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The Propagation of Discontinuities Along Characteristic Surfaces in Non-Equilibrium Fluids*

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1. INTRODUCTION

We will determine the ordinary differential equation which governs the growth and decay, along the bicharacteristic curves, [1, p. 597] of the discontinuities in the derivatives of the solutions to systems of quasi-linear partial differential equations. The technique will then be used to study the growth of discontinuities in the physical variables appearing in the equations of a compressible nonequilibrium fluid.

Our results will be stated for solutions which are in the Lichnerowicz class (C^1, C^3), [2, p. 3], but the technique applies to other classes also. The method consists of using Coburn's representation [3, Definition 5] for the derivatives of the unknown functions in order to modify an approach due to Courant [1, p. 618] for the linear case.

The growth of discontinuities in the equilibrium case has been studied by Kaul [4] for a shallow liquid, and by Gopalkrishna [5] for a collisionless plasma, each author using the methods of T. Y. Thomas. The linearized equations for a nonequilibrium system have been analyzed by Gelfand [6], Stupochenko and Stakanov [7]. For the one-dimensional equilibrium case, see Jeffrey and Tanuiti [8].

In this paper, we will apply our general results to the nonlinear equations of a nonequilibrium fluid to obtain sufficient conditions for the growths of the discontinuities along the bicharacteristics corresponding to a particular speed of propagation.

2. THE ORDINARY DIFFERENTIAL EQUATION FOR THE GROWTH OF DISCONTINUITIES

Throughout this section, we will consider quasi-linear systems of the form[1]

$$A^{\tau}(x, u)\, \partial_{\tau} u + B(x, u)\, u + C(x, u) = 0, \qquad \tau = 0, 1, \dots n, \tag{2.1}$$

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[1] We use the form (2.1) rather than $A^{\tau}(x, u)\, \partial_{\tau} u + E(x, u) = 0$ in order to compare results more easily with Courant [1].

where $\partial_\tau u \equiv \partial u/\partial x_\tau$ and x_τ is an orthogonal coordinate system in the usual Euclidean metric; u is a $k \times 1$ matrix of unknown scalar functions, each of which will be denoted by $\overset{a}{u}$, $a = 1,\ldots, k$; $A^\alpha(x, u)$, $\alpha = 0,\ldots, n$ is a $k \times k$ matrix with entries $(A^\alpha)_{ij}$ such that for each fixed i, j the collection $(A^\alpha)_{ij}$, $\alpha = 0,\ldots, n$ transforms as a vector; B and C are, respectively, $k \times k$ and $1 \times k$ matrices of known scalar functions. We assume A^α, B, $C \in C^2$ in all their variables.

Motivated by the desire to study (C^1, C^3) solutions of (2.1), we introduce the system [see Eq. (2.2)] obtained from Eq. (2.1) by differentiation. Before doing this, we make the following convenient definition:

Definition 1A. The operation of differentiation by the variable $\overset{a}{u}$ will be denoted by D_a:

$$D_a \equiv \partial_{\overset{a}{u}}, \qquad a = 1,\ldots, k.$$

With this definition, differentiation of Eq. (2.1) leads to

$$\begin{aligned} A^\tau\partial_\tau\partial_\lambda u &+ \Big(\partial_\lambda A^\tau + \sum_a (D_a A)\, \partial_\lambda \overset{a}{u}\Big)\, \partial_\tau u + B\partial_\lambda u \\ &+ \Big(\partial_\lambda B + \sum_a (D_a B)\, \partial_\lambda \overset{a}{u}\Big)\, u + \partial_\lambda C + \sum_a (D_a C)\, \partial_\lambda \overset{a}{u} = 0. \end{aligned} \tag{2.2}$$

Definition 1. Following Courant [1, p. 597], we denote by l, r respectively any nonzero solution of $lA^\alpha n_\alpha = 0$, $A^\alpha n_\alpha r = 0$ and refer to l, r, respectively, as left, right null vectors of $\mathcal{A} \equiv A^\alpha n_\alpha$.

Remark 1. If $u \in (C^1, C^3)$ is a solution of Eq. (2.2), then by a known result [3, Definitions 5, 8, Theorems 2, 3] there are scalars $\overset{a}{J}$, $a = 1,\ldots, k$ such that

$$[\partial_\alpha\partial_\beta \overset{a}{u}] = n_\alpha n_\beta \overset{a}{J}, \qquad a = 1,\ldots, k. \tag{2.3}$$

Definition 2. The tuple $\overset{a}{J}$ is called the jump tuple for the solution u, and will be denoted by J.

Proposition 1. *If $u \in (C^1, C^3)$ is a solution of Eq. (2.2) with nonzero jump tuple J, then there is a scalar σ and a right null vector $r \equiv (\overset{1}{r}, \overset{2}{r},\ldots, \overset{k}{r})^T$ such that (where T indicates the transposed matrix)*

$$J = \sigma r, \qquad (\overset{1}{r})^2 + \cdots + (\overset{k}{r})^2 = 1.$$

Proof. Forming the jump of Eq. (2.2) and noting $u \in C^1$, we obtain

$$A^\alpha[\partial_\alpha\partial_\lambda u] = 0. \tag{2.4}$$

Using Definition 2 and (2.3), this becomes

$$A^{\alpha}n_{\alpha}J = 0 \tag{2.5}$$

and the result follows, by taking the unit vector r which we defined by

$$\frac{J}{((\overset{1}{J})^2 + \cdots + (\overset{k}{J})^2)^{1/2}}.$$

Remark 2. As in Courant's theory, it is clear from Eq. (2.5) that at each point of the discontinuity surface where $J \neq 0$, the normal n_{α} satisfies the determinant relation $\| A^{\alpha}n_{\alpha} \| = 0$.

Remark 3. The operator $lA^{\alpha}\partial_{\alpha}$ is a tangential operator since $lA^{\alpha}n_{\alpha} = 0$.

DEFINITION 3. The integral curves of $b^{\alpha} \equiv lA^{\alpha}r$ are called bicharacteristics of $\mathfrak{A}$ (cf. [1, p. 597]).

DEFINITION 4. Let $\overset{a}{u} \in E_k$ and $u \to f(u)$ be a real-valued function. If $u = u(x_0, \ldots, x_n)$, then let $\bar{f}$ be the function on E_{n+1} defined by

$$\bar{f}(x_0, \ldots, x_n) \equiv f(\overset{1}{u}(x_0, \ldots, x_n), \ldots, \overset{k}{u}(x_0, \ldots, x_n)).$$

For example

$$\overline{\partial_{\lambda}A^{\tau}} \equiv \partial_{\lambda}A^{\tau}(x_0, \ldots, x_n, \overset{1}{u}(x_0, \ldots, x_n), \ldots \overset{k}{u}(x_0, \ldots, x_n)).$$

In the proof of the main result, we will use the following:

PROPOSITION 2. *Let r be a right null vector of $\mathfrak{A}$ defined on a neighborhood $\mathcal{N}$ of Σ and suppose the first partials of r exist on Σ. If l is a left null vector of $\mathfrak{A}$ on Σ, then on Σ*

$$-\, ln^{\lambda}(\partial_{\lambda}\bar{A}^{\beta})\, n_{\beta}r = n^{\epsilon}b^{\tau}\partial_{\epsilon}n_{\tau}. \tag{2.6}$$

Proof. By hypothesis $\bar{A}^{\beta}n_{\beta}r = 0$ on a neighborhood of Σ. Thus we find

$$0 = \partial_{\alpha}(\bar{A}^{\beta}n_{\beta}r) = (\partial_{\alpha}\bar{A}^{\beta})\, n_{\beta}r + \bar{A}^{\beta}(\partial_{\alpha}n_{\beta})\, r + \bar{A}^{\beta}n_{\beta}\partial_{\alpha}r. \tag{2.7}$$

Operating with l on each side of Eq. (2.7) and using $lA^{\beta}n_{\beta} = 0$ on Σ, we obtain

$$0 = l(\partial_{\alpha}\bar{A}^{\beta})\, n_{\beta}r + l\bar{A}^{\beta}(\partial_{\alpha}n_{\beta})\, r. \tag{2.8}$$

Rearranging (2.8) and using Definition 3, we obtain

$$-\, l(\partial_{\alpha}\bar{A}^{\beta})\, n_{\beta}r = b^{\beta}\partial_{\alpha}n_{\beta}. \tag{2.9}$$

Taking the scalar product with n^α

$$-\, l n^\alpha (\partial_\alpha \bar{A}^\beta)\, n_\beta r = n^\alpha b^\beta \partial_\alpha n_\beta \tag{2.10}$$

which proves the assertion.

PROPOSITION 3. *Suppose* $u \in (C^1, C^3)$ *is a solution of Eq.* (2.2) *and* l *is a left null vector of* $\mathcal{Cl}$ *on* Σ. *Furthermore, suppose* r *satisfies* $A^\alpha n_\alpha r = 0$ *on a neighborhood* $\mathcal{N}$ *of* Σ *and that the first partials of* r *exist on* Σ, *and* $J = \sigma r$ *on* Σ. *Then* σ *satisfies the following ordinary differential equation along the bicharacteristics:*

$$\frac{d\sigma}{db} + M\sigma = 0, \tag{2.11}$$

where

$$\begin{aligned} M \equiv l \Big\{ \bar{A}^\beta \partial_\beta r + \sum_a (\overline{D_a A^\tau})\, \overset{a}{r} \partial_\tau u + \sum_a (\overline{D_a B})\, \overset{a}{r}\, u + \sum_a (\overline{D_a C})\, \overset{a}{r} + \bar{B} r \Big\} \\ -\, 2 n^\epsilon b^\tau \partial_\epsilon n_\tau \end{aligned} \tag{2.12}$$

and b *is the parameter along the bicharacteristic.*

Proof. From Eq. (2.3) and Proposition 1, we obtain

$$[\partial_\alpha \partial_\lambda u] = n_\alpha n_\lambda \sigma r. \tag{2.13}$$

Operating with $l\bar{A}^\beta \partial_\beta$ on Eq. (2.13), we find

$$l\bar{A}^\beta \partial_\beta [\partial_\alpha \partial_\lambda u] = l\bar{A}^\beta \partial_\beta (n_\alpha n_\lambda \sigma r). \tag{2.14}$$

Since $l\bar{A}^\beta \partial_\beta$ is a tangential operator, we have by Hadamard's theorem (i.e., $t^\alpha [\partial_\alpha F] = t^\alpha \partial_\alpha [F]$)

$$l[\bar{A}^\beta \partial_\beta \partial_\alpha \partial_\lambda u] = l\bar{A}^\beta \partial_\beta (n_\alpha n_\lambda \sigma r) \tag{2.15}$$

$$n^\alpha n^\lambda l [\bar{A}^\beta \partial_\beta \partial_\alpha \partial_\lambda u] = n^\alpha n^\lambda l \bar{A}^\beta \partial_\beta (n_\alpha n_\lambda \sigma r). \tag{2.16}$$

Since u is piecewise C^3, we can interchange the order of differentiation yielding,

$$l[\bar{A}^\beta \partial_\beta \partial_\alpha \partial_\lambda u] = l[\bar{A}^\beta \partial_\alpha \partial_\beta \partial_\lambda u]. \tag{2.17}$$

By use of the product rule for differentiation this becomes

$$l[\bar{A}^\beta \partial_\beta \partial_\alpha \partial_\lambda u] = l[\partial_\alpha (\bar{A}^\beta \partial_\beta \partial_\lambda u) - (\partial_\alpha \bar{A}^\beta)\, \partial_\beta \partial_\lambda u]. \tag{2.18}$$

The function u satisfies Eq. (2.2). Hence if we rewrite the first term on the right of Eq. (2.18), we obtain

$$
\begin{aligned}
l[\bar{A}^{\beta}\partial_{\beta}\partial_{\alpha}\partial_{\lambda}u] = -\, l \Big[\partial_{\alpha} \Big\{\Big(\overline{\partial_{\lambda}A^{\tau}} + \sum_{a} (\overline{D_{a}A^{\tau}})\, \partial_{\lambda}\overset{a}{u}\Big)\, \partial_{\tau}u \\
+ \bar{B}\partial_{\lambda}u + \Big(\overline{\partial_{\lambda}B} + \sum_{a} (\overline{D_{a}B})\, \partial_{\lambda}\overset{a}{u}\Big)\, u + \overline{\partial_{\lambda}C} + \sum_{a} (\overline{D_{a}C})\, \partial_{\lambda}\overset{a}{u}\Big\} \\
+ (\partial_{\alpha}\bar{A}^{\beta})\, \partial_{\beta}\partial_{\lambda}u\Big] . \qquad (2.19)
\end{aligned}
$$

Carrying out the differentiation by ∂_{α}, and using the assumptions on the class of A^{α}, B, C, and u, we have

$$
\begin{aligned}
l[\bar{A}^{\beta}\partial_{\beta}\partial_{\alpha}\partial_{\lambda}u] = -\, l \Big\{\overline{\partial_{\lambda}A^{\tau}}[\partial_{\alpha}\partial_{\tau}u] \\
+ \sum_{a} (\overline{D_{a}A^{\tau}})\, \partial_{\lambda}\overset{a}{u}[\partial_{\alpha}\partial_{\tau}u] + \sum_{a} (\overline{D_{a}A^{\tau}})\, [\partial_{\alpha}\partial_{\lambda}\overset{a}{u}]\, \partial_{\tau}u \\
+ \bar{B}[\partial_{\alpha}\partial_{\lambda}u] + \sum_{a} (\overline{D_{a}B})\, [\partial_{\alpha}\partial_{\lambda}\overset{a}{u}]\, u \qquad (2.20) \\
+ \sum_{a} (\overline{D_{a}C})\, [\partial_{\alpha}\partial_{\lambda}\overset{a}{u}] + (\partial_{\alpha}\bar{A}^{\beta})\, [\partial_{\beta}\partial_{\lambda}u]\Big\} .
\end{aligned}
$$

By use of the chain rule we see the sum of the first two terms on the right of Eq. (2.20) is $\partial_{\lambda}\bar{A}^{\tau}[\partial_{\alpha}\partial_{\tau}u]$. Thus Eq. (2.20) becomes

$$
\begin{aligned}
l[\bar{A}^{\beta}\partial_{\beta}\partial_{\alpha}\partial_{\lambda}u] = -\, l \Big\{(\partial_{\lambda}\bar{A}^{\tau})\, [\partial_{\alpha}\partial_{\tau}u] \\
+ \sum_{a} (\overline{D_{a}A^{\tau}})\, [\partial_{\alpha}\partial_{\lambda}\overset{a}{u}]\, \partial_{\tau}u + \bar{B}[\partial_{\alpha}\partial_{\lambda}u] \qquad (2.21) \\
+ \sum_{a} (\overline{D_{a}B})\, [\partial_{\alpha}\partial_{\lambda}\overset{a}{u}]\, u + \sum_{a} (\overline{D_{a}C})\, [\partial_{\alpha}\partial_{\lambda}\overset{a}{u}] + \partial_{\alpha}\bar{A}^{\beta}[\partial_{\beta}\partial_{\lambda}u]\Big\} .
\end{aligned}
$$

Applying Proposition 1 to Eq. (2.21), we obtain

$$
\begin{aligned}
l[\bar{A}^{\beta}\partial_{\beta}\partial_{\alpha}\partial_{\lambda}u] = -\, l\sigma n_{\alpha} \Big\{(\partial_{\lambda}\bar{A}^{\tau})\, n_{\tau}r + \sum_{a} (\overline{D_{a}A^{\tau}})\, n_{\lambda}\overset{a}{r}\partial_{\tau}u + \bar{B}n_{\lambda}r\Big\} \\
\qquad (2.22) \\
-\, l\sigma n_{\lambda} \Big\{\sum_{a} (\overline{D_{a}B})\, n_{\alpha}\overset{a}{r}\, u + \sum_{a} (\overline{D_{a}C})\, n_{\alpha}\overset{a}{r} + (\partial_{\alpha}\bar{A}^{\beta})\, n_{\beta}r\Big\} .
\end{aligned}
$$

Taking the scalar product of (2.22) with $n^\alpha n^\lambda$ we have

$$\begin{aligned} n^\alpha n^\lambda l[\bar{A}^\beta \partial_\beta \partial_\alpha \partial_\lambda u] = &- l\sigma \Big\{ n^\lambda (\partial_\lambda \bar{A}^\tau)\, n_\tau r + \sum_a (\overline{D_a A^\tau})\, \overset{a}{r} \partial_\tau u + \bar{B} r \\ &+ \sum_a (\overline{D_a B})\, \overset{a}{r}\, u + \sum_a (\overline{D_a C})\, \overset{a}{r} + n^\alpha (\partial_\alpha \bar{A}^\beta)\, n_\beta r \Big\} . \end{aligned} \tag{2.23}$$

Now rewriting the first and last terms with the aid of Proposition 2 and rearranging, we obtain

$$\begin{aligned} n^\alpha n l[\bar{A}^\beta \partial_\beta \partial_\alpha \partial_\lambda u] = &- l\sigma \Big\{ \sum_a (\overline{D_a A^\tau})\, \overset{a}{r} \partial_\tau u \\ &+ \sum (\overline{D_a B})\, \overset{a}{r}\, u + \sum (\overline{D_a C})\, \overset{a}{r} + \bar{B} r \Big\} + 2\sigma n^\epsilon b^\tau \partial_\epsilon n_\tau . \end{aligned} \tag{2.24}$$

Returning to (2.16) and expanding the right member, we obtain

$$n^\alpha n^\lambda l \bar{A}^\beta \partial_\beta (n_\alpha n_\lambda \sigma r) = l\bar{A}^\beta (\partial_\beta \sigma)\, r + l\sigma \bar{A}^\beta \partial_\beta r \tag{2.25}$$

or

$$n^\alpha n^\lambda l \bar{A}^\beta \partial_\beta (n_\alpha n_\lambda \sigma r) = \frac{d\sigma}{db} + l\sigma \bar{A}^\beta \partial_\beta r \tag{2.26}$$

where the last equality follows from Definition 2.

The equality of the right members of (2.24) and (2.26) follows from (2.16) and leads after simplification to

$$\begin{aligned} &\frac{d\sigma}{db} + \sigma l \Big\{ \bar{A}^\beta \partial_\beta r + \sum_a (\overline{D_a A^\tau})\, \overset{a}{r} \partial_\tau u + \sum_a (\overline{D_a B})\, \overset{a}{r} u + \sum_a (\overline{D_a C})\, \overset{a}{r} + \bar{B} r \Big\} \\ &\quad - 2\sigma n^\epsilon b^\tau \partial_\epsilon n_\tau = 0. \end{aligned} \tag{2.27}$$

Remark 4. The same technique can be used to derive an ordinary differential equation from (C^k, C^{k+m}) solutions to systems of order $k + 1$, where $k \geqslant 0$. The resulting ordinary differential equation may be linear or nonlinear depending on the hypotheses made. For example if $u \in (C^0, C^2)$ is a solution of (2.1) such that $\partial_\tau u = 0$ on one side of Σ and $\overline{D_a A^\beta}$, $\overline{\partial_\tau C}$, $\overline{\partial_\tau B}$ are all continuous then σ satisfies

$$\frac{d\sigma}{db} + M\sigma + N\sigma^2 = 0$$

where

$$M = l \Big\{ \bar{A}^\beta \partial_\beta r + \bar{B} r + \sum_a (\overline{D_a B})\, \overset{a}{r} u + \sum_a (\overline{D_a C})\, \overset{a}{r} \Big\} - 2 n^\epsilon b^\tau \partial_\epsilon n_\tau$$

$$N = l \sum (\overline{D_a A^\beta})\, \overset{a}{r} n_\beta r.$$

3. Applications to Non-Equilibrium Hydrodynamics

When referred to a Newtonian coordinate system $t \equiv x_0$, x_i, $i = 1, 2, 3$ (i.e., an orthonormal coordinate system in Euclidean 4-space), the equations of compressible, non-equilibrium fluids ([7]) are:

$$A^\alpha \partial_\alpha u + C = 0, \qquad \alpha = 0, 1, 2, 3. \tag{3.1}$$

The unknown u has entries $\overset{1}{u} = \rho$, $\overset{2}{u} = v_1$, $\overset{3}{u} = v_2$, $\overset{4}{u} = v_3$, $\overset{5}{u} = q$, $\overset{6}{u} = s$, where ρ is the invariant mass density, p is the pressure, v_i are the components of the velocity vector referred to a Newtonian coordinate system and q, s are, respectively, the relaxation variable and the specific entropy. Also, the matrices

$$A^0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A^1 \equiv \begin{pmatrix} v_1 & \rho & 0 & 0 & 0 & 0 \\ A & \rho v_1 & 0 & 0 & I & b \\ 0 & 0 & \rho v_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho v_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & v_1 \end{pmatrix}, \tag{3.2}$$

where $A \equiv \partial p/\partial \rho$, $b \equiv \partial p/\partial s$, $I \equiv \partial p/\partial q$ and A^2, A^3 have expressions similar to that of A^1. Finally, $TC \equiv (0, 0, 0, 0, TK(\partial e/\partial q), - K(\partial e/\partial q)^2)^T$, where $e = e(\rho, q, s)$ is the internal energy per gram of mixture and the function K is called the relaxation scalar. Throughout this section we will assume (as in section 2) that $u \in (C^1, C^3)$ and A^α, C are C^2 in all their arguments.

There are basic differences between the systems (3.2) and (2.1) (i.e., the entries $(A^\alpha)_{21}$ do not transform as components of a vector, and the entries of u are not all independent of the choice of coordinate systems). However, the techniques of section two will be useful in obtaining an ordinary differential equation of the form (2.26). We will work in a fixed coordinate system.

As in section two we introduce the following differential system:

$$A^\tau \partial_\tau \partial_\lambda u + \sum_a (D_a A^\tau)\, \partial_\lambda \overset{a}{u} \partial_\tau u + \sum_a (D_a C)\, \partial_\lambda \overset{a}{u} = 0. \tag{3.3}$$

In the remainder of this paper, we will consider only one space variable, i.e., $\alpha = 0, 1$. With this in mind, (2.3) and Remark 2 imply that the characteristics are determined by

$$0 = \| A^\alpha n_\alpha \| = \begin{Vmatrix} L & \rho n_1 & 0 & 0 \\ A n_1 & \rho L & I n_1 & b n_1 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \end{Vmatrix} = \rho L^2 (L^2 - A n_1{}^2), \tag{3.3a}$$

where

$$L \equiv n_0 + v^1 n_1 .$$

For the remainder of this paper, we will assume that $L = 0$ at each point of the discontinuity surface Σ. This assumption is justified by the following:

PROPOSITION 3A. *At each point of the discontinuity surface where $Q \neq 0$ or $S \neq 0$, we have $L = 0$.*

Proof. Taking the jumps of the third and fourth rows of (3.3), respectively, and noting the classes of the functions we obtain

$$LQ = 0 \qquad \text{and} \qquad LS = 0,$$

which proves the assertion.

DEFINITION 5. We will denote the operation of differentiation by the thermodynamic variables ρ, q, s by ∂_ρ, ∂_q, ∂_s, respectively, where there is no danger of confusion the symbol "∂" will be omitted. According to this convention $e_q \equiv \partial_q e$.

PROPOSITION 4. *Suppose that the discontinuity surface satisfies $L = 0$. Then the following hold:*

(i) *Any normalized right null vector of $\bar{A}^\alpha n_\alpha$ has the form*

$$r = (a\alpha + c\beta, 0, a, c)^T \tag{3.3b}$$

where

$$\alpha \equiv -\frac{I}{A}, \qquad \beta \equiv -\frac{b}{A}, \qquad (a\alpha + c\beta)^2 + a^2 + c^2 = 1;$$

(ii) *There are scalars P, Q, S such that the jump matrix J of a (C^1, C^3) solution of* (3.3) *is*

$$J = (P, 0, Q, S)^T; \tag{3.4}$$

(iii) *For each scalar e, f*

$$l \equiv (0, 0, e, f) \tag{3.5}$$

is a left null vector of $\bar{A}^\alpha n_\alpha$.

Proof. First, we will prove (i). The general solution of $\bar{A}^\alpha n_\alpha X = 0$, for $L = 0$, is $(\bar{a}\alpha + \bar{c}\beta, 0, \bar{a}, \bar{c})^T$. Normalizing this last matrix, we obtain (i). Next, we will prove (ii).

By a known result ((3.4) [3]), we have

$$[\partial_\alpha \partial_\beta v_1] = n_\alpha V_{\beta 1} \tag{3.6}$$

and

$$[\partial_\beta \partial_\alpha v_1] = n_\beta V_{\alpha 1} . \tag{3.7}$$

Since v_1 is piecewise C^3, the left members of (3.6), (3.7) are equal. Therefore, we obtain

$$V_{\beta 1} = n_\beta n^\tau V_{\tau 1} . \tag{3.8}$$

By another known result (Theorem 3, [3]), there are scalars P, Q, S such that

$$[\partial_\alpha \partial_\beta p] = n_\alpha n_\beta P, \qquad [\partial_\alpha \partial_\beta q] = n_\alpha n_\beta Q, \qquad [\partial_\alpha \partial_\beta s] = n_\alpha n_\beta S. \tag{3.9}$$

It follows from substituting (3.8), (3.9) in (2.3) that

$$J = (P, n^\tau V_{\tau 1}, Q, S)^T. \tag{3.10}$$

Since both r and J satisfy $\bar{A}^\alpha n_\alpha X = 0$, it follows from (i) that $n^\alpha V_{\alpha 1} = 0$ which proves (ii). Finally, we prove (iii). The matrix $l \equiv (0, 0, e, f)$ is the general solution of $X \bar{A}^\alpha n_\alpha = 0$ for $L = 0$.

PROPOSITION 5. *Let $u \in (C^1, C^3)$ be a solution of* (3.3) *such that along a given bicharacteristic*

$$a \neq 0, \tag{3.10a}$$

$$n^\epsilon \partial_\epsilon n_\tau \neq 0, \tag{3.10b}$$

$$L = 0. \tag{3.10c}$$

Furthermore, suppose the right null vector r defined on Σ by (3.3b) *can be extended to a field of null vectors of $\mathfrak{A}$ which is defined and has first partial derivatives on a neighborhood of Σ. Then*

$$\frac{d\sigma}{dt} + M\sigma = 0, \tag{3.11}$$

where

$$M = \frac{1}{a}\frac{da}{dt} + \partial_q(Ke_q) + \frac{c}{a}\partial_s(Ke_q). \tag{3.12}$$

Proof. From Proposition 4 and the results of Section 2, we have

$$[\partial_\alpha \partial_\lambda u] = n_\alpha n_\lambda J = n_\alpha n_\lambda \sigma r, \tag{3.13}$$

where r is given by (3.3b). Operating with $l\bar{A}^{\beta}\partial_{\beta}$ on each side of (3.13), we obtain

$$l\bar{A}^{\beta}\partial_{\beta}[\partial_{\alpha}\partial_{\lambda}u] = l\bar{A}^{\beta}\partial_{\beta}(n_{\alpha}n_{\lambda}\sigma r) \tag{3.14}$$

the usual rule $\bar{v}_{\alpha} \equiv v_{\beta}\, \partial x^{\beta}/\partial\bar{x}^{\alpha}$ where $v_{\beta} \equiv (1, v_1)$, α, $\beta = 0$, 1 where

$$l \equiv \frac{1}{ae + fc}(0, 0, e, f) \tag{3.15}$$

Remark 3 and Hadamard's theorem imply the relation (3.14) becomes

$$l[\bar{A}^{\beta}\partial_{\beta}\partial_{\alpha}\partial_{\lambda}u] = l\bar{A}^{\beta}\partial_{\beta}(n_{\alpha}n_{\lambda}\sigma r). \tag{3.16}$$

Then proceeding exactly as in the proof of Proposition 3, we find that σ satisfies

$$\frac{d\sigma}{db} + M\sigma = 0, \tag{3.17}$$

where

$$M = l\left\{\bar{A}^{\beta}\partial_{\beta}r + \sum_{a}(\overline{D_a A^{\tau}})\overset{a}{r}\partial_{\tau}u + \sum_{a}(\overline{D_a C})\overset{a}{r}\right\} + 2n^{\epsilon}b^{\tau}\partial_{\epsilon}n_{\tau} \tag{3.18}$$

and $b^{\tau} \equiv l\bar{A}^{\tau}r = (1, v_1)$. This means that the bicharacteristics are parameterized by time in Newtonian coordinate systems. By hypothesis, $n^{\epsilon}\partial_{\epsilon}n_{\tau}$ vanishes in that coordinate system and so the last term in (3.18) vanishes. To find the explicit value of M, take r as given in (3.3b) and choose $l = (0, 0, 1, 0)/a$. Then direct calculation from (3.18) gives

$$M = \frac{1}{a}\frac{da}{dt} + \partial_q(Ke_q) + \frac{c}{a}\,\partial_s(Ke_q). \tag{3.19}$$

Remark 5. The terms in (3.12) involving the scalar

$$a = \frac{Q}{(P^2 + Q^2 + S^2)^{1/2}}$$

are unknown. Our next result will use the freedom in choosing the scalars e, f in (3.5) in order to derive a differential equation which determines a. For simplicity, we will suppose $\alpha = \beta = 0$. However, the same procedure will work for all permissible α, β.

PROPOSITION 6. *Let u be a solution of* (3.3) *satisfying all the hypotheses of Proposition 5 and in addition suppose that along a given bicharacteristic*

$$\alpha = \beta = 0 \tag{3.19a}$$

and

$$c \neq 0, \qquad t \in (t_1, t_2). \tag{3.19b}$$

Then on the interval (t_1, t_2) *the scalar "a" satisfies*

$$\frac{1}{a(1-a^2)}\frac{da}{dt} \pm \frac{(1-a^2)^{1/2}}{a}\frac{\partial(Ke_q)}{\partial s} \pm \frac{a}{(1-a^2)^{1/2}}\frac{\partial\left(\frac{K}{T}e_q^2\right)}{\partial q} + \frac{\partial(Ke_q)}{\partial q} - \frac{\partial\left(\frac{K}{T}e_q^2\right)}{\partial s} = 0 \tag{3.20}$$

where the plus (minus) sign is used according to whether $c = (1-a^2)^{1/2}$ *or* $c = -(1-a^2)^{1/2}$.

Proof. For each pair of scalars e, f not both zero define

$$l(e,f) = \frac{(0, 0, e, f)}{ae} + cf.$$

Then from (3.18) and (3.10b) we have

$$M(e,f) = \frac{1}{ae+cf}\left\{e\frac{da}{dt} + f\frac{dc}{dt} + e\left(a\frac{\partial(Ke_q)}{\partial q} + c\frac{\partial(Ke_q)}{\partial s}\right) - fa\frac{\partial}{\partial q}\left(\frac{K}{T}e_q^2\right) - c\frac{\partial}{\partial s}\left(\frac{K}{T}e_q^2\right)\right\}. \tag{3.21}$$

Direct calculation shows

$$l(e,f)\,\bar{A}^{\alpha}r = (1, v) \equiv b^{\alpha} \tag{3.22}$$

for all e, f. Therefore, if we put $e = 1, f = 0$ we obtain from (3.17),

$$\frac{d\sigma}{dt} + M\sigma = 0, \tag{3.23}$$

where from (3.18) and (3.21)

$$M = M(1,0) = \frac{1}{a}\left\{\frac{da}{dt} + a\frac{\partial}{\partial q}(Ke_q) + c\frac{\partial}{\partial s}(Ke_q)\right\}. \tag{3.24}$$

Similarly, if we put $e = 0, f = 1$, we find (3.23) holds with

$$M = M(0,1) = \frac{1}{c}\left\{\frac{dc}{dt} - a\frac{\partial}{\partial q}\left(\frac{K}{T}e_q^2\right) - c\frac{\partial}{\partial s}\left(\frac{K}{T}e_q^2\right)\right\}. \tag{3.25}$$

Since σ depends only on J and r and not on l, we conclude

$$M(0, 1) = M(1, 0). \tag{3.26}$$

Substituting $c = \pm (1 - a^2)^{1/2}$ into (3.24), (3.25) and then using (3.26) yields the result.

The following can be easily proved.

PROPOSITION 7. *Let $u \in (C^1, C^3)$ be a solution of* (3.3) *satisfying all the hypotheses of Propositions 5 and 6 with a, $c > 0$, and in addition, suppose that along the given bicharacteristic*

$$\lim_{t\to\infty} K \geqslant 0, \tag{3.27}$$

$$\lim_{t\to\infty} K_q > 0, \qquad \lim_{t\to\infty} K_s > 0, \tag{3.28}$$

$$\lim_{t\to\infty} T_q \leqslant 0, \qquad \lim_{t\to\infty} T_s \leqslant 0, \qquad \lim_{t\to\infty} e_{qq} \leqslant 0, \tag{3.29}$$

$$\lim_{t\to\infty} e_q < 0, \tag{3.29a}$$

then

$$\lim_{t\to\infty} Q = \lim_{t\to\infty} S = \infty.$$

Remark 6. In Proposition 7, we have taken $a > 0$, $c > 0$ for all t sufficiently large. For the other possibilities (namely $a > 0, c < 0$; $a < 0, c < 0$) conditions on the limiting behavior of the thermodynamic functions can also be formulated which are sufficient to imply that Q, S becomes infinite as t approaches infinity.

Finally, let us consider the possibility of an infinite discontinuity developing at some finite time t_2.

PROPOSITION 8. *If $t_2 < \infty$ and $Q(t)$ or $S(t)$ become infinite as t approaches t_2, then the domain $a_1 < x < \bar{a}_1$, $a_0 < t < \bar{a}_0$ on which the solution of* (3.3) *is defined must be finite in space and in time.*

Remark 7. The basic equations (3.2), in particular the linear phenomenological relation $dq/dt = - Ke_q$, is intended to describe physical systems which are "near" equilibrium. Objections may be raised that our hypotheses $\lim_{t\to\infty} e_q < 0$ [see Eq. (3.29a)], $\lim_{t\to\infty} e_{qq} \leqslant 0$ [see Eq. (3.29)], contradicts the "near" equilibrium postulate. Equilibrium is usually taken to imply $e_q = 0$, $e_{qq} > 0$.

Addendum

The case $L \neq 0$ will be discussed in a future paper in which the methods of the present paper are extended to the case where the jump equations corresponding to the system (2.2) are non-homogeneous.

Papers [4] and [5] assume that the medium in front of the wave is at rest. In Sections 1-3 of this paper and in a forthcoming paper no such assumption is made.

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