3.7) can be further refined to the form

$$\begin{cases} \dot{x} = f(x,t) + Cx + Ku & (3.8) \\ y = h(x,t) & (3.9) \end{cases}$$

where x is the composite system state vector, f is a column vector of the f_i's and C is the partitioned matrix

$$C = \left[\begin{array}{c|c|c|c|c} C_{11} & C_{12} & C_{13} & \cdots & C_{1m} \\ \hline C_{21} & C_{22} & C_{23} & \cdots & C_{2m} \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline C_{m1} & C_{m2} & C_{m3} & \cdots & C_{mm} \end{array}\right] \tag{3.10}$$

and K is the partitioned matrix

$$K = \left[\begin{array}{c} K_1 \\ \hline K_2 \\ \hline \vdots \\ \hline K_m \end{array}\right] . \tag{3.11}$$

Since $C_{ij} = D_i B_{ij} H_j$, the C matrix defined in (3.10) will serve the same purpose as the B matrix of (3.3) in indicating the type of interconnections present in the composite system.

CHAPTER 4
STABILITY DEFINITIONS AND THEOREMS

Precise definitions of the various types of stability to be used throughout this thesis will now be given. Sufficient conditions for the existence of these various types of stability are then obtained using LSM. The stability definitions are given in terms of regions in the state space. This generalization of the usual definitions will be useful in Chapter 5.

4.1 Stability and Boundedness of Solutions of Differential Equations

The stability and boundedness definitions to be given refer to solutions of the ordinary differential equations which are models for the systems (transfer systems and composite systems) of interest in this report.[7] The definitions given below are a generalization of the standard stability definitions [Ref. 12] which characterize the behavior of solutions of the differential equation in the neighborhood of the null solution $x = 0$. (The behavior in the vicinity of any fixed solution can be reduced to this problem by a change of variables. See Section 2.3). The generalization is the characterization of solution behavior in the neighborhood of a fixed set M or set of solutions rather than a fixed equilibrium point $x = 0$ or a fixed solution. This generalization of the

[7] No attempt is made to give a complete list of commonly used stability definitions and the omission of specific definitions does not imply that the present results cannot be extended to these forms. Those chosen are representative of the stability concepts found valuable in a majority of applications.

stability definitions (and the associated stability theorems) leads to the introduction of several new concepts in Section 5 and appears to have possibilities of further applications not considered in this thesis. When $M = \{0\}$, these definitions reduce to the familiar forms for the behavior in the neighborhood of (stability of) the equilibrium solution, $x = 0$ [Ref. 12].

Consider the solutions of the vector differential equation

$$\dot{x} = f(x, t) \qquad x(t_o) = x_o \tag{4.1}$$

where $f(0, T) \equiv 0$. This equation can be considered as a model for either an unforced system ($u(t) = 0$) or a forced system with a fixed $u(t)$ that is included in $f(x, t)$.

Def. 4.1: The solution $x(t;x_o, t_o)$ of (4.1) is bounded on a bounded set M if $x(T_o;x_o, t_o) \subseteq M$.

Def. 4.2: A bounded set M in the state space of (4.1) is stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $x(T_o;S_\delta(M), t_o) \subset S_\epsilon(M)$.[8]

The following theorem shows that Def. 4.2 implies Def. 4.1.

Theorem 4.1: If a bounded set M in the state space of (4.1) is stable, then, the solutions of (4.1), starting on $\overline{M}$, are bounded on $\overline{M}$.

[8] Recall that $x(T_o;S_\delta(M), t_o)$ is the set of all points $x(t;x_o, t_o)$ for $(t, x_o) \in T_o \times S_\delta(M)$ or, in other words, the set of all solution paths $x(t;x_o, t_o)$ for $t \geq t_o$ obtained with $x_o \in S_\delta(M)$.

Proof: For any $(x_o, t_o) \epsilon \overline{M} x T$, let $d_M(x_o, t_o) = \sup_{t \epsilon T_o} [d(x(t; x_o, t_o), M)]$. If $d_M(\overline{M}, T) \equiv 0$, the Lemma is proved. If $d_M(x'_o, t'_o) \not\equiv 0$ for some $(x'_o, t'_o) \epsilon \overline{M} x T$, then let $\epsilon = \frac{1}{2} d_M(x'_o, t'_o)$. From the stability of M there is then a $\delta > 0$ such that $x_o \epsilon S_\delta(M)$ implies $d_M(x_o, t_o) < \epsilon$. But $x'_o \epsilon \overline{M}$ implies $x'_o \epsilon S_\delta(M)$ and thus $d_M(x'_o, t'_o) < \epsilon = \frac{1}{2} d_M(x'_o, t'_o)$. Since $d_M \geq 0$, this implies that $d_M(x_o, t_o) \equiv 0$ on $\overline{M} x T$. Thus, the solutions starting on $\overline{M}$ are bounded on $\overline{M}$.

Def. 4.3: A solution $x(t; x_o, t_o)$ of (4.1) approaches a set M if $\lim_{t \to \infty} d(x(t; x_o, t_o), M) = 0$.

If all the solutions starting in some $S_\delta(M)$ approach M, then the set M might be called quasi-asymptotically stable. This property in itself does not insure that the solutions are uniformly bounded.

Def. 4.4: A bounded set M in the state space of (4.1) is asymptotically stable (AS) if it is stable and if there is some $\gamma > 0$ such that every solution $x(t; x_o, t_o)$ of (4.1) with $x_o \epsilon S_\gamma(M)$ approaches M; that is $\lim_{t \to \infty} d(x(t; x_o, t_o), M) = 0$ for every $x_o \epsilon S_\gamma(M)$.

The above definitions consider behavior in an arbitrarily small neighborhood of M. For applications this is frequently unsatisfactory and neighborhoods of reasonable size must be considered. One approach to this problem is to consider stability-in-the-large.

Def. 4.5: A bounded set M in the state space of (4.1) is asymptotically stable-in-the-large (ASL) if it is stable and for every $(x_o, t_o) \in E^n \times T$, $\lim_{t \to \infty} d(x(t;x_o, t_o), M) = 0$.

A stronger form of ASL is ultimate boundedness [Ref. 45].

Def. 4.6: The solutions of (4.1) are ultimately bounded (UB) on a bounded set M if for every $(x_o, t_o) \in E^n \times T$ there is a $\tau > t_o$ such that $t > \tau$ implies that $x(t;x_o, t_o) \in M$.

If $M = \{0\}$, then the above definitions (with the exception of Def. 4.6) reduce to the usual definitions [Refs. 12, 16 or Section 2 above] for stability of the equilibrium solution, $x = 0$.

Def. 4.7: The equilibrium solution $x = 0$ of (4.1) is said to be stable, asymptotically stable, or asymptotically stable-in-the-large if the set $M = \{0\}$ is stable, asymptotically stable, or asymptotically stable-in-the-large, respectively.

Two further implications of ASL are shown by the following theorems.

Theorem 4.2: If a set M in the state space of (4.1) is ASL, then the solutions of (4.1) are bounded for $t \geq t_o$; that is, there is a constant $b(x_o, t_o)$ such that $\|x(t;x_o, t_o)\| \leq b(x_o, t_o)$ for all $t \geq t_o$.

Proof: Given any $\epsilon > 0$ there is a $\tau > t_o$ such that $t > \tau$ implies $d(x(t;x_o, t_o), M) < \epsilon$. Thus, for all $t > \tau$, the solutions are bounded in $S_\epsilon(M)$. Then choose any fixed $\tau_1 > \tau$. On the closed interval $[t_o, \tau_1]$, $x(t;x_o, t_o)$ must be bounded because it is a continuous function of t (continuous on the left at t_o). Thus, the solutions are bounded for all $t \geq t_o$.

Theorem 4.3: Let M be a bounded set in the state space of (4.1) containing the origin. Then, for any $(x_o, t_o) \in E^n \times T$,

$$\|x(t;x_o, t_o)\| \leq d(x(t;x_o, t_o), M) + s[M]$$

and, in addition, if M is ASL

(a) $\lim_{t\to\infty} d(x(t;x_o, t_o), M) = 0$ for all $(x_o, t_o) \in E^n \times T$.

(b) $d(x(t;x_o, t_o), M) = 0$ for all $(t, x_o, t_o) \in T_o \times 0 \times T$.

(c) $d(x(t;x_o, t_o), M)$ is continuous in t, x_o, and t_o at all points in $T_o \times E^n \times T$.

Proof: The bound on the norm of $x(t;x_o, t_o)$ is a restatement of the triangle inequality while property (a) follows immediately from the definition of ASL. Property (b) holds because $x = 0$ is an equilibrium point that is contained in M. Property (c) follows from the continuity of $d(x, M)$ in x and the continuity of $x(t;x_o, t_o)$ in t, x_o, and t_o.

4.2 Sufficient Conditions for Stability and Boundedness of Solutions of Differential Equations

In this section, sufficient conditions for the several types of stability and boundedness defined in Section 4.1 will be obtained through the application of LSM. Two different approaches will be employed to obtain two sets of sufficient conditions for each of the definitions in Section 4.1.

The first approach is essentially a straightforward application of the original procedures attributed to Lyapunov [Refs. 12, 27] where sign definite (or semi-definite) functions (Lyapunov functions) are used to establish solution behavior in terms of a generalized metric on the state space.

Let $R \subseteq S_\epsilon(0)$ for some $\epsilon > 0$ with $0 \in R$.

Def. 4.8: A real valued function $v(x)$, defined in E^n, is said to be positive definite [positive semi-definite] on R^c (recall that R^c is the complement of R in E^n) if $x \in R^c$ implies that $v(x) > 0$ [$v(x) \geq 0$] and $v(0) = 0$.

Def. 4.9: A real valued function $v(x, t)$ defined on $E^n \times T$ is said to be positive definite on $R^c \times T$ if $v(0, T) = 0$ and there is a function $w(x)$, that is positive definite on R^c and such that $v(x, t) \geq w(x)$ on $R^c \times T$.

Note that the behavior of $v(x)$, $v(x, t)$ or $w(x)$ inside R is immaterial.

Def. 4.10: A real valued function $v(x, t)$ defined on $E^n \times T$ is said to be positive semi-definite on $R^c \times T$ if $v(x,t) \geq 0$ on $R^c \times T$ and $v(0, T) = 0$.

Def. 4.11: The functions $v(x)$ and $v(x, t)$ are said to be negative definite [negative semi-definite] if $-v(x)$ and $-v(x, t)$ are positive definite [positive semi-definite].

If $R = \{0\}$, then replace R^c in the above definitions by E^n. For instance, a real valued function $v(x)$, defined in E^n, is said to be positive definite on E^n if $x \in E^n$ implies that $v(x) > 0$ for $x \neq 0$ and $v(0) = 0$. These are the usual definitions of sign definite functions [Ref. 12]. In their domain of definition, all of the definite and semi-definite functions defined above are assumed to be continuous and to have continuous first partial derivatives with respect to all arguments. In all cases where $v(x, t)$ is a Lyapunov function, $\mathring{v}(x, t)$ denotes the total derivative of $v(x,t)$ with respect to the differential equation under consideration (see Section 2.3).

Theorem 4.4: Let $v(x)$ be positive definite on E^n. If R_h (as defined in Section 3.1) is a bounded subset of E^n and $\mathring{v}(x, t)$ is negative semi-definite on $R_h^c \times T$, then all solutions of (4.1) starting on $\overline{R}_h$ are bounded on $\overline{R}_h$.

Proof: With $(x_o, t_o) \in \overline{R}_h \times T$, assume that for some $t_2 > t_o$ the solution $x(t_2;x_o, t_o) \in \overline{R}_h^c$ (note that $\overline{R}_h^c \triangleq (\overline{R}_h)^c$). Since the solutions are assumed continuous in t and $v(x)$ is continuous in x, there is a time t_1 where $t_o \leq t_1 \leq t_2$ such that $x(t_1) \in B(R_h)$.

Since $x(t_2) \epsilon \overline{R}_h^c$, then $v(x(t_2)) > h = v(x(t_1))$ in a region where $\dot{v}(x,t) \leq 0$. This contradiction shows that the solutions remain in $\overline{R}_h$ for all $t > t_o$.

<u>Theorem 4.5:</u> Under the hypothesis of Theorem 4.4, the set R_h is stable.

<u>Proof:</u> Given any $\epsilon > 0$, choose a $\gamma > 0$ such that $\overline{R}_{h+\gamma} \subset S_\epsilon(R_h)$. This is always possible due to the continuity of v(x) in x. Then choose $\delta > 0$ so that $S_\delta(R_h) \subset \overline{R}_{h+\gamma}$. Now, if $x_o \epsilon S_\delta(R_h) \subset \overline{R}_{h+\gamma}$, the solution $x(T_o; S_\delta(R_h), t_o)$ is bounded on $\overline{R}_{h+\gamma} \subset S_\epsilon(R_h)$ by Theorem 4.4. Thus, R_h is stable.

<u>Theorem 4.6:</u> Let v(x) be positive definite on E^n and assume that R_h is a bounded set. If $\dot{v}(x,t)$ is negative definite on $\overline{R}_h^c \times T$ and negative semi-definite on $B(R_h) \times T$, then every bounded solution of (4.1) approaches R_h.

<u>Proof:</u> If $(x_o, t_o) \epsilon \overline{R}_h \times T$, then by Theorem 4.4, $x(t; x_o, t_o)$ is bounded on $\overline{R}_h$ and Def. 4.3 is satisfied trivially. If $x(t; x_o, t_o)$ is any other bounded solution, then there is a constant P such that $\|x(t; x_o, t_o)\| < P$ and a closed, bounded set D where $s[D] > P$ such that $D \supset \overline{R}_h$ and $x(t; x_o, t_o)$ is bounded on D. Choose $\delta_1 > 0$ such that $R_{h+\delta_1} \subset D$ and $x_o \epsilon (D - R_{h+\delta_1})$. Since the region $(D - R_{h+\delta_1}) \subset \overline{R}_h^c$ is closed and bounded (and hence compact), $\dot{v}(x,t) \leq -w(x) < 0$ (w(x) is positive definite on E^n) takes on a maximum value, say $-\alpha < 0$ therein. Along the solution $x(t; x_o, t_o)$

in this region,

$$v(x) = v(x_o) + \int_{t_o}^{t} \mathring{v}\, dt \leq v(x_o) - \alpha(t-t_o).$$

Since $v(x) > 0$ for all $x \neq 0$, the solution can remain in this region for only a finite length of time. Since the solution is bounded on D, it must then approach R_h. Thus, given any $\delta > 0$, there is a $\tau_1 > t_o$ such that $t > \tau_1$ implies that $x(t;x_o, t_o) \in R_{h+\delta}$. It is now only necessary to show that this last statement implies that $d(x(t), R_h)$ becomes arbitrarily small. Given any $\epsilon > 0$, choose $\delta > 0$ so that $R_{h+\delta} \subset S_\epsilon(R_h)$. Then, for any $\epsilon > 0$ there is a $\tau > t_o$ such that $t \geq \tau$ implies $x(t;x_o, t_o) \in R_{h+\delta} \subset S_\epsilon(R_h)$. Thus, $t > \tau$ implies that $d(x(t;x_o, t_o), R_h) < \epsilon$, and the solution approaches R_h.

Theorem 4.7: Let $v(x)$ be positive definite on E^n and R_h be a bounded subset of E^n. If $\mathring{v}(x,t)$ is negative definite on $\overline{R}_h^c \times T$, and negative semi-definite on $B(R_h) \times T$, then the set R_h is asymptotically stable (AS).

Proof: Stability follows from Theorem 4.5. Since R_h is stable, the solutions may be bounded in some $S_\epsilon(R_h)$ by choosing x_o in some $S_\delta(R_h)$. Then choose D as some closed bounded set such that $S_\epsilon(R_h) \subseteq D$. All solutions with $x_o \in S_\delta(R_h)$ are then bounded on the closed bounded set $D \supset R_h$. The remaining hypotheses of Theorem 4.6 are then satisfied so all solutions starting in $S_\delta(R_h)$ approach R_h and this set is AS.

Theorem 4.8: Let $v(x)$ be positive definite on E^n. If $\dot{v}(x,t)$ is negative definite on $\overline{R}_h^c \times T$, negative semi-definite on $B(R_h) \times T$, and $\lim_{\|x\| \to \infty} v(x) = \infty$, then the set R_h is ASL.

Proof: The set R_h is stable by Theorem 4.5. Given any $(x_o, t_o) \in E^n \times T$, there is a scalar $\infty > h_1 > v(x_o)$ such that R_{h_1} is bounded, $R_{h_1} \supset R_h$, $x_o \in R_{h_1}$ (the selection of such an h_1 is always possible because $\lim_{\|x\| \to \infty} v(x) = \infty$), and the solution $x(t;x_o,t_o)$ of (4.1) is bounded on $\overline{R}_{h_1}$ by Theorem 4.4. Since $\overline{R}_{h_1}$ is a closed bounded set in which the solution $x(t;x_o,t_o)$ is bounded, it follows from Theorem 4.6 that this solution approaches R_h. Moreover, this is true for every solution $x(t;x_o,t_o)$ with $(x_o,t_o) \in E^n \times T$.

Theorem 4.9: Let $v(x)$ be positive definite on E^n with $\lim_{\|x\| \to \infty} v(x) = \infty$. If $\dot{v}(x,t)$ is negative definite on $R_h^c \times T$, then the solutions $x(t;E^n, T)$ of (4.1) are ultimately bounded (UB) on $\overline{R}_h$.

Proof: If $(x_o,t_o) \in \overline{R}_h \times T$, then by Theorem 4.4, the solution $x(t;x_o,t_o)$ is bounded on $\overline{R}_h$. For every $(x_o,t_o) \in R_h^c \times T$ there is a scalar $h_1 > v(x_o)$ such that $\overline{R}_{h_1}$ is bounded, $\overline{R}_h \subset R_{h_1}$, and the solution $x(t;x_o,t_o)$ is bounded on $\overline{R}_{h_1}$ by Theorem 4.4. Thus, the solution is bounded and lies initially in the closed region $(\overline{R}_{h_1} - R_h) \subset R_h^c$ where $\dot{v}(x,t) \leq -w(x) < 0$. In this region the continuous function $w(x)$ must take on a maximum value $-\alpha < 0$, and along the solution in this region

$$v(x) = v(x_o) + \int_{t_o}^{t} \mathring{v}\, dt \leq v(x_o) - \alpha(t-t_o) \;.$$

Since v(x) is positive for $x \neq 0$, the solution cannot remain in this region for more than a finite length of time. Thus, there is some $\tau \geq t_o$ such that $t > \tau$ implies $x(t;x_o,t_o) \in (\overline{R}_{h_1} - R_h)^c = \overline{R}_{h_1}{}^c \cup R_h$. Since the solution is bounded in $\overline{R}_{h_1}$, it must enter R_h for $t > \tau$. As noted above, it then remains in $\overline{R}_h$ for all $t > \tau$.

The second approach to this problem of obtaining sufficient conditions for stability and boundedness follows Conti [Refs. 5, 9] who views the positive definite functions as dependent variables in a first-order auxiliary equation (Brauer calls this a comparison equation). For example, let v(x, t) be a positive definite function on $E^n \times T$ and $\mathring{v}$ be the total derivative of v with respect to (4. 1). If there is a function $\omega(v, t)$ such that along the solutions of (4. 1)

$$\mathring{v} \leq \omega(v, t) \tag{4. 2}$$

then with quite mild restrictions it can be shown that (4. 1) has the same stability and boundedness properties as the first-order (scalar) differential equation

$$\mathring{r} = \omega(r, t) \tag{4. 3}$$

when $r_o \triangleq r(t_o) = v(x(t_o)) \triangleq v_o$. This reduces the problem of determining the stability of an n^{th}-order system to that of determining the stability of a first-order system--a very significant simplification.

(However, it is still necessary and frequently difficult, to choose the proper Lyapunov function to obtain (4.2).)

An important tool in this approach is the following lemma:

Lemma 4.1: Let $\omega(r,t)$ be a real valued continuous function on T x T with $\omega(0,T) \geq 0$ and satisfying conditions sufficient[9] to insure that for all $(r_o, t_o) \in T \times T$ the first-order differential equation $\mathring{r} = \omega(r,t)$ has a unique solution $r(t;r_o,t_o)$ that is continuous in t, r_o, and t_o. Let $v(t;v_o,t_o)$ be a real valued function on T x T x T such that

(a) $v(t;v_o,t_o)$ is continuous in t, v_o, and t_o

(b) $v(t_o;v_o,t_o) = v_o$

(c) for any $(v_o, t_o) \in T \times T$, the function $v(t;v_o,t_o)$ satisfies the differential inequality $\mathring{v} \triangleq \frac{dv}{dt} \leq \omega(v,t)$ for all $t \in T_o$.

Then, for any (v_o,t_o) and (r_o,t_o) in T x T with $0 \leq v_o \leq r_o$ it follows that $r(t;r_o,t_o) \geq 0$ and $r(t;r_o,t_o) \geq v(t;v_o,t_o)$ for all $t \in T_o$.

Proof: Since $\omega(0,T) \geq 0$, no solution $r(t;r_o,t_o)$ of (4.3) with $r_o \geq 0$ can cross the $r = 0$ axis into the region where r is negative. Thus, $r_o \geq 0$ implies that $r(t;r_o,t_o) \geq 0$ for all $t \in T_o$.

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[9] See footnote 6, page 43.

The fact that $v(t;v_o, t_o) \leq r(t;r_o, t_o)$ follows from a point-by-point consideration of the implications of the facts that $v_o \leq r_o$ and $\dot{v}(t) \leq \dot{r}(t)$ whenever $v = r$. To see this assume that $v_o \leq r_o$ but $v(t_2;v_o, t_o) > r(t_2;r_o, t_o)$ for some $t_2 > t_o$. Since both v and r are continuous functions of time, there must be some $t_1 \geq t_o$ such that $v(t_1;v_o, t_o) = r(t_1;r_o, t_o)$ and $v(t;v_o, t_o) > r(t;r_o, t_o)$ for $t > t_1$. But this means that $\dot{v}(t_1) > \dot{r}(t_1)$ which is a contradiction. Thus, $v(t;v_o, t_o) \leq r(t;r_o, t_o)$ for all $t \in T_o$.

Using Lemma 4.1, one may obtain the following theorem:

Theorem 4.10: Let v(x) be positive definite on E^n and R_h be a bounded set. Assume that along the solutions of (4.1), v(x(t)) satisfies the inequality $\dot{v} \leq \omega(v, t)$ with $\omega(v, t)$ satisfying the hypotheses of Lemma 4.1. Then

(a) the solutions of (4.1) will be bounded on $\overline{R}_h$ if the solutions of (4.3) are bounded on the set $\overline{P}_h = \{ r \mid 0 \leq r \leq h \}$

(b) the set R_h in the state space of (4.1) will be stable, AS, or ASL if the set $P_h = \{ r \mid 0 \leq r < h \}$ is stable, AS, or ASL in the state space of (4.3).

Proof: If the solutions of (4.3) are bounded on the set P_h for $r(t_o) \in \overline{P}_h$, it follows by Lemma 4.1 and the positive definiteness of v that $0 \leq v(x(t)) \leq h$ and thus $x(t) \in \overline{R}_h = \{ x \mid v(x) \leq h \}$.

Similarily, if the set P_h is stable, then for every $\epsilon_r > 0$ there is a $\delta_r > 0$ such that $r(t_o) \leq h + \delta_r$ implies $r(t;r_o,t_o) \leq h + \epsilon_r$. Now, given any $\epsilon > 0$, choose ϵ_r so that $\overline{R}_{h+\epsilon_r} \subset S_\epsilon(R_h)$ and $\delta > 0$ so that $x_o \in S_\delta(R_h)$ insures that $v(x_o) \leq h + \delta_r$. Then $v(x(t)) \leq r(t;r_o,t_o) \leq h + \epsilon_r$ for all $t \in T_o$ and therefore, $x(T_o;x_o,t_o) \subset S_\epsilon(R_h)$. The proof of AS and ASL follows in an equivalent fashion.

Example 4.1: Consider a transfer system modeled by the first-order equation

$$\mathring{x} = -f(x) + bu(t) \tag{4.4}$$

where $b > 0$, $u(t)$ is some fixed input function in the normed linear space C, $0 < cx^2 \leq xf(x) < \infty$ for $x \neq 0$ where c is a positive constant and $f(0) = 0$. Choose a positive definite function $v(x) = \frac{1}{2}x^2$. Then

$$\mathring{v} = -xf(x) + bxu(t) \leq -cx^2 + bxu(t) = -cx^2\left[1 - \frac{bu(t)}{cx}\right]$$

or

$$\mathring{v} \leq -cx^2\left[1 - \frac{b\|u\|_c}{cx}\right]. \tag{4.5}$$

Now let $h = \dfrac{b^2}{2c^2}\|u\|_c^2$. Then if

$$R_h = \{x \mid v(x) < h\} = \{x \mid |x| < \frac{b\|u\|_c}{c}\},$$

it is clear from (4.5) that $\mathring{v}$ is negative definite on $\overline{R}_h^c \times T$ and negative semi-definite on $B(R_h) \times T$. Thus, the hypotheses of Theorem 4.8 are satisfied and the set R_h is ASL.

This example is particularly interesting because the same region R_h is ASL for all $u(t) \in C$ that have the same norm $\|u\|_c$. This suggests that for such systems a "gain" could be meaningfully defined relating the size of R_h to the norm $\|u\|_c$. In the case considered, the size of R_h is $\frac{b}{c}\|u\|_c$ so the gain would be $\frac{b}{c}$. This concept of gain is explored further in Section 5.1.

<u>Example 4.2:</u> Again consider (4.4), but now note that (see Lemma 5.1)

$$\mathring{v} \leq -\frac{c}{2}x^2 + \frac{b^2}{2c}\|u\|_c^2$$

or

$$\mathring{v} \leq -cv + \frac{b^2}{2c}\|u\|_c^2 .$$

Thus, the auxiliary equation is

$$\mathring{r} = -cr + \frac{b^2}{2c}\|u\|_c^2$$

and for this system the region $0 \leq r < \frac{b^2}{2c^2}\|u\|_c^2$ is ASL since

$$r(t) = \left[r(t_o) - \frac{b^2}{2c^2}\|u\|_c^2\right] e^{-c(t-t_o)} + \frac{b^2}{2c^2}\|u\|_c^2 .$$

Thus, by Theorem 4.10, the region $R_h = \{x \mid v(x) < \frac{b^2}{2c^2}\|u\|_c^2\}$ where $h = \frac{b^2}{2c^2}\|u\|_c^2$ is ASL in the state space of (4.4). But

$$v(x) = \tfrac{1}{2}x^2 < \frac{b^2}{2c^2}\|u\|_c^2$$

implies that

$$|x| < \frac{b}{c}\|u\|_c \quad ,$$

which is the same result as Example 4. 1.

In conclusion, note that at the expense of considerable complication of the proofs, all of the theorems of this section may be stated with much weaker hypotheses. For example, relaxation of requirements of uniqueness of the solutions of (4. 1) is considered by Conti [Ref. 9] while relaxation of requirements of continuity in the Lyapunov functions is considered by Massera [Ref. 31]. Other generalizations are also possible, but the added complication of the proofs does not appear to be necessary for the present purposes and is therefore omitted.

CHAPTER 5

PROPERTIES OF TRANSFER SYSTEMS

With certain restrictions, the stability definitions and theorems of Chapter 4 can be applied to the study of transfer systems. This study leads to the definition of special classes of transfer systems which are important in the development of a stability theory to follow.

5.1 Gains for Transfer Systems

The stable regions discussed in Chapter 4 offer a method for characterizing certain important features of transfer systems. However, this problem is complicated by the fact that transfer systems are input-output devices and the characterization must include the relation between input and output.

In the discussion of stability of regions, a fixed differential equation was assumed and fixed regions were considered. If u(t) may be one of a class of input functions in the differential equation

$$\mathring{x} = f(x, t, u(t)) , \qquad (5.1)$$

then the existence and location of regions M exhibiting some of the stability properties described in Section 4.1, will depend on the specific input. In general, when (5.1) exhibits stable regions they must be denoted M(u(t)) to indicate this dependence on the input.

For example, in a transfer system modeled by (5.1) it may happen that for each u(t) in some set U there are corresponding regions M(u(t)) that are ASL. There are several interesting special cases of this situation.

Def. 5.1: A transfer system modeled by (5.1) is said to be in Class C if $u(t) \in C$ implies[10] that $x(t;x_0, t_0, u(t)) \in C$ for all $(x_0, t_0) \in E^n \times T$.

Thus, for a system in Class C a continuous bounded input gives a continuous bounded output. This is similar to the bounded input implies bounded output stability defined in Ref. 14.

Def. 5.2: A transfer system modeled by (5.1) is said to be in Class C_0 if $u(t) \in C_0$ implies $x(t;x_0, t_0) \in C_0$ for all $(x_0, t_0) \in E^n \times T$.

A situation of particular interest occurs when there is one fixed set in the state space that is ASL for all u(t) in some set U. For instance, suppose that for every u(t) in a set $U_\alpha \subset C$ there is one bounded set $M(U_\alpha)$ containing the origin that is ASL. [Since any set $N \supseteq M(U_\alpha)$ will also be ASL, let $M(U_\alpha)$ be the smallest of all those that are ASL for $u(t) \in U_\alpha$]. Then, by Theorem 4.3, for each $u(t) \in U_\alpha$

$$\|x(t;x_0, t_0, u(t))\| \leq d(x(t;x_0, t_0, u(t)), M(U_\alpha)) + s[M(U_\alpha)] \quad . \qquad (5.2)$$

In order that the right-hand side of (5.2) be independent of u(t), except through U_α, assume that $d(x(t;x_0, t_0, u(t)), M(U_\alpha))$ can be bounded above by a scalar function $\gamma(t, x_0, t_0)$ that is independent of u(t) and approaches 0 as t increases (the distance d(x, M) has this last property for each

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10 Here C refers to the normed linear space of continuous, bounded functions on T. See Section 3.1.

fixed u(t)--see Theorem 4.3). The inequality (5.2) then becomes

$$\|x(t;x_0,t_0,u(t))\| \leq \gamma(t,x_0,t_0) + s[M(U_\alpha)] \quad . \tag{5.3}$$

The solutions $x(t;x_0,t_0,u(t))$ now approach a region $M(U_\alpha)$ whose size is determined by u(t) through U_α. This suggests that for certain classes of systems it is possible to define a "gain" relating the size of the input to the size of this region approached by the output. The following class of transfer systems has the necessary properties.

Def. 5.3: Let $\gamma(s,x,t)$ be defined on $T \times E^n \times T$ and continuous in s, x, and t at all points in $T \times E^n \times T$ with $\gamma(T,0,T) \equiv 0$ and $\lim_{s\to\infty} \gamma(s,E^n,T) = 0$. A transfer system modeled by (5.1) is said to be in Class G if

(a) for every $u(t) \in C$ there is a bounded region M(u) containing the origin that is ASL.

(b) $u(t) \equiv 0$ implies $M(u) = \{0\}$.

(c) there exists a function $\gamma(s,x,t)$ with the above properties such that for all $u(t) \in C$ and all $(x_0,t_0) \in E^n \times T$, $d(x(t;x_0,t_0,u(t)), M(u)) \leq \gamma(t,x_0,t_0)$.

It is clear from the statement of Def. 5.3 that Class G is a subclass of Class C.

Now if $U_\alpha = \{u(t) \mid u(t) \in C \text{ and } \|u\|_c = \alpha\}$, for transfer systems in Class G, a plot of $s[M(U_\alpha)]$ versus α appears as shown in Fig. 8.

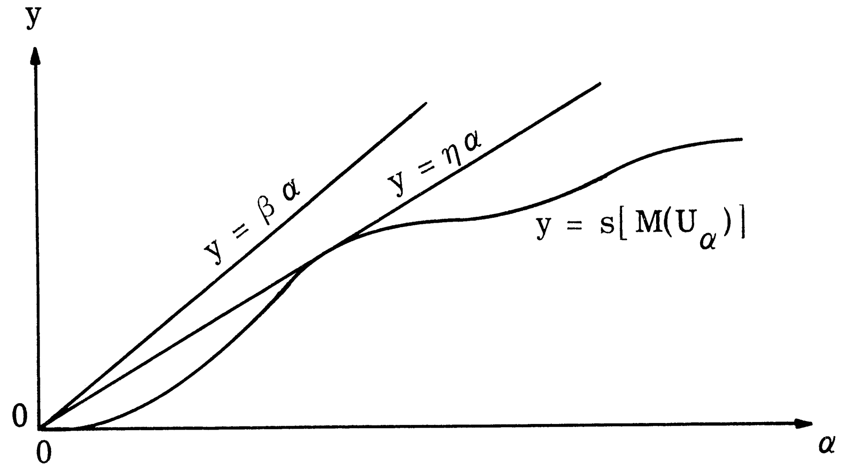

Fig. 8. Graph of $s[M(U_\alpha)]$ versus α for a system in Class G.

The fact that this curve goes through the origin follows from part (b) of Def. 5.3. Clearly then, there are constant β such that $\beta\,\alpha \geq s[M(u)]$. This leads to a definition of what will be called the gain of a system in Class G.

Def. 5.4: The gain η of a transfer system in Class G is defined as

$$\eta = \inf\{\beta \mid \beta\,\alpha \geq s[M(U_\alpha)] \text{ for all } u(t) \in C\} .$$

Theorem 5.1: For every transfer system in Class G there is a gain η and a function $\gamma(t;x_0, t_0)$ such that for all $u(t) \in C$

$$\|x(t;x_0, t_0, u(t))\| \leq \gamma(t, x_0, t_0) + \eta\|u\|_C \quad . \tag{5.4}$$

Proof: This follows immediately from Defs. 5.3 and 5.4 and Theorem 4.3.

Note that Class G and the definition of gain represent, to a certain extent, attempts to extend single-input, single-output, linear system concepts to a wider class of transfer systems. The term $\gamma(t, x_o, t_o)$ in (5.4) is similar to the transient response of a linear system while the term $\eta \|u\|_c$ corresponds to the steady state solution. The gain η is then a constant relating the size of the steady state response to the size (here the norm $\|u\|_c$) of the input. The following examples illustrate these ideas.

<u>Example 5.1:</u> Consider a single-input, single-output, first-order linear transfer system modeled by

$$\begin{cases} \dot{x} = -ax + bu(t) & x(t_o) = x_o \\ y = x \end{cases} \tag{5.5}$$

where a and b are positive constants. The solution is

$$x(t;x_o, t_o, u(t)) = e^{-a(t-t_o)} x_o + \int_{t_o}^{t} e^{-a(t-\tau)} bu(\tau)\, d\tau$$

giving

$$\|x(t;x_o, t_o, u(t))\| \leq \| e^{-a(t-t_o)} x_o\| + \left[b \int_{t_o}^{t} \|e^{-a(t-\tau)}\| \, d\tau \right] \|u\|$$

or

$$\|x(t;x_o, t_o, u(t))\| \leq \|e^{-a(t-t_o)} x_o\| + \frac{b}{a} \|u\|_c \, . \tag{5.6}$$

To see if this system is in Class G, check Def. 5.3. For part (a), it is clear from (5.6) that for every $u(t) \in C$ the bounded region $M(u(t)) =$

$\{x \mid \|x\| \leq \frac{b}{a} \|u\|_c\}$ is ASL and contains the origin. Since the null solution $M = \{0\}$ is ASL when $u(t) \equiv 0$, part (b) is satisfied. For part (c) note that

$$d(x(t;x_o, t_o, u(t)), M(u(t)) = \|x(t;x_o, t_o, u(t))\| - \frac{b}{a}\|u\|_c \leq \|e^{-a(t-t_o)} x_o\|$$

which has all the required properties of a function $\gamma(s, x, t)$. Thus, the transfer system modeled by (5.5) is in Class G. If $U_\alpha = \{u(t) \mid u(t) \in C$ and $\|u\|_c = \alpha\}$, then it is clear from (5.6) that $s[M(U_\alpha)] \leq \frac{b}{a}\alpha$ and so the gain of this system is less than or equal to $\frac{b}{a}$. (This is a typical case where only an upper bound on the gain can be obtained.) Note that this is just the maximum gain of the transfer function

$$H(s) = \frac{X(s)}{U(s)} = \frac{b}{s+a}$$

which occurs at zero frequency ($s = 0$).

Example 5.2: Consider the general linear, time-invariant system modeled by

$$\begin{cases} \dot{x} = A x + B u(t) \qquad x(t_o) = x_o \\ y = H x \end{cases} \tag{5.7}$$

where A, B, and H are constant matrices and it is assumed that all of the eigenvalues of A have negative real parts. The well-known solution to (5.7) is [Ref. 8]

$$x(t;x_o, t_o, u(t)) = e^{A(t-t_o)} x_o + \int_{t_o}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau . \tag{5.8}$$

Because all the eigenvalues of A have negative real parts, there are positive constants β and γ such that $\|e^{At}\| \leq \beta e^{-\gamma t}$ (see Ref. 6 or use the Gronwall-Bellman lemma, Ref. 4). Thus,

$$\|x(t;x_0, t_0, u(t))\| \leq \beta e^{-\gamma(t-t_0)} \|x_0\| + \left[\beta \int_{t_0}^{t} \|e^{-\gamma(t-\tau)} B\| \, d\tau \right] \|u\|_c$$

or

$$\|x(t;x_0, t_0, u(t)\| \leq \beta e^{-\gamma(t-t_0)} \|x_0\| + \frac{\beta\|B\|}{\gamma} \|u\|_c \, .$$

Reasoning as in Example 5.1 shows that this system is in Class G, and the set

$$\{ x | \; \|x\| \leq \frac{\beta\|B\|}{\gamma} \alpha \}$$

is ASL for all $u(t) \in U_\alpha = \{ u(t) \mid u(t) \in C \text{ and } \|u\|_c = \alpha \}$. Thus,

$$M(U_\alpha) \subseteq \{ x \mid \|x\| \leq \frac{\beta\|B\|}{\gamma} \alpha \}$$

and the gain η is less than or equal to $\frac{\beta}{\gamma} \|B\|$.

5.2 Exponential Stability and Class E

The above discussion has led to the definition of a class of transfer systems (Class G) having special properties that are important in the study of composite systems. In this section, an important subclass of Class G is defined and it is shown that Class G is "large enough" to contain many systems of practical significance.

The systems in this subclass of Class G are related by the property of exponential stability-in-the-large.

Def. 5.5: The null solution of the equation

$$\dot{x} = f(x, t) \qquad f(0, T) \equiv 0 \tag{5.9}$$

is said to be exponentially stable-in-the-large (ESL) if there are two positive constants α and β such that $\|x(t;x_o, t_o)\| \leq \beta \|x_o\| e^{-\alpha(t-t_o)}$ for all $(x_o, t_o) \in E^n \times T$.

Def. 5.6: A vector function f(x, t) on $E^n \times T$ is said to be in Class B (denoted $f \in B$) if, for all $(x, t) \in E^n \times T$, f(x, t) is continuous in x and t and has continuous first partial derivatives with respect to $x_1, x_2, \ldots, x_n$ such that $\left|\frac{\partial f_i}{\partial x_j}\right| < L$ (L is a constant and $i, j = 1, \ldots, n$).

Theorem 5.2: Assume that $f(x, t) \in B$ in (5.9). Then (5.9) is ESL if and only if there is a positive definite function v(x, t) such that

(a) $c_1 \|x\|^2 \leq v(x, t) \leq c_2 \|x\|^2$,

(b) $\dot{v}(x, t) \leq -c_3 \|x\|^2$,

(c) $\|\nabla v\| \leq c_4 \|x\|$,

where c_1, c_2, c_3, and c_4 are positive constants. If (5.9) is autonomous, then v can be chosen independent of t.

Proof: See Appendix or Ref. 18, p. 59.

Using this theorem, it is possible to show that an n^{th}-order system modeled by

$$\dot{x} = f(x, t) + Du(t) \tag{5.10}$$

is in Class G if $\dot{x} = f(x,t)$ is ESL and $f(x,t) \in B$.

Lemma 5.1: If $a > 0$, then for all $z \in T$

$$-az^2 + bz \leq -\frac{a}{2}z^2 + \frac{b^2}{2a} . \tag{5.11}$$

Proof:

$$-az^2 + bz \leq -\frac{a}{2}z^2 + \frac{b^2}{2a}$$

$$\text{iff } -\frac{a}{2}z^2 + \frac{b^2}{2a} - \frac{a}{2}z^2 + bz - \frac{b^2}{2a} \leq -\frac{a}{2}z^2 + \frac{b^2}{2a}$$

$$\text{iff} \left(-\frac{a}{2}z^2 + \frac{b^2}{2a} \right) - \frac{1}{2a}(az-b)^2 \leq -\frac{a}{2}z^2 + \frac{b^2}{2a}$$

which is obviously true.

Theorem 5.3: If $\dot{x} = f(x,t)$ is ESL and $f(x,t) \in B$, then a transfer system modeled by (5.10) is in Class G with gain $\eta \leq \left(\frac{c_4}{c_3}\right) \sqrt{\frac{c_2}{c_1}} \|D\|$ where c_1, c_2, c_3, and c_4 are the constants occuring in Theorem 5.2.

Proof: Since $\dot{x} = f(x,t)$ is ESL and $f(x,t) \in B$, there is a positive definite function $v(x,t)$ satisfying (a), (b), and (c) of Theorem 5.2. The total derivative of this function with respect to (5.10) is then

$$\dot{v} = \dot{v}_1(x,t) + \nabla v' Du \leq -c_3 \|x\|^2 + c_4 \|x\| \, \|D\| \, \|u(t)\|$$

where $\dot{v}_1$ is the total derivative of $v(x,t)$ with respect to (5. 9).

Use of Lemma 5. 1 yields

$$\dot{v} \leq -\frac{c_3}{2}\|x\|^2 + \frac{c_4^2\|D\|^2}{2c_3}\|u(t)\|^2$$

or

$$\dot{v} \leq -\frac{c_3}{2c_2} v + \frac{c_4^2\|D\|^2}{2c_3}\|u(t)\|^2 . \tag{5. 12}$$

This is a differential inequality in v and by Lemma 4. 1,

$$v(t) \leq v(t_o)\, e^{-\frac{c_3}{2c_2}(t-t_o)} + \frac{c_4^2\|D\|^2}{2c_3}\int_{t_o}^{t} e^{-\frac{c_3}{2c_2}(t-\tau)}\|u(\tau)\|^2\, d\tau . \tag{5. 13}$$

If $u(t) \in C$ with norm $\|u\|_c$, then

$$v(t) \leq v(t_o)\, e^{-\frac{c_3}{2c_2}(t-t_o)} + \frac{c_2 c_4^2\|D\|^2}{c_3^2}\|u\|_c^2$$

and from (a) of Theorem 5. 2

$$\|x\|^2 \leq \frac{c_2}{c_1}\|x(t_o)\|^2\, e^{-\frac{c_3}{2c_2}(t-t_o)} + \frac{c_2 c_4^2\|D\|^2}{c_1 c_3^2}\|u\|_c^2$$

or

$$\|x\| \leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}}\|x(t_o)\|\, e^{-\frac{c_3}{4c_2}(t-t_o)} + \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} \frac{c_4}{c_3}\|D\|\,\|u\|_c .$$

This result can now be compared with the requirements of Def. 5.3 to show that the system is in Class G. Requirements (a) and (b) are satisfied since the region

$$M(u) = \{x \mid \|x\| \leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} \frac{c_4}{c_3} \|D\| \, \|u\|_c \}$$

contains the origin, is ASL for every $u(t) \in C$, and reduces to $\{0\}$ when $u(t) \equiv 0$. Requirement (c) is satisfied since

$$d(x(t;x_o, t_o, u(t)), M(u)) = [\, \|x(t;x_o, t_o, u(t))\| - \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} \frac{c_4}{c_3} \|D\| \, \|u\|_c \,]$$

$$\leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} \|x(t_o)\| \, e^{-\frac{c_3}{4c_2}(t-t_o)}$$

and this last term on the right has all the required properties of a function $\gamma(s, x, t)$. Thus, all of the requirements of Def. 5.3 are satisfied and this system is in Class G. Moreover, from Def. 5.4, this system has a gain

$$\eta \leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} \frac{c_4}{c_3} \|D\| \; .$$

Def. 5.7: A system modeled by (5.10) will be said to be in Class E if the unforced model $[(5.10)$ with $u(t) = 0]$ is ESL and $f(x, t) \in B$.

The gain estimate determined in Theorem 5.3 depends on the constants c_1, c_2, c_3 and c_4 and thus on the particular Lyapunov function chosen. This is a reoccurrence of the old problem of choosing the proper metric. The significance of this problem in the present context will become more apparent in Chapter 6.

From Theorem 5.3 it is clear that Class E is a subclass of Class G. In addition, it can be shown that Class E (and therefore Class G) contains many systems of practical importance. This follows from the fact that many important equations are ESL.

Theorem 5.4: The following ordinary differential equations are ESL:

(a) The linear constant coefficient equation $\dot{x} = Ax$ when A is stable (i.e., all of its eigenvalues have negative real parts);

(b) The linear time-varying equation $\dot{x} = A(t)x$ where $A(t)$ is continuous and bounded and $\dot{x} = A(t) + u(t)$ is in Class C;

(c) The equation $\dot{x} = f(x, t)$ where $f(0, T) = 0$ and $x'f(x, t) \leq -c_3 \|x\|^2 < 0$ for all $(x, t) \epsilon E^n \times T$ with $x \neq 0$.

Proof: (a) is a well-known result [Ref. 12], (b) see Ref. 32, p. 518, and (c) see Example 5.3 below.

The list given in Theorem 5.4 is far from exhaustive. A more detailed description of Class E and Class G should be the subject of future study. The following example shows a nonlinear equation that is ESL.

Example 5.3: Consider the n^{th}-order equation

$$\dot{x} = f(x, t) \qquad f(0, T) = 0$$

where $x'f(x,t) \leq -c_3\|x\|^2 < 0$ for $x \neq 0$. Let $v(x) = \frac{1}{2}\|x\|^2$ so that

$$\dot{v}(x,t) = -x'f(x) \leq -c_3\|x\|^2 = -2c_3 v \ .$$

Then, by Lemma 4.1,

$$v(t) \leq v(t_o)\ e^{-2c_3(t-t_o)} \ ,$$

and thus,

$$\|x(t;x_o, t_o)\| \leq \|x_o\|\ e^{-c_3(t-t_o)}$$

showing that this equation is ESL.

CHAPTER 6

STABILITY OF COMPOSITE SYSTEMS

Using concepts developed in the previous chapters, one can now develop an approach to the basic problem, the stability of composite systems.

6.1 Two Simple Composite Systems

The tools developed earlier will first be applied in the analysis of the stability of the two simple composite systems shown in Fig. 9.

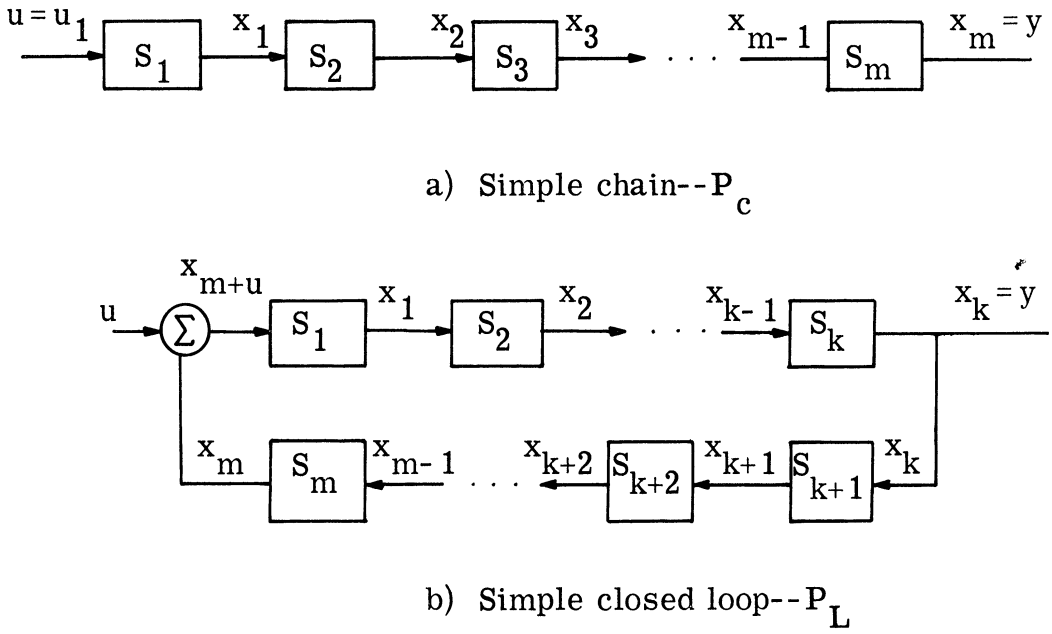

Fig. 9. Two simple composite systems.

Note that both composite systems are shown with an input u and an output y suggesting that they may be transfer systems in a larger composite. However, only the stability of the unforced composite systems ($u = 0$) will be considered.

The following lemmas will be of major importance in this section.

<u>Lemma 6.1:</u> Let $x(t;x_o, t_o)$ be a solution of the differential inequality[11]

$$\dot{x} \leq Ax \tag{6.1}$$

with $x(t_o;x_o, t_o) = x_o$ and let $y(t;y_o, t_o)$ be a solution of the differential equation

$$\dot{y} = Ay . \tag{6.2}$$

If all the elements a_{ij} $(i, j = 1, \ldots, n)$ of A are nonnegative, and $x_o = y_o$, then $x(t;x_o, t_o) \leq y(t;y_o, t_o)$ for all $t \in T_o$.

<u>Proof:</u> First note that the solution to (6.2) is $y(t;y_o, t_o) = e^{A(t-t_o)} y_o$. The inequality (6.1) can be rewritten as

$$\dot{x} = Ax - p(t)$$

where p(t) is a vector whose elements are nonnegative functions on T; i.e., $p(t) \geq 0$. Then,

$$x(t;x_o, t_o) = e^{A(t-t_o)} x_o - \int_{t_o}^{t} e^{A(t-\tau)} p(\tau)\, d\tau .$$

But the integral on the right represents a nonnegative vector because each element of $e^{A(t-\tau)}$ is nonnegative as long as $t > \tau$. Then, since $x_o = y_o$,

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[11] Throughout this section, the notation $x \leq y$ where x and y are n-vectors means that $x_i \leq y_i$ for $i = 1, \ldots, n$.

$$x(t;x_o, t_o) = y(t;y_o, t_o) - g(t)$$

where g(t) denotes the integral and hence $g(t) \geq 0$. Therefore,

$$x(t;x_o, t_o) \leq y(t;y_o, t_o)$$

for all $t \in T_o$.

Lemma 6. 2: Let B be a matrix with negative diagonal elements and nonnegative off-diagonal elements. If $x(t;x_o, t_o)$ and $y(t;y_o, t_o)$ are solutions of

$$\dot{x} \leq Bx \qquad (6.3)$$

$$\dot{y} = By \qquad (6.4)$$

and $x_o = y_o$, then $x(t;x_o, t_o) \leq y(t;y_o, t_o)$ for all $t \in T_o$.

Proof: Let -d be the smallest of the diagonal elements of B. Application of the transformations $v = e^{+dt}x$ and $w = e^{+dt}y$ changes (6. 3) and (6. 4) to

$$\dot{v} \leq (B + dI)\, v \quad ,$$

$$\dot{w} = (B + dI)\, w \quad ,$$

and B + dI has all nonnegative elements. Then by Lemma 6. 1, since $v_o = w_o$, it follows that $v(t;v_o, t_o) \leq w(t;w_o, t_o)$ and thus $x(t;x_o, t_o) \leq y(t;y_o, t_o)$ for all $t \in T_o$.[12]

[12] The author is indebted to Dr. J. K. Hale for this proof of Lemma 6. 2.

Lemma 6.3: The null solution of the system of differential equations

$$\begin{cases} \dot{x}_1 = -a_1 x_1 + b_1 x_n \\ \dot{x}_2 = -a_2 x_2 + b_2 x_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \dot{x}_n = -a_n x_n + b_n x_{n-1} \end{cases}$$

with a_i and b_i real, $a_i > 0$ and $b_i \geq 0$, $(i = 1, \ldots, n)$, is ASL if and only if

$$\prod_{i=1}^{n} \frac{b_i}{a_i} < 1 \ .$$

Proof: The characteristic equation for this system is

$$\prod_{i=1}^{n} (\lambda + a_i) - \prod_{i=1}^{n} b_i = 0 \ .$$

Now replace b_1 with μb_1 and consider the root locus problem [Ref. 43] for $\mu \geq 0$. If $\mu = 0$, there are n negative real roots, $\lambda = -a_i$. As μ increases, the largest root (one closest to the origin) moves to the right along the real axis reaching the origin when

$$\prod_{i=1}^{n} a_i - \mu \prod_{i=1}^{n} b_i = 0 \ .$$

It is only necessary to ascertain that no complex root has crossed the imaginary axis for a smaller value of μ.

When b_1 is replaced by μb_1, the characteristic equation can be solved for μ giving

$$\mu = \prod_{i=1}^{n} \frac{(\lambda+a_i)}{b_i}$$

as the value of μ corresponding to each point on the root locus. At the origin $\lambda = 0$ and

$$\mu = \prod_{i=1}^{n} \frac{a_i}{b_i} ,$$

while if the root locus crosses the imaginary axis at $\lambda = \lambda_1 \neq 0$ the value of μ at this point is

$$\mu_1 = \prod_{i=1}^{n} \frac{|\lambda_1+a_i|}{b_i} > \prod_{i=1}^{n} \frac{a_i}{b_i} .$$

Thus, the point where the locus crosses the imaginary axis with the smallest value of μ is at the origin and the root moving along the real axis must be first to cross as μ increases. If $\mu = 1$, the necessary and sufficient condition for stability becomes

$$\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i > 0 \quad \text{or} \quad \prod_{i=1}^{n} \frac{b_i}{a_i} < 1 .$$

It is now possible to consider the stability of the simple chain P_c of Fig. 9(a). It is intuitively obvious that an interconnection of individual stable systems will be stable because of the "weak" interconnections involved (no loading assumed, see Section 3. 2). The following theorem gives a simple proof of this fact.

Theorem 6.1: Consider the simple chain P_c with $u = u_1 = 0$ and $u_i = x_{i-1}$ for $i = 2, \ldots, m$. If the individual transfer systems S_i, $(i = 1, \ldots, m)$, are in Class E, then the null solution of the composite system is ASL.

Proof: If the individual transfer systems are in Class E, then for the i^{th} transfer system there is an inequality [see (5.12)]

$$\dot{v}_1 \leq -\alpha_i v_i + \gamma_i \|u_i\|^2 ,$$

where

$$\alpha_i = \frac{c_{i3}}{2c_{i2}} > 0, \ \gamma_i = \frac{c_{i4}{}^2 \|D_i\|^2}{2c_{i3}} > 0 .$$

When the interconnections are made, $u_1 \equiv 0$ and $\|u_i\|^2 = \|x_{i-1}\|^2 \leq \frac{1}{c_{i-1,1}} v_{i-1}$ for $i = 2, 3, \ldots, m$. With $\beta_i = \frac{\gamma_i}{c_{i-1,1}}$, $(i = 2, \ldots, m)$, the resulting system of differential inequalities becomes

$$\begin{cases} \dot{v}_1 \leq -\alpha_1 v_1 \\ \dot{v}_2 \leq -\alpha_2 v_2 + \beta_2 v_1 \\ \cdots\cdots\cdots\cdots\cdots \\ \dot{v}_n \leq -\alpha_m v_m + \beta_m v_{m-1} \end{cases} \tag{6.5}$$

where $v_i \geq 0$. Since $\beta_1 = 0$, the null solution of the system of equations (6.5) with inequalities replaced by equalities (a system of auxiliary equations) is ASL (Lemma 6.3). Then by Lemma 6.2, the system (6.5) is also ASL which means that each v_i approaches 0. Since $\|x_i\|^2 \leq \frac{1}{c_{i1}} v_i$, the equilibrium solution $x = 0$ of the composite system is therefore ASL.

A more impressive result is obtained for the simple closed loop P_L. In this case, the stability of the individual systems is not sufficient. The additional requirement is that the loop gain estimated from the chosen Lyapunov functions be less than unity--again an intuitively plausible requirement.

Theorem 6.2: Consider the simple closed loop P_L with $u = 0$ (no external input) and $u_1 = x_m$, and $u_i = x_{i-1}$ for $i = 2, \ldots, m$. If the individual transfer systems S_i, are in Class E with gain estimates η_i $(i = 1, \ldots, m)$, then the null solution of the composite system will be ASL if $\prod_{i=1}^{m} \eta_i < 1$.

Proof: As in the proof of Theorem 6.1, there is a system of differential inequalities

$$\left\{\begin{array}{l} \dot{v}_1 \leq -\alpha_1 v_1 + \beta_1 v_n \\ \dot{v}_2 \leq -\alpha_2 v_2 + \beta_2 v_1 \\ \cdots\cdots\cdots\cdots\cdots \\ \dot{v}_m \leq -\alpha_m v_m + \beta_m v_{m-1} \end{array}\right. \qquad (6.6)$$

where

$$\alpha_i = \frac{c_{i3}}{2c_{i2}} > 0 \quad i = 1, \ldots, m$$

$$\beta_1 = \frac{c_{14}^{\,2}\|D_1\|^2}{2c_{13}c_{m1}} > 0$$

$$\beta_i = \frac{c_{i4}^{\,2}\|D_i\|^2}{2c_{i3}c_{i-1,1}} > 0 \quad i = 2, \ldots, m ,$$

where the $v_i \geq 0$, for $i = 1, \ldots, m$. The null solution of the composite system will be ASL if the system of auxiliary equations [(6. 6) with inequalities replaced by equalities] is ASL. But, by Lemma 6. 3, this will occur if

$$\prod_{i=1}^{m} \frac{\beta_i}{\alpha_i} < 1 .$$

Now

$$\frac{\beta_i}{\alpha_i} = \frac{c_{i2}c_{i4}^{\,2}\|D_i\|^2}{c_{i-1,1}c_{i3}^{\,2}} = \frac{c_{i1}}{c_{i-1,1}}\,\eta_i^{\,2} ,$$

(where $c_{i-1,1} = c_{m1}$ when $i = 1$) and so the requirement that

$$\prod_{i=1}^{m} \eta_i^{\,2} < 1 \quad \text{or} \quad \prod_{i=1}^{m} \frac{\beta_i}{\alpha_i} < 1 ,$$

thereby giving ASL.

The possibility of requiring only that the individual transfer systems be in Class G is suggested by the following theorem.

Theorem 6. 3: If the transfer systems in Theorem 6. 2 are only required to be in Class G with $\prod_{i=1}^{m} \eta_i < 1$, then the null solution of the composite system P_L is stable.

<u>Proof:</u> This follows easily from Theorem 5. 4 and the fact that the loop gain is less than one.

It seems reasonable to expect that further study of Class G will allow an extension of this result to ASL or a clarification of the relation between Classes G and E.

6. 2 Complex Composite Systems

In Section 3. 2, it was noted that when the transfer systems of a composite system are individually modeled by

$$\begin{cases} \dot{x}_i = f_i(x_i, t) + D_i u_i(t) \; , & (6.7) \\ y_i = H_i x_i \; , & (6.8) \end{cases}$$

for $i = 1, \ldots, m$, then the composite system modeled has the particularly simple form

$$\begin{cases} \dot{x} = f(x, t) + Cx + Ku & (6.9) \\ y = h(x, t) \; . & (6.10) \end{cases}$$

Since transfer systems in Class E have models of the form (6. 7), (6. 8), this situation is of particular interest in this analysis.

Since $C_{ij} = D_i B_{ij} H_j$, (see Section 3. 2), the C matrix of (6. 9) serves the same purpose as the B matrix of (3. 3) in indicating the topology of the interconnections in the composite system. For example, in the case of the simple chain, P_c, the C matrix has the partitioned form

$$C_c = \left[\begin{array}{c|c|c|c|c|c}
0 & 0 & 0 & \cdots & 0 & 0 \\ \hline
C_{21} & 0 & 0 & \cdots & 0 & 0 \\ \hline
0 & C_{32} & 0 & \cdots & 0 & 0 \\ \hline
0 & 0 & C_{43} & \cdots & 0 & 0 \\ \hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
0 & 0 & 0 & \cdots & C_{m,m-1} & 0
\end{array}\right]$$

and for the simple closed loop, P_L, the C matrix has the form

$$C_L = \left[\begin{array}{c|c|c|c|c|c}
0 & 0 & 0 & \cdots & 0 & C_{1m} \\ \hline
C_{21} & 0 & 0 & \cdots & 0 & 0 \\ \hline
0 & C_{32} & 0 & \cdots & 0 & 0 \\ \hline
0 & 0 & C_{43} & \cdots & 0 & 0 \\ \hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
0 & 0 & 0 & \cdots & C_{m,m-1} & 0
\end{array}\right]$$

More general interconnections will result in more general patterns of nonzero elements (submatrices) in the C matrix. It is now possible to give sufficient conditions for ASL of the $x \equiv 0$ solution of a composite system with arbirary interconnections; that is, a system where C has an arbitrary number of nonzero elements. It will, however, be assumed that $C_{ii} = 0$ so that there is no direct feedback around any individual transfer system.

Theorem 6. 4: Let P be a general composite system made up of m transfer systems S_i of order n_i $(i = 1, \ldots, m)$ and modeled by (6. 9) with $C_{ii} = 0$. Assume that each transfer system S_i

is in Class E and has a Lyapunov function $v_i(x_i t)$ satisfying the bounds listed in Theorem 5.2 with coefficients $c_{i1}, c_{i2}, c_{i3}, c_{i4}$ (all coefficients are positive). Consider the m^{th}-order linear system of auxiliary equations

$$\dot{r} = Ar \tag{6.11}$$

where A is an m x m matrix of elements

$$a_{ij} = \begin{cases} \dfrac{c_{i3}}{2c_{i2}} & \text{for } i = j \\[2ex] \dfrac{c_{i4}{}^2 \sum\limits_{j=1}^{m} \|C_{ij}\|^2}{2c_{i3}\, c_{j1}} & \text{for } i \neq j \end{cases}$$

with C_{ij} an element (submatrix) of C. The equilibrium solution $x \equiv 0$ of the unforced composite system model [(6.9) with $u \equiv 0$] will be ASL if the equilibrium solution $r \equiv 0$ of the m^{th}-order linear auxiliary system (6.11) is ASL.

<u>Proof:</u> The i^{th} transfer system can be modeled by

$$\begin{cases} \dot{x}_1 = f_i(x_i, t) + D_i u_i(t) , & (6.12) \\ y_i = H_i x_i , & (6.13) \end{cases}$$

and due to the interconnections

$$u_i = \sum_{j=1}^{m} B_{ij}\, y_j \quad . \tag{6.14}$$

Since this i^{th} transfer system is in Class E, there is a

Lyapunov function $v_i(x, t)$ such that

$$\dot{v}_i \leq -c_{i3} \|x_i\|^2 + c_{i4} \|x_i\| \, \|D_i u_i\| \, . \tag{6.15}$$

Now a substitution (6.13) and (6.14) into (6.15) and application of Lemma 5.1 gives

$$\dot{v}_i \leq -\frac{1}{2} c_{i3} \|x_i\|^2 + \frac{c_{i4}^2}{2c_{i3}} \left(\sum_{\substack{j=1 \\ j \neq i}}^{m} \|D_i B_{ij} H_j\| \, \|x_j\| \right)^2$$

which, by Holder's inequality, shows that

$$\dot{v}_i \leq -\frac{1}{2} c_{i3} \|x_i\|^2 + \frac{c_{i4}^2}{2c_{i3}} \sum_{\substack{j=1 \\ j \neq i}}^{m} \|D_i B_{ij} H_j\|^2 \sum_{\substack{j=1 \\ j \neq i}}^{m} \|x_j\|^2 \, .$$

Now a reintroduction of the inequalities

$$c_{i1} \|x_i\|^2 \leq v_i(x_i, t) \leq c_{i2} \|x_i\|^2 \quad i = 1, \ldots, m$$

and a use of the relation $C_{ij} = D_i B_{ij} H_j$ yields

$$\dot{v}_i \leq -\frac{c_{i3}}{2c_{i2}} v_i + \frac{c_{i4}^2}{2c_{i3}} \sum_{j=1}^{m} \|C_{ij}\|^2 \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{v_j}{c_{j1}} \, .$$

The resulting system is

$$\left\{ \begin{array}{lll} \dot{v}_1 \leq -\dfrac{c_{13}}{2c_{12}} v_1 + & \left[\left(\dfrac{c_{14}^2}{2c_{13}} \displaystyle\sum_{j=1}^{m} \|C_{1j}\|^2 \right) \left(\displaystyle\sum_{j=2}^{m} \dfrac{v_j}{c_{j1}} \right) \right. \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{v}_m \leq -\dfrac{c_{m4}^2}{2c_{m2}} v_m + & \left. \left(\dfrac{c_{m4}^2}{2c_{m3}} \displaystyle\sum_{j=1}^{m} \|C_{mj}\|^2 \right) \left(\displaystyle\sum_{j=1}^{m-1} \dfrac{v_j}{c_{j1}} \right) \right] \end{array} \right.$$

or

$$\dot{v} \leq A v \ , \tag{6.16}$$

where $v = \mathrm{col}[v_1, v_2, \ldots, v_m]$ and $\leq$ means each component is $\leq$. Now by Lemma 6.2, $v(t) \leq r(t)$. Thus,

$$\|x_i(t;x_o, t_o)\|^2 \leq \frac{1}{c_{i1}} v(t) \leq r(t) \quad i = 1, \ldots, m$$

and so ASL of the solution $r \equiv 0$ of (6.11) implies ASL of the solution $x \equiv 0$ of the composite system.

This result does not have the intuitive appeal found in the results of Theorems 6.1 and 6.2. That is, there is nothing like a loop gain to suggest that the resulting stability criteria is reasonable. However, this is not surprising since the same difficulty is encountered even in a linear composite system with arbitrary interconnections. Theorems 6.1 and 6.2 are now corollaries to Theorem 6.4 when the C matrix has the form C_c or C_L.

6.3 Examples

As noted in the introduction, there is no known previous work that has expressed the viewpoint suggested in this report. On the other hand, many previous results and some interesting new results can be obtained using the techniques developed above. A few examples are given in this section.

Example 6.1: Aizerman's Conjecture

As noted in several previous examples (Examples 4.1, 4.2, and 5.3), a transfer system modeled by the first-order equation

$$\dot{x} = -f(x) + bu(t) \ ,$$

where $0 < cx^2 \leq xf(x) < \infty$ for $x \neq 0$, $f(0) = 0$ and $b \geq 0$ is in Class E and has a gain $\eta \leq \frac{b}{c}$. Now consider a simple closed loop of n such systems modeled by the first-order (scalar) equations

$$\dot{x}_i = -f_i(x_i) + b_i u_i(t) \qquad 0 < c_i x_i^2 \leq x_i f_i(x_i) < \infty \tag{6.17}$$

for $i = 1, 2, \ldots, n$. Here it is required that for each of the system models

$$c_i = \max \left\{ \alpha \,\middle|\, |\alpha x_i| \leq |f_i(x_i)| \text{ for all } x_i \in E^1 \right\} \tag{6.18}$$

so that an accurate gain estimate is obtained. The individual gains are then $\frac{b_i}{c_i}$ and the loop will be ASL if

$$\prod_{i=1}^{n} \frac{b_i}{c_i} < 1 \ .$$

Aizerman's Conjecture (see Section 2.4), in its most general form, implies the following problem: Consider an n^{th}-order equation

$$\dot{x} = f(x) \qquad f(0) = 0 \tag{6.19}$$

and the related parametric family of linear equations

$$\dot{y} = J(x)\, y \tag{6.20}$$

where J(x) is the Jacobian Matrix $\left[\frac{\partial f_i}{\partial x_j}\right]$. When does the asymptotic stability of (6. 20) for all values of $x \in E^n$ imply asymptotic stability-in-the-large of the null solution of (6. 19)? This question has received a considerable amount of attention in the control and applied mathematics literature [Refs. 2, 12, 15, 16, 18, 33] The best results to date consider only fourth-order systems In the following example, Theorem 6. 2 is applied in obtaining an answer for an n^{th}-order system.

In the composite system which is a loop of the subsystems modeled by (6. 17), the composite system model corresponding to (6. 19) is

$$\begin{cases} \dot{x}_1 = -f_1(x_1) + b_1 x_n , \\ \dot{x}_2 = -f_2(x_2) + b_2 x_1 , \\ \cdots\cdots\cdots\cdots \\ \dot{x}_n = -f_n(x_n) + b_n x_{n-1} , \end{cases}$$

and the related linear equations corresponding to (6. 20) are

$$\dot{y} = \begin{bmatrix} -\frac{\partial f_1(x_1)}{\partial x_1} & 0 & 0 & \cdots & 0 & b_1 \\ b_2 & -\frac{\partial f_2(x_2)}{\partial x_2} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & b_n & -\frac{\partial f_n(x_n)}{\partial x_n} \end{bmatrix} y .$$

The stability of the linear system depends on the roots of the characteristic equation

$$\prod_{i=1}^{n} \left[- \frac{\partial f_i(x_i)}{\partial x_i} - \lambda \right] + (-1)^{n+1} \prod_{i=1}^{n} b_i = 0 ,$$

or

$$\prod_{i=1}^{n} (\lambda + \frac{\partial f_i}{\partial x_i}) - \prod_{i=1}^{n} b_i = 0 . \tag{6.21}$$

This equation will have LHP roots if and only if (see Lemma 6.3):

$$\frac{\partial f_i(x_i)}{\partial x_i} > 0 \quad (i = 1, \ldots, n) \text{ for all } x \in E^n \tag{6.22}$$

and

$$\prod_{i=1}^{n} \frac{\partial f_i(x_i)}{\partial x_i} - \prod_{i=1}^{n} b_i > 0 \text{ for all } x \in E^n . \tag{6.23}$$

Now (6.18) and (6.22) together imply that for each $i = 1, \ldots, n$ there is a $x_i' \in E^1$ such that $\frac{\partial f_i(x_i')}{\partial x_i} = c_i > 0$ and thus,

$$\prod_{i=1}^{n} \frac{\partial f_i(x_i)}{\partial x_i} \geq \prod_{i=1}^{n} c_i > 0 \quad \text{for all } x \in E^n ,$$

$$\prod_{i=1}^{n} \frac{\partial f_i(x_i')}{\partial x_i} = \prod_{i=1}^{n} c_i > 0 \quad \text{for } x' = (x_1', x_2', \ldots, x_n').$$

Equation (6.23) implies that

$$\prod_{i=1}^{n} \frac{\partial f_i(x_i)}{\partial x_i} > \prod_{i=1}^{n} b_i > 0 \quad \text{for all } x \in E^n ,$$

and thus (6. 18), (6. 22), and (6. 23) together imply that

$$\prod_{i=1}^{n} c_i > \prod_{i=1}^{n} b_i \quad ,$$

or

$$\prod_{i=1}^{n} \frac{b_i}{c_i} < 1 \ .$$

That is, the stability of the family of linear equations (6. 20) implies that the estimated loop gain of the composite system (6. 19) is less than unity. By Theorem 6. 2, the composite system is ASL and Aizerman's Conjecture holds.

This is apparently the first verification of Aizerman's Conjecture for an n^{th}-order system. Moreover, (6. 22) suggests a root locus approach to the stability of this nonlinear system. To see this replace b_1 with μb_1, where $\mu \geq 0$, and write

$$\prod_{i=1}^{n} \left[\lambda + g_i(x_i) \right] - \mu \prod_{i=1}^{n} b_i = 0 \qquad (6.24)$$

where $g_i(x_i) = \dfrac{\partial f_i(x_i)}{\partial x_i}$. Normal root locus procedures can then be used on (6. 24) if one notes that the pole positions depend on x (also note that this root locus provides only a sufficient condition for stability). This is still a formidable task if all the f_i's are nonlinear. However, if only a few of the f_i's are nonlinear, a useful stability criterion for (6. 19) is obtained.

Example 6.2: A Complex Composite System

In a recent paper, Markus and Yamabe [Ref. 30] have considered the n^{th}-order system of equations

$$\dot{x} = f(x) \tag{6.25}$$

where

$$\begin{aligned}
&(a)\quad \frac{\partial f_i}{\partial x_j}(x) = 0 \quad \text{for } j > i \text{ and for all } x \in E^n,\\
&(b)\quad \frac{\partial f_i}{\partial x_i}(x) < 0 \quad \text{for } i = 1,\dots,n \text{ and for all } x \in E^n,\\
&(c)\quad f(x) = 0 \quad \text{if and only if } x = 0 .
\end{aligned} \tag{6.26}$$

They have shown that the null solution, $x = 0$, of the system of equations (6.25) is ASL under the restrictions provided in (6.26).

When (6.25) is viewed as an interconnection of first-order transfer systems, this result is quite reasonable. The condition $\frac{\partial f_i}{\partial x_j} = 0$ for $j > i$ shows that the composite system is basically a chain with inputs to each transfer system coming only from the preceding system as shown in Fig. 10.

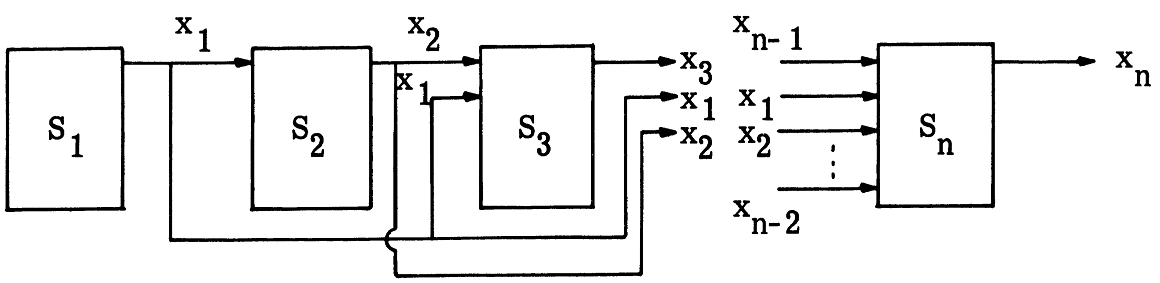

Fig. 10. A complex chain.

The stability of each of the transfer systems is ensured by the second and third assumptions. Since there are no closed loops, one would expect this very simple stability criteria.

While it is not presently possible to prove this theorem as stated using the techniques suggested in this paper, a proof of a slightly weaker version is given below. Moreover, the validity of the general result suggests that the viewpoint adopted of the present paper can be extended to a broader class of problems.

Theorem 6.5: If each of the first-order equations in (6.25) is of the form

$$\dot{x}_i = b_{i1}x_1 + b_{i2}x_2 + \dots + b_{i,i-1}x_{i-1} + f_i(x_i) \qquad i = 1, \dots, n \tag{6.27}$$

where $b_{10} \equiv 0$, then the null solution of the (6.25) system will be ASL under the restrictions

$$\begin{array}{lllll} \text{(a)} & \dfrac{\partial f_i}{\partial x_j}(x_i) & = & 0 & \text{for } j > i \text{ and for all } x \in E^n \ , \\ \text{(b)} & \dfrac{\partial f_i}{\partial x_i}(x_i) & \leq & -c_i < 0 & \text{for } i = 1, \dots, n \text{ and for all } x \in E^n \ , \\ \text{(c)} & f(x) & = & 0 & \text{if and only if } x = 0 \ . \end{array} \tag{6.28}$$

Proof: Although Theorem 6.4 applies in this case, a step-by-step construction of an auxiliary system will be used to illustrate the procedure. For the i^{th} subsystem take the Lyapunov function

$v_i(x_i) = \frac{1}{2} x_i^2$ so that

$$\dot{v}_i = x_i b_{i1} u_{i1} + \dots + x_i b_{i,i-1} u_{i,i-1} + x_i f_i(x_i) \quad i = 1, \dots, n$$

where u_{ij} are the inputs to i^{th} subsystems ($u_{ij} = x_j$ when the interconnections are made) and $u_{10} \equiv 0$. Let $b = \max\limits_{i,j} b_{ij}$ and note that restrictions (b) and (c) imply that $x_i f_i(x_i) \leq -c_i x_i^2$. Thus,

$$\dot{v}_i \leq -c_i x_i^2 + b(u_{i1} + \dots + u_{i,i-1})\, x_i \qquad i = 1, \dots, n$$

or

$$\dot{v}_i \leq -\frac{c_i}{2} x_i^2 + \frac{b^2(u_{i1} + \dots + u_{i,i-1})^2}{2c_i}$$

when Lemma 5.1 is used. Now note that

$$(u_{i1} + \dots + u_{i,i-1})^2 \leq n u_{i1}^2 + \dots + n u_{i,i-1}^2 ,$$

so that

$$\dot{v}_i \leq -\frac{c_i}{2} x_i^2 + \frac{nb}{c_i} \left(\frac{u_{i1}^2}{2} + \dots + \frac{u_{i,i-1}^2}{2} \right) .$$

When the interconnections are made $u_{ij} = x_j$ so $\frac{u_{ij}^2}{2} = v_j$ and there results a system of linear differential inequalities

$$\begin{cases} \dot{v}_1 \leq -c_1 v_1 , \\ \dot{v}_2 \leq -c_2 v_2 + \frac{nb}{c_2} v_1 , \\ \dot{v}_3 \leq -c_3 v_3 + \frac{nb}{c_3} v_1 + \frac{nb}{c_3} v_2 , \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{v}_n \leq -c_n v_n + \frac{nb}{c_n} v_1 + \dots + \frac{nb}{c_n} v_{n-1} . \end{cases} \tag{6.29}$$

It is obvious that the corresponding system of auxiliary equations (equations obtained when the inequalities are replaced by equalities) has a null solution that is ASL. Thus, the null solution of (6. 2) is ASL by Lemma 6. 2. Since $v_i(x_i) = \frac{1}{2} x_i^{\,2}$, the null solution of the system of equation (6. 27) will also be ASL.

Example 6. 3: Another Complex Composite System

Consider the ninth-order composite system shown in Fig. 11.

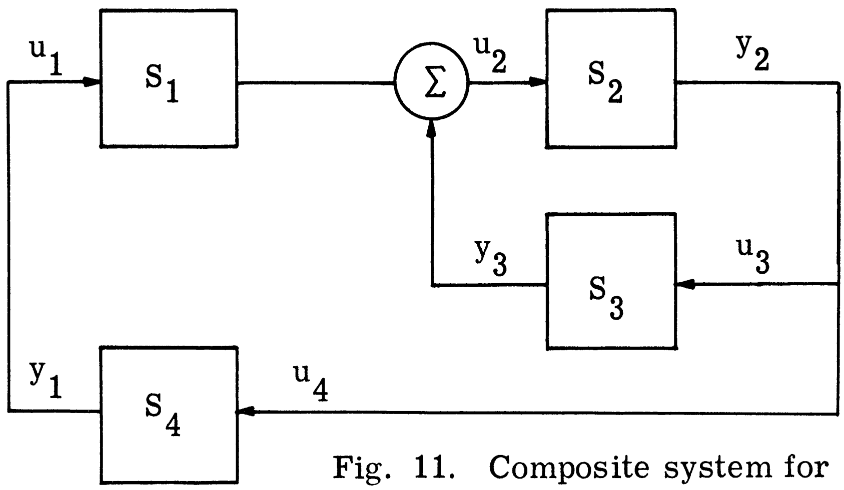

Fig. 11. Composite system for Example 6. 3.

The individual transfer systems are assumed to have the following models:

S_1 : Linear, constant-coefficient, third-order

$$\begin{cases} \dot{x}_1 = A_1 x_1 + D_1 u_1(t) \ , \\ y_1 = H_1 x_1 \ , \end{cases} \tag{6.30}$$

where

$$A_1 = S \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix} S^{-1} \ , \tag{6.31}$$

and S is any nonsingular 3 x 3 matrix.

S_2: Linear, variable coefficient, second-order

$$\begin{cases} \dot{x}_2 = A_2(t)\, x_2 + D_2\, u_2(t) , \\ y_2 = H_2 x_2 , \end{cases} \tag{6.32}$$

where

$$A_2(t) = \begin{bmatrix} 0 & a(t) \\ -1 & -2a(t) \end{bmatrix} , \tag{6.33}$$

and a(t) is a continuous real valued function. In addition, it is assumed that $a^{-1}(t)$ exists for all $t \in T$,

$$0 \leq \frac{d\, a^{-1}(t)}{dt} \leq 1 , \tag{6.34}$$

and

$$.5 \leq a^{-1}(t) \leq 1 . \tag{6.35}$$

S_3: Nonlinear, first-order

$$\begin{cases} \dot{x}_3 = f_3(x_3) + D_3\, u_3(t) , \\ y_3 = H_3\, x_3 , \end{cases} \tag{6.36}$$

where

$$f_3(x_3) = -x_3 - \frac{1}{2} \sin 2\, x_3 . \tag{6.37}$$

S_4: Nonlinear, third-order

$$\begin{cases} \dot{x}_4 = f_4(x_4) + D_4\, u_4(t) , \\ y_4 = H_4\, x_4 , \end{cases} \tag{6.38}$$

where $f_4(0) = 0$.

The problem here is to determine a value of a positive constant k that will insure that the composite system is ASL if

$$x_4' f_4(x_4) \leq -k \|x_4\|^2 < 0 \quad \text{for } x_4 \neq 0 . \tag{6.39}$$

The interconnections suggested in Fig. 11 are assumed to be

$$\begin{aligned} u_1 &= y_4 , \\ u_2 &= y_1 + B_{23} y_3 , \\ u_3 &= y_2 , \\ u_4 &= y_2 , \end{aligned}$$

and the resulting interconnection matrix B [see (3.3)] has the partitioned form

$$B = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & I \\ \hline I & 0 & B_{23} & 0 \\ \hline 0 & I & 0 & 0 \\ \hline 0 & I & 0 & 0 \end{array}\right] , \tag{6.40}$$

where I is an identity matrix. Because of the form of the models for S_1 through S_4 it is possible to describe these interconnections with the matrix C whose partitioned elements are $C_{ij} = D_i B_{ij} H_j$. In this case, C has the partitioned form

$$C = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & C_{14} \\ \hline C_{21} & 0 & C_{23} & 0 \\ \hline 0 & C_{32} & 0 & 0 \\ \hline 0 & C_{42} & 0 & 0 \end{array}\right] , \qquad (6.41)$$

and the composite system model is

$$\dot{x} = f(x,t) + Cx , \qquad (6.42)$$

where $x = \mathrm{col}\,[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{31}, x_{41}, x_{42}, x_{43}]$ is the composite system state vector. When writtén out in detail, the composite system model has the form

$$\left\{\begin{aligned}
\dot{x}_{11} &= a_{11}x_{11} + a_{12}x_{12} + a_{13}x_{13} + c_{14}^{11}x_{41} + c_{14}^{12}x_{42} + c_{14}^{13}x_{43} , \\
\dot{x}_{12} &= a_{21}x_{11} + a_{22}x_{12} + a_{23}x_{13} + c_{14}^{21}x_{41} + c_{14}^{22}x_{42} + c_{14}^{23}x_{43} , \\
\dot{x}_{13} &= a_{31}x_{11} + a_{32}x_{12} + a_{33}x_{13} + c_{14}^{31}x_{41} + c_{14}^{32}x_{42} + c_{14}^{33}x_{43} , \\
\dot{x}_{21} &= a(t)x_{22} + c_{21}^{11}x_{11} + c_{21}^{12}x_{12} + c_{21}^{13}x_{13} + c_{23}^{11}x_{31} , \\
\dot{x}_{22} &= -x_{21} - 2a(t)x_{22} + c_{21}^{21}x_{11} + c_{21}^{22}x_{12} + c_{21}^{23}x_{13} + c_{23}^{21}x_{31} , \\
\dot{x}_{31} &= f_3(x_{31}) + c_{32}^{11}x_{21} + c_{32}^{12}x_{22} , \\
\dot{x}_{41} &= f_{41}(x_{41}, x_{42}, x_{43}) + c_{42}^{11}x_{21} + c_{42}^{12}x_{22} , \\
\dot{x}_{42} &= f_{42}(x_{41}, x_{42}, x_{43}) + c_{42}^{21}x_{21} + c_{42}^{22}x_{22} , \\
\dot{x}_{43} &= f_{43}(x_{41}, x_{42}, x_{43}) + c_{42}^{31}x_{21} + c_{42}^{32}x_{22} ,
\end{aligned}\right. \qquad (6.43)$$

where the c_{ij}^{mn}'s are the mn^{th} elements of C_{ij}, the ij^{th} submatrix in the partition of C.

A straightforward approach to this problem would involve choosing a Lyapunov function involving the 9 state variables, evaluating its derivative with respect to (6. 43), and studying this derivative to obtain conditions under which it is negative definite. Clearly, this approach would involve the solution of some very difficult problems. On the other hand, the techniques developed in Section 6. 2 provide a method for obtaining a solution to this problem rather quickly.

The first step is to find Lyapunov functions of the type described in Theorem 5. 2 for each of the transfer systems. This will, of course, be possible if and only if these transfer systems are in Class E.

S_1: Since this is constant-coefficient linear system, standard techniques can be used (see Theorem 2. 1) to obtain the Lyapunov function $v(x_1) = x_1' P x_1$ where

$$P = (S^{-1})' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1} ,$$

which has a derivative $v(x_1) = -x_1' Q x_1$ where

$$Q = (S^{-1})' \begin{bmatrix} 4 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 30 \end{bmatrix} S^{-1} .$$

For this Lyapunov function the following inequalities are satisfied

$$\|x_1\|^2 \leq v_1(x_1) \leq 3\|x_1\|^2 ,$$

$$\dot{v}_1(x_1) \leq -4\|x_1\|^2 ,$$

$$\|\nabla v_1(x_1)\| \leq 2\|P\| \; \|x_1\| = 6\|x_1\| .$$

S_2: In this case, choose $v_2(x_2, t) = x_2' P(t)\, x_2$ where

$$P(t) = \begin{bmatrix} 2 + a^{-1}(t) & 1 \\ 1 & 1 \end{bmatrix} .$$

Then $\dot{v}_2(x_2, t) = -x_2' Q(t)\, x_2$ where

$$Q(t) = \begin{bmatrix} 2 & 0 \\ 0 & 4a(t) \end{bmatrix} .$$

For this Lyapunov function the inequalities are

$$.5\|x_2\|^2 \leq v_2(x_2, t) \leq 3.5\|x_2\|^2 ,$$

$$\dot{v}_2(x_2, t) \leq -2\|x_2\|^2 ,$$

$$\|\nabla v_2(x_2, t)\| \leq 7\|x_2\| .$$

S_3: Since this model if first-order, $x_3 = x_{31}$. Choose $v_3(x_3) = \frac{1}{2} x_{31}^2$ and then $\dot{v}_3(x_3) = x_{31} f_3(x_{31})$. The inequalities are now

$$\frac{1}{2} x_{31}^{2} \leq v_3(x_3) \leq \frac{1}{2} x_{31}^{2} ,$$

$$v_3(x_3) \leq .56\, x_{31}^{2} ,$$

$$\|\nabla v_3(x_3)\| \leq \|x_{31}\| = |x_{31}| .$$

S_4: In this case, choose $v_4(x_4) = \frac{1}{2} x_4' x_4$ and note that $x_4' f(x_4) \leq -k\|x_4\|^2$. Then

$$\frac{1}{2}\|x_4\|^2 \leq v_4(x_4) \leq \frac{1}{2}\|x_4\|^2 ,$$

$$\dot{v}_4(x_4) \leq -k\|x_4\|^2 ,$$

$$\|\nabla v_4(x_4)\| \leq \|x_4\| .$$

With Lyapunov functions chosen for each of the transfer systems, it is possible to apply Theorem 6. 4. The system of auxiliary equations is, in this case, the fourth-order system

$$\begin{cases} \dot{r}_1 = -\alpha_1 r_1 + \beta_1 r_4 , \\ \dot{r}_2 = -\alpha_2 r_2 + \beta_2 r_1 + \gamma_2 r_3 , \\ \dot{r}_3 = -\alpha_3 r_3 + \beta_3 r_2 , \\ \dot{r}_4 = -\alpha_4 r_4 + \beta_4 r_2 . \end{cases} \tag{6.44}$$

Here,

$$\alpha_1 = \frac{2}{3}, \quad \alpha_2 = \frac{1}{3.5}, \quad \alpha_3 = .56 \quad \alpha_4 = k$$

and

$$\beta_1 = 18 \, \|C_{14}\| \, ,$$

$$\beta_2 = \frac{49}{2} (\|C_{21}\| + \|C_{23}\|) \, ,$$

$$\beta_3 = \frac{2}{.56} \|C_{32}\| \, ,$$

$$\beta_4 = \frac{2}{k} \|C_{42}\| \, ,$$

$$\gamma_2 = 49 (\|C_{21}\| + \|C_{23}\|) \, ,$$

where C_{ij} are submatrices of C. (The numerical values of the above constants are obtained from the bounds on the Lyapunov functions chosen for the individual transfer systems. See Theorem 6. 4 and Eq. 6. 11) According to Theorem 6. 4, the null solution of the composite system (Fig. 11) will be ASL if the null solution of the system of auxiliary equations is ASL. The latter will occur if and only if the roots of the characteristic equation

$$[-\alpha_4(k)-\lambda] [-\alpha_1-\lambda] [(-\alpha_1-\lambda)(-\alpha_3-\lambda)-\gamma_2\beta_3] + \beta_4(k)[(-\alpha_3-\lambda)\beta_1\beta_2] = 0 \tag{6. 45}$$

all have negative real parts. [Here the dependence of α_4 and β_4 on k is indicated as $\alpha_4(k)$ and $\beta_4(k)$.] Thus the original problem has been reduced to a much simpler problem that can be solved by root locus or numerical techniques.

CHAPTER 7

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

7.1 Conclusions

Lyapunov's Second Method is a powerful tool applicable to a broad class of stability problems regardless of order, complexity, or existence of nonlinearities. Unfortunately, the full promise of this important tool has not been realized because of formidable problems encountered in the actual application of LSM to high-order, nonlinear systems. These problems are encountered at basically two points: the selection of a suitable Lyapunov function and the determination of suitable properties in the total derivative of this Lyapunov function. It is noted that the difficulty of these two problems is strongly dependent on the order of the system under study. Thus, low-order (third, fourth, even fifth) problems have received a considerable amount of attention while high-order problems are still inaccessible. These same order dependent limitations are present in the several general techniques that have been proposed for constructing Lyapunov functions (Krasovskii, Lur'e, etc.). While most of these techniques are also theoretically applicable to problems of arbitrary order, practical limitations due to algebraic complexity, etc., grow rapidly with the order of the systems being treated.

With these problems in mind, this thesis has described a circumvention of these two points of difficulty in the application of LSM by noting that many high-order systems are actually, or effectively, an interconnection of lower order systems for which both points of difficulty hold less significance. The recognition of this interconnection structure has led to the definition of a composite system as an interconnection of simpler systems, termed transfer systems, and to a study of the properties of these transfer systems with the aid of LSM and the associated concept of the auxiliary equation. These transfer systems are input-output systems and, by using LSM, it has been possible to define a broad class of transfer systems that have what is called a "gain" relating the size of state space regions that are ASL to the size of the input. It has also been found that for composite systems made up of transfer systems belonging to a reasonably broad class, the auxiliary equations for the individual transfer systems can be interconnected following the interconnections existing in the given composite system. There results a system of auxiliary equations whose solutions have the same stability properties as those of the given composite system. Moreover, in the construction of this system of auxiliary equations it has been necessary to find Lyapunov functions only for the lower order transfer systems. The construction of one Lyapunov function for the high-order composite system has actually been avoided. This system of auxiliary equations gives a simple method of determining asymptotic stability-in-the-large for a class of high-order composite systems.

As might be suspected, this simplification of the original problem is not obtained without some attendant sacrifice. The main disadvantage of this approach and a fault common to most attempts to obtain general sufficient conditions for stability is that it is sometimes overly restrictive (overly sufficient) due to a failure to make best use of available information about the detailed structure of the system under analysis. This fine structure is "washed out" at points where matrix norms or absolute values are used. On the other hand, the introduction of transfer systems and interconnection information into stability analysis has increased the class of problems to which LSM has practical application. This is indicated by the proof of the special case of Aizerman's Conjecture and other examples given in Section 6.3.

7.2 Suggestions for Future Research

During the course of the investigation reported here, numerous areas for further investigation have become evident. A few of the more promising are mentioned briefly below.

First is the general concept of vector Lyapunov functions and their application. The proofs of Chapter 6 actually involve the development and use of vector Lyapunov functions whose components are scalar Lyapunov functions for the individual transfer systems. An obvious question is whether these vector Lyapunov functions have other applications. A first trial might be along the lines that scalar Lyapunov functions have proved valuable; for example, optimization problems. The vector Lyapunov functions may also be of value as an aid to finding single scalar Lyapunov functions when such are more desirable.

A second area involves a direct extension of the above approach to a larger class of composite systems. For example, it is important to be able to handle composite systems where one or more of the transfer systems is unstable when isolated from the composite system. This is the well-known problem of feedback stabilization which has been considered by other authors using conventional techniques [Refs. 25,37]. Another important extension would be a strengthening of Theorem 6.2, to obtain stability criteria for composite systems where the loop gain is greater than unity. The theory of linear, constant-coefficient systems suggests that this might be an important class of problems. Both of these extensions will probably require the inclusion of a larger amount of interconnection information since the "polarity" of the individual interconnections will now become important.

A third area for future research involves a study of the importance of Class G and Class E and the relation between them. For example, Class G appears to be larger than Class E. Thus, there may be theorems similar to Theorems 6.1, 6.2, and 6.4 that can be developed for systems in Class G. There may be even broader classes than E or G for which the transfer systems have auxiliary equations of the general form $\mathring{r} = \omega(r, t, u(t))$. The extension of the basic approach used in Chapter 6 to such broader classes would then depend on an extension of Lemma 6.2 to cover the differential inequalities that were encountered.

Finally, the special Lyapunov function $v(x) = \|x\|$ leads to a useful auxiliary equation that has been employed by Sell [Ref. 44], Rosen [Ref. 37] and others. Using this approach, Rosen has reduced certain stability problems to nonlinear programming problems amenable to numerical solution. It would be of interest to consider an extension of this type of procedure to a study of composite systems.

APPENDIX

PROOF OF THEOREM 5.2

Consider the n^{th}-order vector differential equation

$$\dot{x} = f(x,t) \qquad f(0,T) \equiv 0 \tag{A.1}$$

where f(x, t) is in Class B (Def. 5.6). The following two lemmas will be needed in the proof of the theorem. The first is due to Krasovskii (Ref. 18).

Lemma A.1: If f(x, t) is in Class B with $\left|\frac{\partial f_i}{\partial x_j}\right| < L \; (i, j = 1, \ldots, n)$ and $x(t; x_o, t_o)$ is a solution of (A.1), then

$$\|x(t;x_o,t_o)\| \geq \|x_o\| \, e^{-nL(t-t_o)} . \tag{A.2}$$

Proof: Since $f \epsilon B$ with $\left|\frac{\partial f_i}{\partial x_j}\right| < L$ it follows easily that $|f_i(x,t)| \leq L\|x\|$. Now note that

$$\frac{d}{dt}\|x\|^2 = 2\sum_{i=1}^{n} x_i \dot{x}_i . \tag{A.3}$$

Since

$$|x_i f_i(x,t)| \leq |x_i| \, L\|x\| \leq L\|x\|^2 ,$$

then

$$x_i \dot{x}_i = x_i f_i(x,t) \geq -|x_i f_i(x,t)| \geq -L\|x\|^2 ,$$

and thus

$$\frac{d}{dt}\|x\|^2 \geq -2\sum_{i=1}^{n} L\|x\|^2 = -2n\,L\|x\|^2 .$$

Then, as in the proof of Lemma 4.1,

$$\|x(t;x_o, t_o)\|^2 \geq \|x_o\|^2\, e^{-2n\,L(t-t_o)}$$

and (A.2) follows immediately.

The second lemma, due to Nemytskii and Stepanov (Ref. 34), shows the continuity of the solutions of (A.1) in the initial state.

Lemma A.2: If $x(t;x_o, t_o)$ is a solution to (A.1), then

$$\left|\frac{\partial x_j(t;x_o, t_o)}{\partial x_{io}}\right| \leq n\, e^{nL(t-t_o)} \qquad \text{(A.4)}$$

for $i = 1, \ldots, n$; $j = 1, \ldots, n$ and $t \geq t_o$ (x_{io} is the i^{th} component of x_o).

Proof: See Ref. 34, p. 14.

It is now possible to prove Theorem 5.2 which is restated below.

Theorem A.1: The null solution of (A.1) is ESL if and only if there is a positive definite function $v(x, t)$ such that

(a) $c_1\|x\|^2 \leq v(x, t) \leq c_2\|x\|^2$,

(b) $\overset{\circ}{v}(x, t) \leq -c_3\|x\|^2$,

(c) $\|\nabla v\| \leq c_4\|x\|$,

where c_1, c_2, c_3, and c_4 are positive constants.

Proof: (1) Sufficiency: if there is a positive definite function satisfying requirements (a), (b), and (c) above, then

$$\dot{v} \leq -\frac{c_3}{c_2} v ,$$

and by Lemma 6.1,

$$v \leq v_o e^{-\frac{c_3}{c_2}(t-t_o)} .$$

Thus

$$\|x(t;x_o, t_o)\|^2 \leq \frac{c_2}{c_1} \|x_o\|^2 e^{-\frac{c_3}{c_2}(t-t_o)} ,$$

or

$$\|x(t;x_o, t_o)\| \leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} \|x_o\| e^{-\frac{c_3}{2c_2}(t-t_o)}$$

which shows that the null solution of (A.1) is ESL.

(2) Necessity: If there are positive constants α and β such that for all $(x_o, t_o) \epsilon E^n \times T$.

$$\|x(t;x_o, t_o)\| \leq \beta\|x_o\| e^{-\alpha(t-t_o)} , \qquad (A.5)$$

then let $\tau = \frac{1}{\alpha} \ln \beta\sqrt{2}$ and choose

$$v(x,t) = \int_t^{t+\tau} \|x(\theta;x,t)\|^2 \, d\theta \, . \tag{A.6}$$

Now substituting (A. 5) in (A. 6) gives

$$v(x,t) \le \int_t^{t+\tau} \beta^2 \|x\|^2 e^{-2\alpha(\theta - t)} \, d\theta = c_2 \|x\|^2$$

which proves the right inequality in (a). To prove the left inequality substitute (A. 2) in (A. 6) giving

$$v(x,t) \ge \int_t^{t+\tau} \|x\|^2 e^{-2nL(\theta - t)} \, d\theta = c_1 \|x\|^2 \, .$$

To prove (b), v(x, t) is differentiated with respect to t along the solutions of (A. 1). Thus, when $x = x(t;x_o, t_o)$

$$\begin{aligned} \frac{d\, v(x(t;x_o,t_o),t)}{dt} &= \frac{d}{dt}\left[\int_t^{t+\tau} \|x(\theta;x(t;x_o,t_o),t\|^2 \, d\theta \right] \\ &= + \|x(t+\tau;\, x(t;x_o,t_o),t)\|^2 \\ &\quad - \|x(t;x(t;x_o,t_o),t\|^2 \\ &\quad + \int_t^{t+\tau} \frac{\partial}{\partial t} \|x(\theta;\, x(t;x_o,t_o),t\|^2 \, d\theta \end{aligned} \tag{A.7}$$

But $x(\theta;\, x(t+\Delta t;\, x_o, t_o),\; t+\Delta t) = x(\theta;x(t;x_o,t_o),t)$ so the derivative inside the integral in the last term on the right in (A. 7) is zero.

From (A. 5) and the fact that $\tau = \frac{1}{\alpha} \ell n\, \beta\sqrt{2}$ it is clear that

$$\|x(t+\tau; x(t;x_o, t_o), t)\|^2 \leq \beta^2 \|x(t;x_o, t_o)\|^2 e^{-2\alpha(t+\tau-t)}$$

$$= \frac{1}{2} \|x(t;x_o, t_o)\|^2$$

Thus, at the point $x(t;x_o, t_o)$,

$$\frac{dv}{dt} \leq \frac{1}{2} \|x(t;x_o, t_o)\|^2 - \|x(t;x_o, t_o)\|^2 = -\frac{1}{2} \|x(t;x_o, t_o)\|^2$$

which proves (b). Now to prove (c) note that

$$\frac{\partial v(x,t)}{\partial x_j} = 2 \int_t^{t+\tau} \sum_{i=1}^{n} x_i(\theta;x,t) \frac{\partial x_i(\theta;x,t)}{\partial x_j} d\theta . \qquad \text{(A. 8)}$$

Substitution of (A. 5) and (A. 4) in (A. 8) gives

$$\left|\frac{\partial v(x,t)}{\partial x_j}\right| \leq \int_t^{t+\tau} 2n^2 \beta \|x\| e^{(nL-\alpha)(\theta-t)} d\theta = k\|x\| .$$

Now

$$\|\nabla v\|^2 = \sum_{j=1}^{n} \left|\frac{\partial v}{\partial x_j}\right|^2 \leq n k^2 \|x\|^2$$

so

$$\|\nabla v\| \leq c_4 \|x\| ,$$

proving (c) and concluding the proof of the theorem.

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